Side branch resonators modelling with Green’s function methods

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Outline

- Introduction
- Impedance matrix and Green’s function formalism
- Simplified models
- Low-frequency applications: Helmholtz and Herschel-Quincke resonators
- Medium-frequency applications: HQ-liner system for inlet fan noise reduction
- Conclusions
Introduction

Side branch resonators are commonly used for engine exhaust noise control: (i) low-frequency applications with a single plane wave mode (automotive) (ii) medium-frequency applications and highly multimodal context (aeronautics).

Figure: Side branch resonator (cavity Ω) with two openings Γ.

Numerical predictions: (i) 1D approximations (with length corrections); (ii) FEM, BEM -> computationally demanding (this includes mesh preparation etc...) especially in a highly multimodal context, lack of physical interpretation.
Impedance matrix

In the frequency domain, the acoustic pressure must obey the Helmholtz equation

\[ \Delta p + k^2 p = 0, \quad k = \omega / c, \]

and \( q = \partial_n p = 0 \) everywhere except on \( \Gamma \). The Green’s function for the rigid-wall cavity is given by the infinite series

\[ G_\Omega(r, r') = \sum_{n=0}^{\infty} \frac{\phi_n(r)\phi_n(r')}{\lambda_n - \lambda} \]

where \( \lambda = \omega^2 \). Eigenfunctions \( \phi_n \) are properly normalized so that application of the Green’s theorem in the cavity yields

\[ p(r) = \int_{\Gamma} G_\Omega(r, r') q(r') \, d\Gamma(r') \]
Impedance matrix

We need to precompute a finite set of eigenfunctions and estimate the truncation error...

Consider the eigenvector $\Phi_n$, the FE discretization of $n$th eigenmode $\phi_n$:

$$\mathbf{A}(\lambda_n)\Phi_n = 0 \quad \text{where} \quad \mathbf{A}(\lambda) = \mathbf{K} - \lambda\mathbf{M}$$

After reduction to the interfacial nodes, we obtain the impedance matrix

$$\mathbf{Z}(\lambda) = \mathbf{I}_I^T \mathbf{A}^{-1}(\lambda) \mathbf{I}_I$$

with

$$\mathbf{A}^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Phi_n \Phi_n^T}{\lambda_n - \lambda} = \Phi \mathbf{D}(\lambda) \Phi^T$$

Finally,

$$\tilde{\mathbf{p}} = \mathbf{Z}(\lambda) \tilde{\mathbf{F}} \mathbf{q} = \mathbf{G}_\Omega \mathbf{q}$$
Truncation

Keeping the first $N$ eigenmodes gives

$$Z(\lambda) = (\tilde{\Phi} D(\lambda) \tilde{\Phi}^T)|_N + R(\lambda).$$

The correction term $R$ remains weakly dependent on the frequency, so we can take the first order Taylor expansion

$$R(\lambda) \approx R(\bar{\lambda}) + (\lambda - \bar{\lambda}) \frac{\partial R}{\partial \lambda} + \ldots$$

The residual matrices are computed via

$$R = I_\Gamma^T A^{-1} I_\Gamma - (\tilde{\Phi} D(\lambda) \tilde{\Phi}^T)|_N$$
$$\partial_\lambda R = I_\Gamma^T A^{-1} M A^{-1} I_\Gamma - (\tilde{\Phi} \partial_\lambda D \tilde{\Phi}^T)|_N$$
Scattering matrix

The theory starts by introducing the hard-walled duct Green’s function

\[ G(r, r') = \sum_{l=0}^{\infty} \frac{\psi_l(x, y)\psi_l^*(x', y')} {2i\beta_l} e^{i\beta_l|z-z'|} \]

The transverse eigenmodes \( \psi_l \) are solution of the boundary value problem

\[ (\partial_{xx}^2 + \partial_{yy}^2)\psi_l + k^2\psi_l = \beta_l^2\psi_l \]

with \( \partial_n\psi_l = 0 \) on the boundary line \( \partial S \). These modes are normalized as

\[ \int_S |\psi_l|^2 \, dS = 1 \]

in particular \( \psi_0 = 1/\sqrt{A_d} \) where \( A_d \) is the cross section area of the main duct. For circular ducts:

\[ \psi_l = N_{m,n} J_m(\alpha_{m,n} r)e^{i m \theta}, \quad \beta_l = \sqrt{k^2 - \alpha_{m,n}^2} \]
The finite element discretization of the integral equation

\[ p(r) = \int_{\Gamma} G(r, r') q(r') \, d\Gamma(r') + p^l(r) \]

gives \( r_i \) is the FE node location

\[ \tilde{p}_i = \sum_{j=1}^{\tilde{N}} G_{ij} q_j + p^l_i \quad \text{with} \quad G_{ij} = \int_{\Gamma} G(r_i, r') \tilde{\phi}_j(r') \, d\Gamma(r') \]

The acoustic velocity is deduced from

\[ q = (G_\Omega - G)^{-1} p^l \]

Note:

i. The computation is not trivial as \( \partial z' G \) is discontinuous at \( z' = z_i \).

ii. The matrix \( G_\Omega - G \) is of small size.
Simplified models : one opening

Starting with one opening only, the impedance matrix relation can be averaged to give

$$\bar{p} = \bar{Z}(\lambda)\bar{q} \quad \text{where} \quad \bar{Z}(\lambda) = \frac{1}{\tilde{N}} \sum_{i=1}^{\tilde{N}} \sum_{j=1}^{\tilde{N}} (Z(\lambda) \tilde{F})_{ij}$$

By the same token,

$$\bar{p} = \frac{1}{W} \bar{q} + \bar{p}' \quad \text{where} \quad W = \frac{2ikA_d}{A}$$

and $A$ denotes the area of the interface. An incident plane wave $p' = e^{ikz}$ produces a transmitted pressure field

$$p = T e^{ikz} \quad \text{with} \quad T = 1 + \frac{1}{W\bar{Z}(\lambda) - 1}$$
Figure: Classical (left); with extended neck (right).
Helmholtz resonator, classical

![Graph showing TL (dB) vs Frequency (Hz) for different models and mode numbers.](image)
Helmholtz resonator, with extended neck
Simplified models : two openings

We consider a symmetric resonator connected to the main duct via two openings located at \( z = z_1 \) and \( z = z_2 \).

\[
\begin{pmatrix}
\bar{p}_1 \\
\bar{p}_2 
\end{pmatrix} = \begin{pmatrix}
\bar{Z}_{11} & \bar{Z}_{12} \\
\bar{Z}_{12} & \bar{Z}_{11}
\end{pmatrix} \begin{pmatrix}
\bar{q}_1 \\
\bar{q}_2
\end{pmatrix}
\]

Moreover,

\[
\begin{pmatrix}
\bar{p}_1 \\
\bar{p}_2
\end{pmatrix} = \frac{1}{W} \begin{pmatrix}
1 & e^{ikL} \\
e^{ikL} & 1
\end{pmatrix} \begin{pmatrix}
\bar{q}_1 \\
\bar{q}_2
\end{pmatrix} + \begin{pmatrix}
\bar{p}_1' \\
\bar{p}_2'
\end{pmatrix}
\]

where \( L = |z_2 - z_1| \). This gives

\[
T = 1 + \frac{2W \bar{Z}_{11} - 2W \bar{Z}_{12} \cos(kL) + (e^{2ikL} - 1)}{(W \bar{Z}_{11} - 1)^2 - (W \bar{Z}_{12} - e^{ikL})^2}
\]

Thus, no acoustic energy is transmitted if

\[
\bar{Z}_{12}^2 - \bar{Z}_{11}^2 - \frac{A \sin(kL)}{kA_d} \bar{Z}_{12} = 0
\]
Herschel-Quincke resonator
Herschel-Quincke resonator
The fan noise is one of the dominant components at take-off and landing for aircraft with modern high bypass ratio turbofan engines: broadband noise + Blade Passing Frequency (BPF) tones.
Fan noise

Actual configuration...

Model...
Validation on a small size configuration

<table>
<thead>
<tr>
<th></th>
<th>Matrix size</th>
<th>CPU time (Matlab)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our model</td>
<td>500</td>
<td>1 h 50 min</td>
</tr>
<tr>
<td>FEM</td>
<td>82 000</td>
<td>31 h 15 min</td>
</tr>
</tbody>
</table>
Optimal configuration (36 HQ tubes)

- Incident
- Liner-HQ system
What does the liner do?

Surface wave modes at 1530 Hz:
Influence of the number of tubes on the first BPF (iso-surface)

<table>
<thead>
<tr>
<th>Number of tubes</th>
<th>Modal power (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (Liner)</td>
<td>0</td>
</tr>
<tr>
<td>0 (Incident)</td>
<td>0</td>
</tr>
<tr>
<td>(7,1)</td>
<td>30</td>
</tr>
<tr>
<td>(8,1)</td>
<td>50</td>
</tr>
<tr>
<td>(2,3)</td>
<td>70</td>
</tr>
<tr>
<td>(-5,2)</td>
<td>90</td>
</tr>
<tr>
<td>(-9,1)</td>
<td>50</td>
</tr>
<tr>
<td>(-3,3)</td>
<td>70</td>
</tr>
</tbody>
</table>

Modes

Number of tubes

Modal power (dB)
Conclusions and prospects

The proposed Green’s function based method allows to reduce the computational effort as only the acoustic velocity at the interface needs to be calculated.

A very high number of propagative modes (few hundreds) can be handled easily on a single PC.

Gives access to physical interpretation in the low-frequency regime.

In prospect:
- could be used for designing taylor-made resonators using optimization procedures.
- viscosity effects should be included.