Understanding the CHAIN RULE

Identifying different types of functions

1. Examples of simple functions:
   \( x, \sin x, \sqrt{x}, \ 3x^4, \ x + 1, \ x^2, \ \ln x, \ 6x^3, \ 4\cos x, \ 5x, \ \tan x, \ x^{-2}, \ e^x, \ 3x + 2\ln x, \ 4\sin x - x^4 \)

   These simple functions can be differentiated straight away.

2. Examples of functions-of-functions:
   \((x + 1)^2, \ \cos^2 x \ [\text{or} (\cos x)^2], \ \sqrt{\ln x}, \ e^{2x+5}, \ \sin 3x, \ x^{\sin x}\)

   To differentiate these, we need the CHAIN RULE. (Although with practice, they can again be differentiated straight away.)

   In each of these examples, there are TWO functions:

   In \((x + 1)^2\), the 1\(^{st}\) function is \(x + 1\), the 2\(^{nd}\) function is “square”

   In \(\sin 3x\), the 1\(^{st}\) function is \(3x\), the 2\(^{nd}\) function is “\(\sin\)”

   The 1\(^{st}\) function is the one you’d work out first if you were going to evaluate the function for a given value of \(x\):
   e.g. to evaluate \((x + 1)^2\), when \(x = 3\), first work out \(3 + 1\), then square, to get 16.

   And to evaluate \(\sin 3x\) when \(x = 30^\circ\), first work out \(3x\), then take the \(\sin e\) of \(90^\circ\), which is 1.

3. Divide the following functions up into two groups, simple functions (ones that can be differentiated straight away) and functions-of-functions (ones that can’t):

   \(\cos x, \ \sin 2x, \ x^3, \ \ln 4x, \ 6x + 5, \ e^{\sin x}, \ 3\tan x, \ 1 + x^3, \ \sin(4x + 3), \ 2\sqrt{x}, \ 7x^6, \ 1 - \ln x, \ 2\sin x + 3\cos x, \ 5e^{-2x}, \ \cos 5x, \ x^{-8}, \ (6x + 3)^5\)

4. Differentiate all of the simple functions

5. To differentiate functions-of-functions, we need the CHAIN RULE:

   Example: to differentiate \(y = (x^3 + 4)^5\)

   We let \(u = \text{‘the 1\(^{st}\) function’}, \text{which is …………\)}

   So now we have \(y = \ldots\)(i.e. a function of \(u\))

   Now work out \(\frac{du}{dx}\), which is………………

   And also work out \(\frac{dy}{du}\), which is ……………………

   Now the CHAIN RULE says: \(\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\)

   (think of multiplying fractions: the ‘\(du\)’s cancel out)

   So we get \(\frac{dy}{dx} = \ldots \times \ldots = \frac{15x^8(x^3 + 4)^4}{4}\) [Answer]

6. Now try to differentiate all of the functions-of-functions that you identified in Q3.
ANSWERS

Q3.

simple functions:

\[ \cos x, \quad x^3, \quad 6x + 5, \quad 3\tan x, \quad 1 + x^3, \quad 2\sqrt{x}, \quad 7x^6, \quad 1 - \ln x, \quad 2\sin x + 3\cos x, \quad x^{-8} \]

functions-of-functions:

\[ \sin 2x, \quad \ln 4x, \quad e^{\sin x}, \quad \sin(4x + 3), \quad 5e^{-2x}, \quad \cos 5x, \quad (6x + 3)^5 \]

Q4. [in the same order as given above]

\[ -\sin x, \quad 3x^2, \quad 6, \quad 3\sec^2x, \quad 3x^2, \quad x^{-\frac{1}{2}}, \quad 42x^5, \quad -\frac{1}{x^2}, \quad 2\cos x - 3\sin x, \quad -8x^{-9} \]

Q5. i) \[ x^3 + 4 \]
     ii) \[ u^5 \]
     iii) \[ \frac{du}{dx} = 3x^2 \]
     iv) \[ \frac{dy}{du} = 5u^4 \]
     v) \[ 5u^4 \times 3x^2 = 15x^2u^4 = 15x^2 (x^3 + 4)^4 \]

Q6.

\[ 2\cos 2x, \quad \frac{1}{x} \text{ or } x^{-1}, \quad \cos x e^{\sin x}, \quad 4\cos(4x + 3), \quad 2\cos x - 3\sin x, \quad -10e^{-2x}, \quad -5\sin 5x, \quad 30(6x + 3)^4 \]