The numerical propagation of scaling symmetries of scale-invariant partial differential equations: the S-property for mass-conserving problems

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Abstract

We consider scale-invariant problems governed by nonlinear partial differential equations on finite moving domains that conserve total mass in time.

We show that, under spatial deformations of the domain driven by a *local* conservation of mass principle, an initial condition that coincides with a self-similar scaling solution is propagated as the self-similar solution for all time, while for spatial deformations generated by the conservation of *distributed* masses, an initial condition that coincides with the piecewise-linear L^2 projection of the self-similar solution is propagated as the piecewise-linear L^2 projection of the self-similar solution for all time, the latter exhibiting a discrete scaling symmetry.

For more general initial conditions we adapt the proofs to obtain related scale-invariant procedures that possess the S-property, i.e. if the initial condition coincides with a self-similar scaling solution (in an appropriate norm), then it is propagated as the self-similar scaling solution exactly in that norm (modulo a projection error in the L^2 case).

Scale-invariant finite-difference and finite-element (piecewise-linear) schemes are constructed for classes of flux-driven mass-conserving problems. The finite-difference scheme possesses the S-property in the l^∞ norm, thus preserving a discrete scaling symmetry, while the finite-element scheme possesses the S-property (in the L^2 norm) modulo the projection error.

The S-property is suggested as a yardstick for establishing confidence in numerical schemes for nonlinear scale-invariant problems, in a similar way to which standard schemes on fixed grids for linear problems are constructed so as to be exact for polynomial solutions of given degree.

Contents

1 Introduction

Many problems governed by partial differential equations (PDEs) that arise in practical applications possess scaling properties which are in a sense more fundamental than the equations themselves [14]. In approximating such problems by numerical schemes it is desirable to construct algorithms that preserve these scaling properties, an objective beyond the reach of conventional numerical schemes using fixed meshes for which the mesh depends on neither time nor the solution. The preservation of such properties is an aspect of geometric integration, which was reviewed by Budd and Piggott in [12, 14] who considered the effectiveness of numerical methods in preserving geometric structures of differential equations, including the necessity for moving meshes. In this paper we relate scaling properties to the conservation of local or distributed masses on moving meshes, leading to procedures that propagate scaling symmetries exactly in time.

We recognise features similar to those in Noether's Theorem (see e.g.[23]) for PDEs derived from a variational principle, that a continuous symmetry implies the existence of conserved quantities. However, in this paper we are concerned with a separate class of PDE problems.

Moving-mesh schemes may be categorised as mapping-based or velocity-based (see [17]). Mapping-based schemes, often based on equidistribution, have been extensively discussed in [11, 17, 15, 13], while velocity-based moving-mesh schemes can be found in [18, 22, 7, 16, 1, 28, 2, 3, 25, 4, 21, 6, 9, 10, 24, 27]. Velocity-based moving-mesh schemes are particularly well adapted to problems posed on finite moving domains with free or moving boundaries whose location depends on the solution. Our approach is velocity-based.

In [25] a velocity-based scale-invariant moving-mesh finite-difference scheme based on local mass conservation was shown to propagate a self-similar solution of a second order nonlinear diffusion problem with a moving boundary exactly to within rounding error. Also, in [9], velocity-based scale-invariant moving-mesh finite-difference and finite-element schemes based on conservation were shown to propagate self-similar solutions of a fourth-order nonlinear moving boundary problem exactly to within rounding error.

In this paper we generalise these results by proving first that for a general class of scale-invariant mass-conserving PDE problems deformations of the domain generated by the conservation of local (or distributed) masses imply the exact propagation of self-similar solutions (or their L^2 projections) in time. We then adapt the steps of the proof to apply to a class of scale-invariant problems with *general* initial conditions, resulting in numerical algorithms that possess the S-property, defined as the exact propagation of a scaling symmetry when the initial condition coincides with the self-similar scaling solution in some norm.

The layout of the paper is as follows. In section 2 we prove that, for scale-invariant time-dependent PDE problems that conserve total mass, conservation of *local* mass implies the propagation of self-similar scaling solutions exactly in time. Then, in section 3 we prove that conservation of *distributed* (piecewise-linear) masses implies the same property in the case of the L^2 projection of a self-similar scaling solution, thus preserving a discrete scaling symmetry in the L^2 norm.

In section 4 these procedures are extended to general initial conditions, yielding algorithms for a class of first-order-in-time flux-driven PDEs that possess the S-property in some norm.

Finite-difference and finite-element algorithms are presented in section 5 for classes of flux-driven problems, again aiming for the S-property. The finite-difference scheme possesses the S-property in the l^∞ norm when a function is interpolated quadratically from adjacent gridpoints, while the finite-element scheme possesses the S-property (in the L^2 norm) but subject to a projection error.

An illustrative example is given in section 6 and the paper summarised in section 7.

We first recall the concepts of scale invariance and similarity.

1.1 Scale invariance

A problem governed by a one-dimensional time-dependent PDE for a scalar function $u(x,t)$ (density) in a moving interval $(a(t), b(t))$ is *scale-invariant* if it is unaltered under the scalings

$$t \to \lambda^\alpha t, \quad x \to \lambda^\beta x, \quad u \to \lambda^\gamma u, \tag{1}$$

where λ is the group parameter. Here α, β, γ are scaling exponents for the particular PDE, (and $a(t), b(t)$ scale in the same way as x). Without loss of generality we take $\alpha = 1$.

Under the transformation (1) the total mass, defined as

$$\theta(t) = \int_{a(t)}^{b(t)} u(x,t)\,\mathrm{d}x, \tag{2}$$

scales as $\lambda^{\gamma+\beta}$. When the total mass is independent of time $\gamma + \beta = 0$.

Similarity variables (themselves scale-invariant) may be defined as

$$\frac{x}{t^\beta} = \xi, \quad \frac{u}{t^{-\beta}} = \eta$$

using $\gamma + \beta = 0$. Also define

$$\xi_a = \frac{a(t)}{t^\beta}, \qquad \xi_b = \frac{b(t)}{t^\beta}$$

1.2 Self-similarity

We define a self-similar scaling solution to be an ansatz of the form

$$u(x,t) = t^\gamma \eta(\xi), \quad \text{where} \quad x = t^\beta \xi \tag{3}$$

The function $\eta(\xi)$ satisfies a reduced order differential equation (see e.g. [8, 11]) in which the partial time derivative of u is

$$\partial_t u = t^{-\beta-1}\{-\beta\eta(\xi) - \beta\xi\eta'(\xi)\} = -\beta t^{-\beta-1}(\xi\eta(\xi))' \tag{4}$$

From (3), for each fixed ξ the time evolution of the x coordinate (written here as $\widehat{x}(t)$) is effected by a similarity velocity

$$v(\widehat{x}(t), t) = \frac{\mathrm{d}\widehat{x}}{\mathrm{d}t} = \beta t^{\beta-1}\xi = \frac{\beta\widehat{x}(t)}{t} \tag{5}$$

2 Propagation of scaling symmetry

2.1 An integral invariant

An invariance property of the self-similar scaling solution (3) is that the local masses between any two coordinates $\widehat{x}_1(t)$ and $\widehat{x}_2(t)$,

$$\int_{\widehat{x}_1(t)}^{\widehat{x}_2(t)} u(\chi, t)\,\mathrm{d}\chi \tag{6}$$

are independent of time for all those $a(t) \leq \widehat{x}_1(t) < \widehat{x}_2(t) \leq b(t)$ that are proportional to $\xi_a \leq \xi_1 < \xi_2 \leq \xi_b$ respectively, by the factor t^β. The result follows by substituting $\chi = t^\beta \zeta$ into (6) to obtain the time-invariant quantity

$$\int_{\xi_1}^{\xi_2} \eta(\zeta)\,\mathrm{d}\zeta$$

where $\xi_1 = \widehat{x}_1(t)/t^\beta$ and $\xi_2 = \widehat{x}_2(t)t^\beta$.

We prove a converse of this property.

2.2 Theorem 1

Theorem 1: Let the density $u(x,t)$ be a strictly positive solution of a time-dependent scale-invariant mass-conserving PDE problem in a moving domain $(a(t), b(t))$.

If

- the points $\widehat{x}_1(t), \widehat{x}_2(t)$ of the domain move in such a way that the local masses

$$\int_{\widehat{x}_1(t)}^{\widehat{x}_2(t)} u(\chi, t)\,\mathrm{d}\chi \quad (= c(\widehat{x}_1, \widehat{x}_2)\,, \text{say}) \tag{7}$$

  are constant in time for all $a(t) \leq \widehat{x}_1(t) < \widehat{x}_2(t) \leq b(t)$,
- the initial condition on $u(x,t)$ coincides with a self-similar scaling solution of the form (3) for all x,

then for any moving coordinate $\widehat{x}(t)$ the solution $u(\widehat{x}(t), t)$ coincides with the self-similar scaling solution (3) in the interval $a(t) \leq \widehat{x}(t) \leq b(t)$ for all t, thus preserving a scaling symmetry, and the induced velocity $v(\widehat{x}(t), t)$ is the similarity velocity (5)..

As a preliminary to the proof we obtain the differential equation satisfied by the velocity $v(x,t)$ induced by the invariance of (7).

Lemma 1:

The velocity $v(x,t)$ induced by (7) satisfies

$$\int_{\widehat{x}_1(t)}^{\widehat{x}_2(t)} \partial_t u(\chi,t)\,\mathrm{d}\chi + [uv]_{\widehat{x}_1(t)}^{\widehat{x}_2(t)} = 0 \qquad (8)$$

for all $a(t) \le \widehat{x}_1(t) < \widehat{x}_2(t) \le b(t)$, where $v(x,t)$ is the induced velocity

Proof: From (7),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\widehat{x}_1(t)}^{\widehat{x}_2(t)} u(\chi,t)\,\mathrm{d}\chi = 0$$

for all $a(t) \le \widehat{x}_1(t) < \widehat{x}_2(t) \le b(t)$, leading by Leibnitz' Integral Rule to

$$\int_{\widehat{x}_1(t)}^{\widehat{x}_2(t)} \partial_t u(\chi,t)\,\mathrm{d}\chi + [uv]_{\widehat{x}_1(t)}^{\widehat{x}_2(t)} = 0$$

for all $a(t) \le \widehat{x}_1(t) < \widehat{x}_2(t) \le b(t)$, as required.

Proof of Theorem 1:

The proof is in six parts.

1. In the first part we show that at the initial time $t = t^0$, say, the velocity induced by (7) is the similarity velocity $v(x,t^0) = \beta x/t^0$ of (5) for all $a(t^0) < x < b(t^0)$.

At time $t = t^0$ the initial condition $u(x,t^0)$ coincides with the self-similar scaling solution $u(x,t^0) = (t^0)^{-\beta}\eta(\xi^0)$ of (3), where $\xi^0 = x/(t^0)^\beta$ and $\partial_t u$ is given by (4). Substituting into (8) at $t = t^0$ we obtain the reduced order equation

$$\int_{\widehat{x}_1(t^0)}^{\widehat{x}_2(t^0)} \left\{-\beta(t^0)^{-\beta-1}\eta(\zeta) - \beta\zeta\partial_\zeta\eta\right\}\,\mathrm{d}\chi + (t^0)^\gamma\left[\eta(\xi)\,v(x,t^0)\right]_{\widehat{x}_1(t^0)}^{\widehat{x}_2(t^0)} = 0 \qquad (9)$$

where now $\xi = x/(t^0)^\beta$, $\zeta = \chi/(t^0)^\beta$.

Changing the integration variable from χ to $\zeta = \chi/t^\beta$, equation (9) reduces to

$$\int_{\xi_1^0}^{\xi_2^0} \left\{-\beta(t^0)^{\beta-1}\partial_\zeta(\zeta\eta(\xi))\right\}\,\mathrm{d}\zeta + \left[\eta(\xi)\,v(x,t^0)\right]_{\xi_1^0}^{\xi_2^0} = 0 \qquad (10)$$

at $t = t^0$, for all $\xi_a^0 \le \xi_1^0 < \xi_2^0 \le \xi_b^0$ such that

$$\xi_1^0 = \frac{\widehat{x}_1(t^0)}{(t^0)^\beta}, \quad \xi_2^0 = \frac{\widehat{x}_2(t^0)}{(t^0)^\beta}, \quad \xi_a^0 = \frac{a(t^0)}{(t^0)^\beta}, \quad \xi_b^0 = \frac{b(t^0)}{(t^0)^\beta},$$

Hence from (10)

$$[-\beta(t^0)^{\beta-1}\xi^0\eta(\xi^0) + \eta(\xi^0)v(x,t^0)]_{\xi_1^0}^{\xi_2^0} = 0 \tag{11}$$

where $\xi^0 = x/(t^0)^\beta$.

In order to solve (8) uniquely for the velocity $v(x, t^0)$ a value is requred at one point. Without loss of generality we take the point at which $v(x, t^0)$ is zero to coincide with the origin of ξ^0 . Thus, putting $\xi_1^0 = 0$ and taking ξ_2^0 as a general point ξ^0 in equation (11), we obtain

$$\eta(\xi^0)\{-\beta(t^0)^{\beta-1}\xi^0 + v(x,t^0)\} = 0$$

Since $\eta(\xi^0) > 0$ (because $u(x, t^0) > 0$) it follows that

$$v(x,t^0) = \beta(t^0)^{\beta-1}\xi^0 = \frac{\beta x}{t^0}, \tag{12}$$

as required.

Remark: Equation (12) does not follow immediately by differentiating the second of (3) with respect to t since (3) holds only at $t = t^0$. On the other hand, the reduced order equation holds at $t = t^0$ with $\partial_t u$ given by (4) at $\xi = \xi^0$.

2. In the second part of the proof we show that under a deformation of the domain initiated by the velocity (12) the similarity variable $\xi^1 = x(t^1)/(t^1)^\beta$ is equal to ξ^0 at time $t^1 = t^0 + h$ to second order in h.

Let $\widehat{x}(t)$ be a moving coordinate, coinciding with x at $t = t^0$ and moving with the velocity $v(x, t^0)$ of (12). Given an increment h in time, a Taylor series expansion of $\widehat{x}(t)$ at $t^1 = t^0 + h$ yields

$$\widehat{x}(t^1) = \widehat{x}(t^0) + h\,v(\widehat{x}(t^0), t^0) + O(h^2) \tag{13}$$

Substituting for $v(\widehat{x}(t^0), t^0)$ from (12), we obtain

$$\widehat{x}(t^1) = \widehat{x}(t^0) + h\frac{\beta\widehat{x}(t^0)}{t^0} + O(h^2) = \left(1 + \frac{\beta h}{t^0}\right)\widehat{x}(t^0) + O(h^2)$$

$$= \left(1 + \frac{h}{t^0}\right)^\beta \widehat{x}(t^0) + O(h^2) = \left(\frac{t^1}{t^0}\right)^\beta \widehat{x}(t^0) + O(h^2),$$

showing that the similarity variable

$$\xi^1 = \frac{\widehat{x}(t^1)}{(t^1)^\beta} = \frac{\widehat{x}(t^0)}{(t^0)^\beta} + O(h^2) = \xi^0 + O(h^2) \tag{14}$$

is invariant to order h^2.

3. In the third part of the proof we demonstrate that under the deformation generated by (12) the similarity variable $\eta(\xi)$ is also invariant to second order in h. From the conservation property (7)

$$\int_{a(t^1)}^{\widehat{x}(t^1)} u(\chi, t^1)\,\mathrm{d}\chi = \int_{a(t^0)}^{\widehat{x}(t^0)} u(\chi, t^0)\,\mathrm{d}\chi \tag{15}$$

Differentiating (15) wrt $\widehat{x}(t^1)$ we obtain

$$u(\widehat{x}(t^1), t^1) = \frac{\mathrm{d}\widehat{x}(t^0)}{\mathrm{d}\widehat{x}(t^1)} \frac{\mathrm{d}}{\mathrm{d}(\widehat{x}(t^0), t^0)} \int_{a(t^0)}^{\widehat{x}(t^0)} u(\chi, t^0)\,\mathrm{d}\chi$$

$$= \frac{\mathrm{d}\widehat{x}(t^0)}{\mathrm{d}\widehat{x}(t^1)} u(\widehat{x}(t^0), t^0)$$

Thus, due to (14)

$$\frac{u(\widehat{x}(t^1), t^1)}{(t^1)^{-\beta}} = \frac{u(\widehat{x}(t^0), t^0)}{(t^0)^{-\beta}} = \eta(\xi^0) + O(h^2),$$

equivalently,

$$\eta(\xi^1) = \frac{u(\widehat{x}(t^1), t^1)}{(t^1)^{-\beta}} = \eta(\xi^0) + O(h^2), \tag{16}$$

using (14) again. Thus the similarity variable $\eta(\xi)$ of (3) is invariant to order h^2.

4. The fourth part of the proof is concerned with repetition of the first three parts over a further time step h. Returning to (11) and using the invariants ξ and $\eta(\xi)$ of (14) and (16) to order h^2, we find that at $t = t^1$

$$\eta(\xi^1)\{-\beta(t^1)^{\beta-1}\xi^1 + v(x, t^1)\} = O(h) \tag{17}$$

where the right hand side of (17) is $O(h)$ rather than $O(h^2)$ since one power of h is lost in the differentiation with respect to ξ in deriving (10) from (8) using (4). Hence by equations (11) through to (12) with t^0 replaced by t^1,

$$v(x,t^1) = \frac{\beta x}{t^1} + O(h)$$

Thence, by the argument from equation (13) through to (16) with t^0 and t^1 replaced by t^1 and t^2, respectively, we obtain

$$\xi^2 = \frac{\widehat{x}(t^2)}{(t^2)^\beta} = \frac{\widehat{x}(t^1)}{(t^1)^\beta} = \frac{\widehat{x}(t^0)}{(t^0)^\beta} = \xi^0, \tag{18}$$

$$\eta(\xi^2) = \frac{u(x(t^2),t^2)}{(t^2)^{-\beta}} = \frac{u(x(t^1),t^1)}{(t^1)^{-\beta}} = \frac{u(x(t^0),t^0)}{(t^1)^{-\beta}} = \eta(\xi^0) \tag{19}$$

together with two terms of order h^2 in each equation (18) and (19). Thus ξ^2 and $\eta(\xi^2)$ are invariant to order h^2.

5. After n time steps of h we find that at $t^n = t^0 + nh$

$$v(x,t^{n-1}) = \frac{\beta x}{t^{n-1}} + O(h)$$

as well as

$$\xi^n = \frac{x(t^n)}{(t^n)^\beta} = \frac{x(t^0)}{(t^0)^\beta} = \xi^0, \tag{20}$$

and

$$\eta(\xi^n) = \frac{u(x(t^n),t^n)}{(t^n)^{-\beta}} = \frac{u(x(t^0),t^0)}{(t^0)^{-\beta}} = \eta(\xi^0), \tag{21}$$

together with n terms of order h^2 in each equation (20) and (21).

6. In the final part of the proof we let $h \to 0$ and $n \to \infty$ in such a way that $nh = t - t^0$, so that $nh^2 = O(h)$. It follows that at time t

$$v(x,t) = \frac{\beta x}{t} + O(h),$$

i.e

$$v(\widehat{x}(t),t) = \frac{\beta \widehat{x}(t)}{t} + O(h)$$

Also

$$\xi = \frac{\widehat{x}(t)}{t^\beta} = \frac{\widehat{x}(t^0)}{(t^0)^\beta} + O(h) = \xi^0 + O(h),$$

$$\eta(\xi) = \frac{u(\widehat{x}(t),t)}{t^{-\beta}} = \frac{u(\widehat{x}(t^0),t^0)}{(t^0)^{-\beta}} + O(h) = \eta(\xi^0) + O(h),$$

In the limit

$$v(\widehat{x}(t),t) = \frac{\beta\widehat{x}(t)}{t}$$

and

$$\xi = \frac{\widehat{x}(t)}{t^\beta} = \xi^0, \qquad \eta(\xi) = \frac{u(\widehat{x}(t),t)}{t^{-\beta}} = \eta(\xi^0),$$

for all $t > t^0$. Hence $u(\widehat{x}(t),t)$ coincides with the self-similar solution (3), and $v(\widehat{x}(t),t)$ concides with the similarity velocity (5) for all $t > t^0$. This completes the proof.

We now seek a similar finite-dimensional result in the L^2 norm.

3 Propagation of scaling symmetry in the L^2 norm

Denote by $U(x,t)$ the strictly positive L^2 projection of a density $u(x,t)$ into an $(N+2)$ dimensional subspace of piecewise linear functions on the subdivision

$$a(t) = X_0(t) < \ldots < X_{N+1}(t) = b(t) \tag{22}$$

of the interval $(a(t), b(t))$, satisfying the projection condition

$$\int_{a(t)}^{b(t)} W_i(\zeta)\,\{U(\chi,t) - u(\chi,t)\}\,\mathrm{d}\chi = 0, \qquad (i = 0,\ldots,N+1) \tag{23}$$

where $\zeta = \chi/t^\beta$, for all W_i belonging to te set of piecewise-linear basis functions on the subdivision (22) constituting a partition of unity. By summing (23) over all i,

$$\int_{a(t)}^{b(t)} u(\chi,t)\mathrm{d}\chi = \int_{a(t)}^{b(t)} U(\chi,t)\mathrm{d}\chi$$

($cf.$ (2)), showing that the total masses are the same.

Scaling invariance holds as in section 1.1.

The L^2 projection of the self-similar scaling solution (3) is defined as

$$U(x,t) = t^{-\beta}\mathcal{N}(\xi) \quad \text{where} \quad \xi = \frac{x}{t^\beta} \tag{24}$$

The function $\mathcal{N}(\xi)$ satisfies a reduced order equation in which the partial time derivative of $U(x,t)$ is

$$\partial_t U = -\beta t^{-\beta-1}\mathcal{N})(\xi) + (-\beta)t^{\gamma-1}\xi\partial_\xi\mathcal{N})(\xi) = -\beta t^{-\beta-1}\partial_\xi(\xi\mathcal{N}) \quad (25)$$

From (24), for each fixed ξ the variation of the spatial coordinate with time (written here as $\widehat{x}(t)$) is effected by a similarity velocity defined as

$$V(\widehat{x}(t),t) = \frac{\mathrm{d}\widehat{x}}{\mathrm{d}t} = \beta t^{\beta-1}\xi = \frac{\beta\widehat{x}(t)}{t}$$

3.1 Invariant integrals

The self-similar scaling solution (3) has the property that for any square-integrable function $W(x/t^\beta)$ the weighted integrals

$$\int_{a(t)}^{b(t)} W_i(\zeta)\, u(\chi,t)\,\mathrm{d}\chi \quad (26)$$

($i = 0,\ldots,N+1$), where $\zeta = \chi/t^\beta$, are invariant in time for all W_i constituting a partition of unity in the interval $(a(t),b(t))$ and all $a(t)$, $b(t)$ proportional to the time-independent coordinates ξ_a, ξ_b by a factor t^β. The result follows by substituting $\chi = t^\beta\zeta$ into (26) to obtain the time-invariant quantity

$$\int_{\xi_a}^{\xi_b} W_i(\zeta)\,\eta(\zeta)\,\mathrm{d}\zeta$$

where $\xi_a = a(t)/t^\beta$ and $\xi_b = b(t)/t^\beta$.

A similar property holds for L^2 projections $U(x,t)$. The weighted integrals

$$\int_{a(t)}^{b(t)} W_i(\zeta)\,U(\chi,t)\,\mathrm{d}\chi$$

($i = 0,\ldots,N+1$), are also time-invariant for all weight functions $W_i(\zeta)$ constituting a partition of unity in the interval $(a(t),b(t))$ and all $a(t)$, $b(t)$ proportional to the time-independent coordinates ξ_a, ξ_b by a factor t^β, the result following from (23) and (26).

We prove a converse of this result.

3.2 Theorem 2

Theorem 2:

If

- the nodes of the partition move such that the weighted masses

$$\int_{a(t)}^{b(t)} W_i(\zeta)\, U(\chi,t)\,\mathrm{d}\chi \quad (= C_i,\ \text{say}), \tag{27}$$

  where $\zeta = \chi/t^{\beta}$, are independent of time for all $i = 0, \ldots, N+1$,
- the piecewise-linear weight functions W_i are advected with a piecewise-linear velocity V induced by (27) (NB: velocities that advect piecewise-linear functions W_i exactly must be piecewise-linear.),
- the initial condition on $U(x,t)$ coincides with the L^2 projection of a self-similar scaling solution of the form (3) for all $a(t) < x < b(t)$,

then for any moving coordinate $\widehat{x}(t)$ the projected solution $U(\widehat{x}(t),t)$ coincides with the L^2 projection of the self-similar scaling solution (3) in the domain $a(t) \leq \widehat{x}(t) \leq b(t)$ for all t, thus exhibiting a discrete scaling symmetry in the L^2 norm, and $V(\widehat{x}(t),t)$ coincides with the similarity velocity (5).

As a preliminary to the proof we obtain the weak form of the differential equation satisfied by a piecewise-linear velocity $V(x,t)$ induced by the invariance of (27).

Lemma 2:
The invariance of (27) together with the advection property of the W_i implies that

$$\int_{a(t)}^{b(t)} W_i(\zeta)\,\{\partial_t U + \partial_\chi (UV)\}\,\mathrm{d}\chi = 0 \tag{28}$$

for all $i = 0, \ldots, N+1$. where $V(x,t)$ is the induced piecewise-linear velocity.

Proof: By the Reynolds Transport Theorem applied to $W(\xi)U(x,t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} W_i(\zeta)\, U(\chi,t)\mathrm{d}\chi = \int_{a(t)}^{b(t)} W_i(\zeta,t)\,\{\partial_t U + \partial_\chi\,(UV)\}\,\mathrm{d}\chi$$

$$+\int_{a(t)}^{b(t)} U(\chi,t)\left\{\partial_t W_i + V(\chi,t)\,\partial_\chi W_i\right\}\mathrm{d}\chi, \tag{29}$$

where $V(x,t)$ is any velocity field consistent with the the boundary velocities.

The advection property of the basis functions W_i gives

$$\partial_t W_i + V\partial_x W_i = 0 \tag{30}$$

reducing (29) to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} W_i(\zeta)\,U(\chi,t)\mathrm{d}\chi = \int_{a(t)}^{b(t)} W_i(\zeta,t)\left\{\partial_t U + \partial_\chi\left(UV\right)\right\}\mathrm{d}\chi \tag{31}$$

where $V(x,t)$ is piecewise-linear.

The Lemma follows from (31) and the time invariance of (27).

We now turn to the proof of Theorem 2.

Proof of Theorem 2

The proof is again in six parts.

1. In the first part we show that the velocity induced by (27) is the similarity velocity (12).

At time $t=t^0$ the initial condition $U(x,t^0)$ coincides with the L^2 projection $(t^0)^\gamma\mathcal{N}(\xi^0)$ of the self-similar scaling solution (24), where $\xi^0 = x/(t^0)^\beta$ and $\partial_t U$ is given by (25).

Substituting into (28) at $t=t^0$ we obtain

$$\int_{a(t^0)}^{b(t^0)} W_i(\zeta)\{-\beta t^{-\beta-1}\partial_\chi(\xi\mathcal{N}) + t^{-\beta-1}\partial_\chi(\mathcal{N}V(\chi,t))\}\mathrm{d}\chi = 0 \tag{32}$$

at $t=t^0$, where $\zeta = \chi/(t^0)^\beta$, $\xi_a^0 = a(t)/t^\beta$ and $\xi_b^0 = b(t)/t^\beta$.

Changing the integration variable from χ to ζ, equation (32) reduces to

$$\int_{\xi_a^0}^{\xi_b^0} W_i(\zeta)\{-\beta(t^0)^{\beta-1}\partial_\zeta(\zeta\mathcal{N}(\zeta)) + \partial_\zeta(\mathcal{N}(\zeta)V(x,t^0))\}\mathrm{d}\zeta = 0,$$

Let

$$\mathcal{Z}(\zeta) = -\beta(t^0)^{\beta-1}\zeta + V(x,t^0) \tag{33}$$

so that (32) can be written

$$\int_{\xi_a^0}^{\xi_b^0} W_i(\zeta)\,\partial_\zeta\{\mathcal{N}(\zeta)\mathcal{Z}(\zeta)\}\mathrm{d}\chi = 0, \qquad (i = 0, \ldots, N+1) \tag{34}$$

at $t = t^0$.

Expanding $\mathcal{Z}(\zeta)$ as

$$\mathcal{Z}(\zeta) = \sum_{j=1}^{J} \mathcal{Z}_j W_j(\zeta, t)$$

equation (34) yields the matrix equation

$$\mathcal{B}(\mathcal{N})\underline{\mathcal{Z}} = 0 \tag{35}$$

at $t = t^0$, where $\underline{\mathcal{Z}} = \{\mathcal{Z}_i\}$ and the matrix $\mathcal{B}(\mathcal{N})$ has entries

$$\int_{\xi_a^0}^{\xi_b^0} W_i(\zeta)\,\partial_\zeta\{\mathcal{N}(\zeta)W_j(\zeta)\}\mathrm{d}\zeta \qquad (i, j = 0, \ldots, N+1)$$

In order to solve for $V(x, t^0)$ uniquely a value of V is required at one point. Without loss of generality we choose the origin of ζ to be the point where V is zero, so that $\mathcal{Z}(0) = 0$. As in Theorem 1 the velocity $V(x, t)$ requires an anchor condition for uniqueness which without loss of generality we take to be zero at $\xi^0 = \xi_a^0 = 0$, so that $\mathcal{Z}(0) = 0$. With this condition imposed, the reduced matrix in (35) is non-singular and it follows that $\underline{\mathcal{Z}} = 0$. Thus from (33)

$$V(x, t^0) = \frac{\beta x}{t^0} \tag{36}$$

($cf.$ (12)) for all x, as in (12).

2. In the second part of the proof, under deformation of the domain generated by the similarity velocity (36) the similarity variable $\xi^1 = \widehat{x}(t^1)/(t^1)^\beta$ coincides with ξ^0 at time $t^1 = t + h$ to second order in h, as in Theorem 1. In particular, the nodes $\widehat{X}_i$ satisfy

$$\frac{\widehat{X}_i(t^1)}{(t^1)^\beta} = \frac{\widehat{X}_i(t^0)}{(t^0)^\beta} + O(h^2), \qquad (i = 0, \ldots, I+1) \tag{37}$$

3. In the third part of the proof we show that under the deformation generated by (37) the L_2 projection $\mathcal{N}(\xi)$ of $\eta(\xi)$ is also invariant

at $t = t^1$ to order h^2 using the conservation law (27) in the form

$$\int_{a(t^1)}^{b(t^1)} W_i(\zeta^1)\, U(\chi, t^1)\, \mathrm{d}\chi = \int_{a(t^0)}^{b(t^0)} W_i(\zeta^0)\, U(\chi, t^0)\, \mathrm{d}\chi \tag{38}$$

($i = 0, \ldots, N+1$), where $\zeta^1 = \chi/(t^1)^\beta,\ \zeta^0 = \chi/(t^0)^\beta$.

Expanding the piecewise-linear functions $U(\widehat{x}(t^1), t^1)$ in terms of the basis functions $W_i(\xi^1)$ as

$$U\left(\widehat{x}(t^1), t^1\right) = \sum_{j=1}^{J} U_j^1 W_j(\xi^1) \tag{39}$$

where $\xi^1 = \widehat{x}(t^1)/(t^1)^\beta$, equation (38) yields the matrix equation

$$\widetilde{\mathcal{M}}(\underline{\widehat{X}}(t^1))\, \underline{U}^1 = \widetilde{\mathcal{M}}(\underline{\widehat{X}}(t^0))\, \underline{U}^0 \tag{40}$$

where $\underline{\widehat{X}} = \{\widehat{X}_i\}$, $\underline{U} = \{U_i\}$, and the $\widetilde{\mathcal{M}}(\underline{\widetilde{X}}(t))$ are standard piecewise-linear mass matrices, each depending on a vector of the nodal differences $\Delta\widehat{X}_i(t)\,(= \widehat{X}_i(t) - \widehat{X}_{i-1})(t)$.

By (37) the $\widetilde{\mathcal{M}}(\underline{\widehat{X}}(t^1))$ and $\widetilde{\mathcal{M}}(\underline{X}(t^0))$ are identical to order h^2 apart from a factor $(t/t^0)^{-\beta}$. It follows from equation (40) that $(t^1)^\gamma \underline{U}^1 = (t^0)^\gamma \underline{U}^0$ and hence from (39)

$$\frac{U(x(t^1), t^1)}{(t^1)^\gamma} = \frac{U(x(t^0), t^0)}{(t^0)^\gamma} + O(h^2) = \mathcal{N}(\xi^0) + O(h^2) \tag{41}$$

4. The fourth part of the proof is concerned with repetition of the first three parts over a further time interval (t^1, t^2), where $t^2 = t^1 + h$, with x^0, x^1 replaced by x^1, x^2, respectively. Returning to (32) and using the invariants (14) and (16) of ξ and $\mathcal{N}(\xi)$ to order h^2, we find at $t = t^1$

$$\int_{\xi_a}^{\xi_b} W_i(\zeta) \left\{ -\beta (t^1)^{\beta-1} \partial_\zeta(\zeta \mathcal{N}(\zeta) + \partial_\xi(\mathcal{N}(\zeta) V(x, t^1)) \right\} \mathrm{d}\chi = O(h),$$

where now $\zeta = \chi/(t^1)^\beta$ and the right hand side is of first order rather than second order in h since one power of h is lost in the differentiation used in deriving (32) from (28) using (25). Also, from (30) the function W_i is advected by the induced velocity to only first order in h, so $W_i(\xi^1) = W_i(\xi^0) + O(h)$ at $t = t^1$.

We then deduce by the argument from equations (32) to (36) with the superfix 0 replaced by 1 that

$$V(x, t^1) = \frac{\beta x}{t^1} + O(h)$$

Further, by the argument from (37) to (41) with the superfixes 0 and 1 replaced by 1 and 2, respectively,

$$\xi^2 = \frac{\widehat{x}(t^2)}{(t^2)^\beta} = \frac{\widehat{x}(t^0)}{(t^0)^\beta} = \xi^0, \tag{42}$$

and

$$\mathcal{N}(\xi^2) = \frac{U(\widehat{x}(t^2), t^2)}{(t^2)^{-\beta}} = \frac{U(\widehat{x}(t^0), t^0)}{(t^0)^{-\beta}} = \mathcal{N}(\xi^0), \tag{43}$$

at $t = t^2$, together with two terms of order h^2 in each equation (42) and (43).

5. After n time steps of h we obtain

$$V(x, t^{n-1}) = \frac{\beta x}{t^{n-1}} + O(h),$$

as well as

$$\xi^n = \frac{\widehat{x}(t^n)}{(t^n)^\beta} = \frac{\widehat{x}(t^0)}{(t^0)^\beta} = \xi^0 \tag{44}$$

and

$$\mathcal{N}(\xi^n) = \frac{U(\widehat{x}(t^n), t^n)}{(t^n)^{-\beta}} = \frac{\widehat{U}_i(x(t^0), t^0)}{(t^0)^{-\beta}} = \mathcal{N}(\xi^0), \tag{45}$$

together with n terms of order h^2 in each equation (44) and (45).

6. In the final part of the proof we let $h \to 0$ and $n \to \infty$ in such a way that $nh = t - t^0$, so that $nh^2 = O(h)$. It then follows that at time t

$$V(x, t) = \frac{\beta x}{t} + O(h),$$

as well as

$$\xi = \frac{\widehat{x}(t)}{t^\beta} = \xi^0 + O(h), \qquad \mathcal{N}(\xi) = \frac{\widehat{U}(t)}{t^{-\beta}} = \mathcal{N}(\xi^0) + O(h),$$

so that in the limit,

$$V(x, t) = \frac{\beta x}{t},$$

equivalent to

$$V(\widehat{x}(t), t) = \frac{\beta \widehat{x}(t)}{t},$$

as well as

$$\xi = \xi^0, \qquad \mathcal{N}(\xi) = \mathcal{N}(\xi^0),$$

for all $t > t^0$. Thus $U(\widehat{x}(t), t)$ coincides with the L_2 projection of the self-similar solution, and $V(\widehat{x}(t), t)$ coincides with the similarity velocity at any time $t > t^0$, for all $a(t) \leq \widehat{x}(t) \leq b(t)$.

This completes the proof.

Corollary: The moving nodes $\widehat{X}_i$ satisfy

$$\frac{\widehat{X}_i(t)}{(t)^\beta} = \frac{\widehat{X}_i(t^0)}{(t^0)^\beta} \qquad (i = 0, \dots, N+1)$$

4 General initial conditions

In the Theorems of the previous two sections the initial conditions coincided with a self-similar solution or its piecewise-linear L^2 projection. However, it is of interest to study methods based on the steps of the proofs for *general* initial conditions, having the property that they propagate self-similar scaling solutions exactly in the event that the initial condition coincides with a self-similar solution, a property referred to here as the S-property.

The main differences for general initial conditions are that $\partial_t u$ is no longer given by (4) nor $\partial_t U$ by (25), hence the velocity is no longer (12) or (36). Consequently, neither the time evolution of the moving coordinate $\widehat{x}(t)$ (or $\widehat{X}(t)$) nor the propagation of the solution $u(\widehat{x}(t), t)$ (or $U(\widehat{x}(t), t)$) are as in the Theorems. Nevertheless, we can adapt the steps of the proofs to construct methods based on conservation for first-order-in-time scale-invariant PDE problems with general initial conditions, leading to formulae for the velocity, moving coordinate, and solution that are consistent with Theorems 1 and 2.

4.1 Calculation of a general velocity

4.1.1 The analytic case

Suppose that a first-order-in-time scale-invariant PDE for the function $u(x,t)$ is written in the form

$$u_t = \mathcal{L}u, \qquad (a(t) < x < b(t)) \tag{46}$$

where $\mathcal{L}$ is a purely spatial operator, with boundary conditions ensuring constant total mass (2).

Then from (8) with $x_1(t) = a(t)$ and $\widehat{x}_2(t) = \widehat{x}(t)$ and (46), the local conservation of mass principle (7) implies that

$$\int_{a(t)}^{\widehat{x}(t)} \mathcal{L}u \mathrm{d}\chi + [uv]_{a(t)}^{\widehat{x}(t)} = 0 \tag{47}$$

which yields the velocity formula

$$v(\widehat{x}(t),t) = \frac{(uv)|_{a(t),t} - \int_{a(t)}^{\widehat{x}(t)} \mathcal{L}u \, \mathrm{d}\chi}{u(\widehat{x}(t),t)} \tag{48}$$

provided that $u(\widehat{x}(t),t) \neq 0$. If $a(t^0)$ is an anchor point at which $v = 0$, the velocity reduces to

$$v(\widehat{x}(t),t) = -\frac{\int_{a(t)}^{\widehat{x}(t)} \mathcal{L}u \, \mathrm{d}\chi}{u(\widehat{x}(t),t)} \tag{49}$$

When u coincides with a self-similar solution the conservation equation (47) reverts to (8) which ensures, as in Theorem 1, that the velocity is the similarity velocity (5).

If the PDE takes the form

$$\partial_t u = \mathcal{L}u = \partial_x f[u], \tag{50}$$

where $f[u]$ is a flux function depending on u and its space derivatives, the velocity (48) can be written

$$v(\widehat{x}(t),t) = \frac{(uv)|_{a(t),t} - [f[u]]_{a(t)}^{\widehat{x}(t)}}{u(\widehat{x}(t),t)} \tag{51}$$

4.1.2 Piecewise linear L^2 projections

From now on we restrict the argument to problems governed by PDEs of the form (50) with zero net flux boundary conditions ensuring that the total mass is constant in time.

Define the weak form of (50) given by

$$\int_{a(t)}^{b(t)} W_i(\zeta) \{\partial_t u - \partial_\chi f[u]\} \,\mathrm{d}\chi = 0 \tag{52}$$

where $\zeta = \chi/t^\beta$, for all piecewise-linear test functions W_i belonging to a partition of unity on the subdivision (22) at time t.

Under the distributed conservation of mass principle (27), from (52) and (28),

$$\int_{a(t)}^{b(t)} W_i(\zeta) \{\partial_\chi f[u] + \partial_\chi (uv)\} \,\mathrm{d}\chi = 0 \tag{53}$$

Integrating (53) by parts using the zero net flux boundary conditions we obtain the weak form

$$\int_{a(t)}^{b(t)} (\partial_\chi W_i)\,(f[u] + uv) \,\mathrm{d}\chi = 0 \tag{54}$$

Taking u and v to be piecewise-linear functions U and V, from (54) the velocity V satisfies

$$\int_{a(t)}^{b(t)} (\partial_\chi W_i)\,(UV + f[U]) \,\mathrm{d}\chi \,\mathrm{d}\chi = \mathcal{E}_i(u, U), \tag{55}$$

where

$$\mathcal{E}_i(u, U) = \int_{a(t)}^{b(t)} (\partial_\chi W_i)\ \{f[U] - f[u]\} \,\mathrm{d}\chi, \tag{56}$$

is a projection error associated with the function $f[u]$.

When $U(x, t)$ coincides with the L^2 projection of a self-similar solution, equation (55) reverts to (28), which ensures that the velocity given by (55) is the similarity velocity (36), as in Theorem 2.

4.2 Evolution of the moving coordinate

In order to find the moving coordinate $\widehat{x}(t)$ for general initial conditions we consider the scale-invariant differential equation problem

$$\frac{\mathrm{d}\widehat{x}}{\mathrm{d}t} = v(\widehat{x}(t), t), \qquad \widehat{x}(t^0) = x,$$

seeking a solution for $\widehat{x}(\mathrm{t})$ that reduces to (5) in the event that v is the similarity velocity. (The often-used explicit Euler scheme does not have this property.)

From the second of (3) we note that in the case of similarity the function $\widehat{y}(t) = \widehat{x}(t)^{1/\beta} = \xi^{1/\beta}t$ is linear in t. Hence the formula

$$\widehat{y}(t) = \widehat{y}(t^0) + (t - t^0)\left(\frac{\mathrm{d}\widehat{y}}{\mathrm{d}t}\right)^0 \tag{57}$$

generates $\widehat{y}(t)$ from $\widehat{y}(t^0)$ exactly.

By the chain rule

$$\frac{\mathrm{d}\widehat{y}}{\mathrm{d}t} = \frac{\mathrm{d}\widehat{y}}{\mathrm{d}\widehat{x}}\frac{\mathrm{d}\widehat{x}}{\mathrm{d}t}$$

so equation (57) can be written as

$$\widehat{y}(t) = \widehat{y}(t^0) + (t - t^0)\beta^{-1}\left\{\widehat{x}(t^0)\right\}^{(1/\beta)-1}v(\widehat{x}(t^0), t^0),$$

or entirely in terms of $\widehat{x}(t)$ as

$$\widehat{x}(t) = \left(1 + \beta^{-1}(t - t^0)\ \frac{v(\widehat{x}(t^0), t^0)}{\widehat{x}(t^0)}\right)^{\beta}\widehat{x}(t^0) \tag{58}$$

Equation (58) is a general formula for the evolution of the spatial coordinate $\widehat{x}(t)$ with the property that is exact in the event that $v(\widehat{x}(t^0), t^0)$ is the similarity velocity $\beta\widehat{x}(t^0)/(t^0)$.

4.3 Solution retrieval

It remains to retrieve the solution on the deformed domain.

4.3.1 The analytic case

The conservation of mass principle (7) implies that

$$\int_{a(t)}^{\widehat{x}(t)} u(\chi, t)\mathrm{d}\chi = \int_{a(t^0)}^{\widehat{x}(t^0)} u(\chi, t^0)\mathrm{d}\chi$$

Differentiating wrt $\widehat{x}(t)$,

$$u(\widehat{x}, t) = \frac{\mathrm{d}}{\mathrm{d}\widehat{x}(t)}\int_{a(t)}^{\widehat{x}(t)} u(\chi, t^0)\mathrm{d}\chi$$

$$= \frac{\mathrm{d}\widehat{x}(t^0)}{\mathrm{d}\widehat{x}(t)} \frac{\mathrm{d}}{\mathrm{d}\widehat{x}(t^0)} \int_{a(t^0)}^{\widehat{x}(t^0)} u(\chi, t^0)\mathrm{d}\chi = \frac{\mathrm{d}\widehat{x}(t^0)}{\mathrm{d}\widehat{x}(t)} u(\widehat{x}, t^0) \tag{59}$$

In the event of an initial condition that coincides with a self-similar solution $u(\widehat{x}(t^0), t^0) = (t^0)^\gamma \eta(\xi^0)$ so, since $\widehat{x}$ is proportonal to t^β, it follows that $u(\widehat{x}(t), t)$ reduces to the self-similar solution (3).

4.3.2 Piecewise-linear L^2 projections

The conservation of distributed mass principle (27) implies that

$$\int_{a(t)}^{b(t)} W_i(\zeta)\, U(\chi, t)\mathrm{d}\chi = \int_{a(t^0)}^{b(t^0)} W_i(\zeta^0)\, U(\chi, t^0)\mathrm{d}\chi \tag{60}$$

where $\zeta = \chi/t^\beta$, $\zeta^0 = \chi/(t^0)^\beta$.

Expanding $U(\chi, t)$ and $U(\chi, t^0)$ as

$$U(\chi, t) = \sum_j U_j(t) W_j(\zeta), \qquad U(\chi, t^0) = \sum_j U_j(t^0) W_j(\zeta^0),$$

equation (60) yields the matrix equation

$$\mathcal{M}(\underline{\widehat{X}}(t))\underline{U}(t) = \underline{C} = \mathcal{M}(\underline{\widehat{X}}(t^0))\underline{U}(t^0)) \tag{61}$$

where $\underline{U}(t) = \{U_i(t)\}$, $\underline{C} = \{C_i\}$, and $\mathcal{M}(\widehat{X}(t))$ is a standard mass matrix for piecewise-linears in terms of the nodal coordinates $\widehat{X}_i(t)$.

When the initial condition coincides with a self-similar solution the components $U_i(t^0)$ are proportional to $(t^0)^\gamma$ and the $\widehat{X}_i(t)$ are proportional to t^β, so equation (61) leads back to (24).

4.4 Summary

Using local and distributed conservation of mass we have constructed two procedures which propagate a scaling symmetry exactly (modulo a projection error in the L^2 case) for a PDE problem of the form (50) with zero net flux at the boundaries ensuring constant total mass.

In the analytic case the combination of steps (48), (58) and (59) yields a scale-invariant procedure possessing the S-property.

In the piecewise linear L^2 case the combination of steps (66), (58) and (60) gives a scale-invariant procedure possessing the S-property in the L^2 norm modulo the projection error (56).

5 Discrete algorithms

In this section, devoted to discrete methods, we continue to focus on flux-driven PDEs of the form (50), with zero net flux boundary conditions ensuring that the total mass is constant in time.

5.1 Semi-discrete velocities

5.1.1 A pointwise approach

From (49) with a zero net flux condition at $a(t)$, the semi-discrete velocity $v_(t)$ at position $\widehat{x}(t)$ is given by

$$v(\widehat{x}(t),t) = -\frac{f[u]_{a(t)}^{\widehat{x}(t)}}{u(\widehat{x}(t),t)} \tag{62}$$

Pointwise, a semi-discrete velocity may be defined in terms of the semi-discrete solution $u_i(t)$ by sampling (62) at mesh points $\widehat{x}_i(t)$, giving

$$v_i(t) = -\frac{[f[u]]_{a(t)}^{\widehat{x}_i(t)}}{u_i(t)} \tag{63}$$

5.1.2 The piecewise-linear L^2 case

In the piecewise-linear L^2 case a semi-discrete velocity $V(x,t)$ may be determined in terms of $U(t)$ from (55) omitting the projection error (56), i.e.

$$\int_{a(t)}^{b(t)} (\partial_\chi W)\,(UV + f[U])\,\mathrm{d}\chi = 0, \tag{64}$$

which is already discrete in space.

Since $V(x,t)$ is piecewise-linear it can be expanded as

$$V(x,t) = \sum_{j=0}^{N+1} V_j(t)\,W_j(\xi)$$

where $\xi = x/t^\beta$. From (64),

$$\int_a^b (\partial_\chi W_i)\,U(\chi,t)\left(\sum_{j=0}^{N+1} V_j(t)\,W_j(\zeta)\right)\mathrm{d}\chi = -\int_a^b (\partial_\chi W_i)\,f[U(\zeta,t)]\mathrm{d}\chi \tag{65}$$

Equation (65) can be written as the matrix equation

$$\mathcal{B}(U)\underline{V} = \underline{b} \tag{66}$$

where $\mathcal{B}(U)$ is the matrix with entries

$$\int_a^b (\partial_\chi W_i(\zeta))\, U(\chi, t)\, W_j(\zeta)\, \mathrm{d}\chi$$

and $\underline{V} = \{V_i\}$, $\underline{b} = \{b_i\}$ in which b_i is the right hand side of (65).

Because the basis functions W_i^n form a partition of unity the matrix equation (69) is not of full rank, but there is a unique solution if the velocity is prescribed at one (anchor) point.

We now give details of two fully discrete scale-invariant algorithms which aim to possess the S-property, beginning with a finite difference scheme.

5.2 Fully discrete velocities

5.2.1 A finite-difference approach

In a finite-difference approach the data representation is pointwise at mesh points x_i with corresponding values u_i say, $(i = 0 \ldots, N+1)$.

In this section we further restrict the argument to PDEs of the form

$$\partial_t u = \partial_x \{u \partial_x p(u)\}, \tag{67}$$

where $p(u) > 0$ is a function of u only, with zero net flux boundary conditions ensuring preservation of the total mass. In certain applications the function $p(u)$ may be identified with a pressure.

From (63) the velocity sampled at the point $\widehat{x}_i^n$ at time t^n is then

$$v_i^n = -\{\partial_x p(u)\}_i^n \tag{68}$$

where $\partial_x p(u)$ is yet to be approximated.

The way in which $\{\partial_x p(u)\}_i$ is estimated from gridpoint values u_i^n is crucial to the preservation of similarity solutions. In the case of similarity the velocity v is linear in $\widehat{x}$ by (5). Thus the S-property may be achieved if the right hand side of (68) is approximated in such a way that it is exact for linears. Such a linear function may be obtained for example by differentiating the interpolating quadratic through values of $p(u)$ at adjacent gridpoints.

5.2.2 A finite-element approach

In the finite element approximation the data representation is piecewise-linear, thus the functions $W(x,t^n)$, $V(x,t^n)$, $X(x,t^n)$, and $U(x,t^n)$ are all piecewise-linear.

At time $t = t^n$ equation (65) can be written as the matrix equation

$$\mathcal{B}(U^n)\underline{V}^n = \underline{b}^n \tag{69}$$

where $\mathcal{B}(U^n)$ is the matrix with entries

$$\int_{a^n}^{b^n} (\partial_\chi W_i(\zeta))\, U(\chi,t^n)\, W_j(\zeta)\, \mathrm{d}\chi$$

where $\zeta = \chi/(t^n)^\beta$, and $\underline{V}^n = \{V_i^n\}$, $\underline{b}^n = \{b_i^n\}$ in which b_i^n from (65) is

$$b_i^n = -\int_{a^n}^{b^n} (\partial_\chi W_i^n)\, f[U(\zeta,t^n)]\mathrm{d}\chi \tag{70}$$

We now consider discretisation in time.

5.3 Time stepping

When the nodal velocities are not similarity velocities, as in the case of general initial conditions, the time evolution (58) from t^0 to t is not exact. Nevertheless, (58) can still be used as one step of a first-order-in-time explicit scheme from t^n to t^{n+1} $(= t^n + h)$, where h is the time step, having the property that it is exact in the case of similarity.

We therefore use the first-order scheme

$$x_i^{n+1} = \left(1 + \beta^{-1} h\, \frac{v_i^n}{\widehat{x}_i^n}\right)^\beta \widehat{x}_i^n \tag{71}$$

where x_i^n and v_i^n are the nodal positions and nodal velocities, respectively, having the property that the x_i^{n+1} are exact in the case of self-similarity. In the finite-element algorithm the nodal positions $\widehat{X}_j^n$ are updated using (71) in the form

$$\widehat{X}_i^{n+1} = \left(1 + \beta^{-1} h\, \frac{V_i^n}{\widehat{X}_i^n}\right)^\beta \widehat{X}_i^n \tag{72}$$

where V_i^n is the nodal velocity.

5.4 Solution retrieval

It remains to retrieve the approximate solutions u_i^{n+1} or U_i^{n+1} at the forward time $t = t^{n+1}$.

5.4.1 Finite-difference solution retrieval

In the finite-difference algorithm for mass-conserving problems of the form (67) equation (59) may be discretised over a time step from t^n to t^{n+1} as

$$u_i^{n+1} = \frac{\Delta \widehat{x}_i^n}{\Delta \widehat{x}_i^{n+1}} u_i^n$$

where $\Delta \widehat{x}_i$ is a spatial difference approximating $\mathrm{d}x$. We use a centred finite-difference discretisation giving

$$\widehat{u}_i^{n+1} = \left(\frac{\widehat{x}_{i+1}^n - \widehat{x}_{i-1}^n}{\widehat{x}_{i+1}^{n+1} - \widehat{x}_{i-1}^{n+1}} \right) \widehat{u}_i^n \tag{73}$$

Boundary values may be specified by Dirichlet boundary conditions.

In the case of similarity the exactness of the node positions and the invariance of the approximate mass ensure that the exactness of the nodal solution values is maintained in time at nodes. If the nodal values are exact at time t^0 they remain exact for all time.

Note that in the finite difference scheme the discretisations (68) and (73) of the two forms of conservation used are equivalent only approximately.

5.4.2 Finite-element solution retrieval

In the L^2 case, for mass-conserving problems of the form (50), using (60) the piecewise-linear solution $U(x, t^{n+1})$ is obtained from $U(x, t^n)$ through

$$\int_{a^{n+1}}^{b^{n+1}} W_i(\zeta^{n+1})\, U(\chi, t^{n+1}) \mathrm{d}\chi = \int_{a^n}^{b^n} W_i(\zeta^n)\, U(\chi, t^n) \mathrm{d}\chi, \tag{74}$$

where $\zeta^n = \chi/(t^n)^\beta$, $\zeta^{n+1} = \chi/(t^{n+1})^\beta$, which generates $U(\widehat{x}, t)$ via (61).

The finite-element solution $U^{n+1}(x)$ is then obtained from (74) via (61) in the form

$$\mathcal{M}(\underline{\widehat{X}}(t))\underline{U}^{n+1} = \mathcal{M}(\underline{\widehat{X}}(t^n))\underline{U}(t^n)) \tag{75}$$

where $\mathcal{M}(\widehat{\underline{X}}(t))$ is a mass matrix and $\underline{U} = \{U_i\}$.

In the case of similarity the exactness of the nodal positions and the invariance of the distributed mass-fractions (27) ensure that the L^2 projection property of the finite element solution is maintained in time.

5.5 Algorithms

We now summarise these algorithms.

5.5.1 The finite-difference algorithm

A scale-invariant finite-difference algorithm for scale-invariant mass-conserving PDE problems of the form (67) (where β is the scaling power for x) with zero net flux boundary conditions is as follows:

Algorithm 1

Given nodes x_i^0 and nodal values u_i^0 sampled from an initial condition at time t^0, then at each time $t^n \geq t^0$,

1. Obtain the velocity v_i^n from (68) using a quadratic interpolation of $p(u)_i$ values
2. Advance $\widehat{x}_i^n$ to $\widehat{x}_i^{n+1}$ using (71)
3. Retrieve u_i^{n+1} from (73)

The algorithm is scale-invariant with the same invariants as the PDE problem and possesses the S-property in the l^∞ norm. A sufficiently small time step is required such that step 2 is stable.

Boundary conditions on u can be imposed in step 3.

A similar algorithm appears in the literature [7, 5, 20, 21, 24, 10] although the time step there is always the explicit Euler scheme rather than that of (71).

5.5.2 The finite-element algorithm

A piecewise-linear finite-element algorithm for scale-invariant mass-conserving PDE problems of the form (50) (where β is the scaling power for x) with zero net flux boundary conditions is as follows:

Algorithm 2

Given nodes X_i^0 and U^0, the L^2 projection of the initial condition $u(x,t^0)$, at time t^0, then at each time $t^n \geq t^0$,

1. Obtain the piecewise-linear velocity V^n from (69)
2. Advance the nodes X_i^n to X_i^{n+1} using (72)
3. Retrieve U_i^{n+1} using (75)

The algorithm is scale-invariant with the same scaling invariants as the PDE problem and possesses the S-property in the L^2 norm, modulo the projection error (56). A small enough time step is required for the time step to be stable.

Boundary conditions on U can be imposed in step 3 but care is required that the family of test functions $W_i(x,t)$ remains a partition of unity (see e.g. [19, 27]).

It is known that the matrix in the reduced form of equation (69) is awkward to invert numerically since the entries oscillate in sign and the matrix $\mathcal{B}(U^n)$ is poorly conditioned

A similar algorithm appears in the literature [1, 2, 3, 4, 9, 27] although the time step there is always the explicit Euler scheme rather than that of (71) and the velocity is obtained indirectly through a velocity potential rather than from (69), avoiding the ill-conditioning of the matrix $\mathcal{B}(U^n)$.

6 Numerical illustrations

6.1 A nonlinear PDE problem

We illustrate the behaviour of the errors in the finite element and finite difference algorithms for the example of a nonlinear diffusion problem governed by the porous medium equation PDE

$$u_t = \partial_x\{u^2(\partial_x u)\} = \partial_x\{u\partial_x(u^2/2)\}, \qquad (a(t) < x < b(t)), \qquad (76)$$

(in which $f[u] = u^2\partial_x u$ and $p(u) = \partial_x(u^2/2)$, where $u = 0$ on the free boundaries $a(t), b(t)$ (so zero net mass flux), which is mass-conserving and scale-invariant with $\beta = 1/4$.

The initial time is $t^0 = 1$ and the initial domain is $(-1 < x < 1)$. We consider the two initial conditions,

$$(a) \qquad u(x,1) = \frac{1}{2}(1-x^2)_+^{1/2} \qquad (77)$$

where the suffix + indicates the positive part, taken from the self-similar solution [8, 26], and

$$(b) \qquad u(x,1) = \frac{1}{2}(1-x^2)_+^{1/2} + \frac{1}{2}(1-x^2)_+ \tag{78}$$

a non self-similar initial condition with the same compact support that does not generate a waiting time.

6.2 Finite differences

In the finite difference algorithm of section 5.5.1 the first step is to obtain the velocity approximation v_i from (68) where for the PDE (76) $p(u) = u^2/2$ and so

$$v_i^n = -u_i(\partial_x u)_i^n = -\{\partial_x(u^2/2)\}_i^n$$

which requires an estimate of the derivative $\partial_x(u^2)$ at each node. For this estimate to be exact at the nodes in the case of similarity it is sufficient to differentiate the quadratic interpolant through nodal values of $p(u)_i$, i.e. $u_i^2/2$, at adjacent nodal values.

Time-stepping is performed using (71) at each node i, and the approximate solution u_i^{n+1} retrieved from (73).

6.2.1 Case (a)

In the initial data case (a), for values of N ranging from 5 to 160, it is found that, as expected, taking a single time step of h results in the errors in both the relative l^∞ norm of the solution and the relative boundary position being at the level of rounding error. After one time step the solution coincides with the self-similar solution at $t^1 = 1 + h$, still proportional to the source term, so the velocity (68) at $t = t^1$ is

$$v_i^1 = \{\partial_x(u^2/2)\}_i^1$$

The derivative is again obtained from the quadratic interpolant through nodal values $(u^2/2)_i^1$ at adjacent nodal values.

Proceeding in this way, the solution at each time step is exact for multiple time steps, exhibiting the same property as a single time step (provided that h chosen to ensure stability). Thus, after 100 time steps (with $h = 1/N^2$ chosen to ensure stability) the errors in both the relative l^∞ norm of the solution and the relative boundary position are found to be of the order of rounding error, for numbers of nodes ranging from 5 to 160.

6.2.2 Case (b)

In the more general case (b), with N ranging from 10 to 80 the errors in both the relative l^∞ norm of the solution and the relative boundary position when compared the solution for 160 nodes (taken to be a very accurate solution) are shown in Table 1. The time step taken to avoid instability is $h = 1/N^2$.

N	Δt	Relative error $e_N(u)$	Relative error $e_N(x)$
10	0.01	1.2×10^{-2}	2.6×10^{-3}
20	0.0025	5.5×10^{-3}	9.0×10^{-4}
40	0.000625	2.4×10^{-3}	3.0×10^{-4}
80	0.00015625	8.7×10^{-4}	7.3×10^{-5}

Table 1: Relative errors $e_N(u)$ in the l^∞ norm of u and $e_N(x)$ in the boundary position, at $t = 2$, when compared with the solution for 160 nodes (taken as a very accurate solution) for the PME (76) when the initial condition is (78).

6.3 Finite elements

In the finite-element algorithm of section 5.5.2 the velocity V^n is given by (69) where in this case b_i^n is defined from (70) by

$$b_i^n = -\int_{a^n}^{b^n} (\partial_\chi W_i)^n (U^2 U_\chi)^n \mathrm{d}\chi, \tag{79}$$

omitting the projection error (56). Since the functions W_i, U are piecewise-linear, the integrand is piecewise quadratic and the integration in (79) can be carried out exactly using a composite Simpson's Rule.

Time-stepping is performed using (72) and the piecewise-linear approximation $U(x, t)$ retrieved from (75).

The finite-element solution is prevented from being exact in the case of similarity by the presence of the projection error (56) in the calculation of the velocity, which is reflected in the results. With $N = 20$, after one step of $h = 0.01$ the relative L^2 error in the the solution is approximately 0.002 and the relative error in the position of the boundary 0.001. Thereafter the build-up of error is very slow, which is not surprising since the major part of the error comes from disregarding the projection error (56). After 1000 time steps of 0.01

(chosen to ensure stability) the relative L^2 norm of the solution is approximately 0.008 and the relative l^∞ norm of the boundary 0.0004. If the exact velocity is used instead of the velocity computed from (69) the errors reduce to the level of rounding error.

Comparative results are given only for the initial condition case (a) where, with N ranging from 10 to 80, errors are shown in Table 2. The time step taken to avoid instability is $h = 1/N^2$.

N	h	Relative error e_N	Relative error X_N
10	0.01	1.3×10^{-2}	1.8×10^{-3}
20	0.0025	8.0×10^{-3}	5.0×10^{-4}
40	0.000625	4.3×10^{-3}	2.5×10^{-4}
80	0.00015625	2.2×10^{-3}	8.7×10^{-5}
160	0.0000390625	1.1×10^{-3}	3.1×10^{-5}

Table 2: Table of relative errors e_N in the L^2 norm of U, and $e_N(X)$ in the absolute value of the boundary position, at $t = 2$, in the case of initial data (a) for the finite-element algorithm.

7 Summary

In this paper we have studied the invariance of scaling symmetry in the evolution of one-dimensional time-dependent scale-invariant mass-conserving PDE problems. It was shown that, under local conservation of mass, initial conditions that coincide with self-similar solutions are propagated exactly in time, while under distributed conservation of mass, piecewise linear L^2 projections of initial conditions that coincide with the piecewise linear L^2 projections of self-similar solutions are propagated as piecewise linear L^2 projections exactly in time.

The steps in the proof were then adapted for general initial conditions in the case of first-order-in-time flux-driven problems, with the aim of obtaining a general procedure that possesses the S-property, i.e. exact propagation of a self-similar solution or its L^2 projection. A deformation velocity was constructed and used to move the nodes via a symmetry-preserving scheme. The solution was then post-processed algebraically from the Lagrangian form of the conservation.

A finite-difference algorithm based on this procedure was constructed for a subclass of problems possessing the S-property in the

l^∞ norm when the velocity is calculated by a special interpolation. A piecewise-linear finite-element algorithm was also described possessing the S-property in the L^2 norm, but subject to a projection error. Numerical illustrations verifying these results were shown for a nonlinear porous medium equation problem with a constant total mass, exhibiting results in accordance with the theory and showing the levels of accuracy in the propagation of relative errors for a non self-similar initial condition.

The S-property can be regarded as a yardstick for confidence in numerical schemes in the case of nonlinear scale-invariant problems, similar to the way in which standard schemes on fixed grids for linear problems based on Taylor series expansions are constructed so as to be exact for polynomial solutions of given degree.

One outcome of this paper is the scale-invariant finite-difference Algorithm 1, for mass-conserving PDE problems of the form (67), possessing the S-property in the l^∞ norm, when the initial condition is sampled from a self-similar solution at the nodes and the velocity is interpolated in a particular way. (The corresponding scale-invariant finite-difference Algorithm 2 does not achieve the same accuracy (in the L^2 norm) due to a projection error.) Comparisons with self-similar solutions are a favourite testing ground for numerical schemes: in this paper Algorithm 1 propagates the solution at the nodes exactly, thus preserving a discrete scaling symmetry.

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