PhD project proposal 2

Eigenfunctions, overdetermined PDEs, and rigidity

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Let U be a bounded Euclidean domain. If we want to consider a spectral problem for the Laplacian on U, we need to impose exactly one boundary condition on the boundary of U, for example a Dirichlet or a Neumann one, in which case we obtain an infinite sequence of eigenvalues and the corresponding eigenfunctions. If we impose more than one boundary condition, we get a so-called overdetermined spectral problem. Generically, it will not have any solutions, but they may exist for some specific geometries and types of the "extra" condition. For example, if U is a ball, then there are some Neumann Laplacian eigenfunctions that depend only on the distance to the center of the ball, and therefore satisfy an extra condition of being constant along the boundary.

The question of what kind of geometries support solutions of overdetermined boundary value problems for PDEs, and related rigidity questions, have been a subject of considerable interest among analysts, partially motivated by physical applications, as in the classical work of J. Serrin. In the context of spectral problems, a conjecture of M. Schiffer states that among all simply connected domains, only the balls may have Neumann eigenfunctions which are constant along the boundary. Hence, in a way, the conjecture says that the existence of Neumann eigenfunctions which are constant along the boundary is somewhat "rigid" — it is not preserved even for small perturbations of a ball.

Schiffer's conjecture is in turn related to a very well-known problem in integral geometry named after D. Pompeiu which has remained unresolved in full generality since 1929: for which domains U there exists a non-zero continuous function on the whole space whose integrals over U and any congruent copy of U are all equal? It is known that any solution to the Pompeiu problem which is not a ball will be a counterexample to Schiffer's conjecture. Another example of a physically motivated spectral overdetermined problem is the existence of the so-called Jones modes in elasticity: that is, the eigenvectors of the linear elasticity operator which are traction-free on the boundary and satisfy an extra condition of having a zero normal displacement.

The project will look at these questions from the rigorous geometric analysis point of view, trying to find, in various situations, geometric conditions allowing the existence of overdetermined eigenfunctions, and analysing their rigidity.

A successful candidate will have a good background in Analysis and differential geometry; some experience in PDEs may be advantageous.