Almost 2-Perfect Max Packings and Min Coverings of $K_n$ with 6-cycles

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An $m$-cycle system of order $n$ is a pair $(X, C)$ where $X$ is a finite set, and $C$ is a collection of edge-disjoint $m$-cycles which partitions the edge-set of the complete undirected graph $K_n$ with vertex set $X$. It is by now well known [1] that the spectrum for $m$-cycle systems (that is, the set of all $n$ such that an $m$-cycle system of order $n$ exists) is the set of all $n$ such that

1. $n \geq m \geq 3$,
2. $n \equiv 1 \pmod{2}$, and
3. $n(n - 1)/2m$ is an integer.

An $m$-cycle system $(X, C)$ of order $n$ is said to be 2-perfect if every pair of vertices is connected by a path of length two in a cycle (and therefore in exactly one cycle) of $C$. There exists quite an extensive collection of works on 2-perfect cycle systems, and the interested reader is referred to [2], [3], [4].

The spectrum for 5-cycle systems was first determined in 1966 [6] to be the set \{ $n : n \equiv 1$ or $5(\text{mod}10)$ \}. Subsequently in 1984, the spectrum for 2-perfect 5-cycle systems has been determined [5] to be the same, except for $n = 15$, for which no 2-perfect 5-cycle system exists.

Now let $(X, C)$ be a 5-cycle system and let $C^*$ be the collection of 5-cycles obtained by replacing each 5-cycle in $C$ with its inside 5-cycle.
Then \((X, C^*)\) is a 5-cycle system if and only if \((X, C)\) is 2-perfect.

For 6-cycles, the situation is somewhat different: unlike 5-cycles, a 6-cycle \((x_1, x_2, x_3, x_4, x_5, x_6)\) has three inside 6-cycles:
\((x_1, x_3, x_5, x_2, x_6, x_4), (x_1, x_3, x_6, x_4, x_2, x_5), (x_1, x_4, x_2, x_6, x_3, x_5)\).
Let \((X, C)\) be a 6-cycle system and let \(C^*\) be a collection of inside 6-cycles, one from each of the cycles in \(C\). (If \((X, C^*)\) is a 6-cycle system, we will say that \((X, C)\) is almost 2-perfect.

At this point, the following question may be asked: Given a 6-cycle system \((X, C)\), is it possible to choose an inside 6-cycle (one out of three possible) for each 6-cycle in \(C\) so that the resulting collection of inside 6-cycles \(C^*\) is a 6-cycle system? As expected, the answer is sometimes yes and sometimes no.

**Example 1** *(An almost 2-perfect 6-cycle system of order 9.)*

Let \(X = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}\).

\[
\begin{align*}
C: & \quad (0, 1, 2, 3, 4, 5) \to (0, 2, 4, 1, 5, 3) \\
& \quad (0, 2, 4, 1, 6, 7) \to (0, 4, 6, 2, 7, 1) \\
& \quad (0, 3, 7, 8, 4, 6) \to (0, 7, 4, 3, 6, 8) \\
& \quad (0, 8, 6, 5, 7, 4) \to (0, 6, 7, 8, 4, 5) \\
& \quad (1, 7, 2, 8, 5, 3) \to (1, 2, 5, 7, 3, 8) \\
& \quad (1, 8, 3, 6, 12, 5) \to (1, 3, 2, 8, 5, 6)
\end{align*}
\]

**Example 2** *(A 6-cycle system of order 9 which is not almost 2-perfect.)*

\[
\begin{align*}
C: & \quad (0, 1, 2, 3, 4, 5), (0, 2, 4, 1, 3, 6), (0, 3, 5, 1, 7, 8), \\
& \quad (0, 4, 6, 8, 2, 7), (1, 6, 5, 7, 3, 8), (2, 5, 8, 4, 7, 6)
\end{align*}
\]

We give a complete survey of the problem of constructing almost 2-perfect 6-cycle systems and extend this to max packings and min coverings (obvious definitions).
References


[2] D. E Bryant, C. C. Lindner . 2-perfect m-cycle systems can be equationally defined for m = 3, 5, and 7 only, Algebra Univ. 35 (1996), 1-7.


