An arithmetical mapping and applications to Ω-results for the Riemann zeta function[1]

Titus Hilberdink
Department of Mathematics, University of Reading, Whiteknights,
PO Box 220, Reading RG6 6AX, UK; t.w.hilberdink@reading.ac.uk

1. Introduction

In this paper we study the linear mapping φ_α which sends a sequence $\{a_n\}_{n\in\mathbb{N}} \to \{b_n\}_{n\in\mathbb{N}}$ where

$$b_n = \frac{1}{n^\alpha}\sum_{d|n} d^\alpha a_d, \tag{0.1}$$

and α is a real parameter. This mapping is just one example of a particular class of 'matrix' mappings, where the matrix is of 'multiplicative Toeplitz' type; that is, with entries a_{ij} of the form $f(i/j)$ where f is a function on the positive rationals (see, for example, [5]). In our case $f(n) = n^{-\alpha}$ for $n \in \mathbb{N}$ and zero otherwise.

First we study φ_α on the spaces l^p $(1 \le p \le \infty)$, giving necessary and sufficient conditions for φ_α to be a bounded mapping from l^p to l^q. We settle the question of boundedness for the 'boundary' cases $p = 1$, $q = \infty$, and $p = q$. For the 'interior' values $1 < p < q < \infty$, the question remains open. Further, for the 'boundary' cases, we show that the operator norm,

$$\|\varphi_\alpha\| = \sup_{\|a\|_p=1} \|\varphi_\alpha(a)\|_q$$

is intimately related to the Riemann zeta function. For example, for $p = q$, $\|\varphi_\alpha\| = \zeta(\alpha)$ for $\alpha > 1$. This result is perhaps implicit in the work of Toeplitz ([14], [15]) who studied related mappings. Various other authors have studied (sometimes indirectly) the mapping (see for example, Wintner [16]). Also of relevant interest are the recent papers [4] and [7].

In section 2, we study the mapping when it is unbounded on l^p by estimating the behaviour of

$$B_{p,q,\alpha}(N) = \sup_{\|a\|_p=1}\left(\sum_{n=1}^N |b_n|^q\right)^{1/q}$$

for large N. We obtain formulas for $B_{1,q,\alpha}(N)$ and $B_{p,\infty,\alpha}(N)$, while for the case $p = q$, we obtain approximate formulas. For example, for the case $p = q = 2$ and writing $B_\alpha(N)$ for $B_{2,2,\alpha}(N)$,

$$\frac{(\log N)^{1-\alpha}}{2(1-\alpha)\log\log N} \lesssim \log B_\alpha(N) \lesssim \frac{(1+(2\alpha-1)^{-\alpha})(\log N)^{1-\alpha}}{2(1-\alpha)\log\log N}. \qquad (\tfrac{1}{2} < \alpha < 1)$$

In the next section, we show that $B_\alpha(N)$ provides a lower bound for $\max_{t\le N}|\zeta(\alpha+it)|$. For the Dirichlet polynomial, $A_N(t) = \sum_{n\le N} a_n n^{it}$, and $\alpha > \frac{1}{2}$

$$\frac{1}{T}\int_0^T |\zeta(\alpha+it)|^2 |A_N(t)|^2\, dt \sim \sum_{m,n\le N} \frac{a_m\overline{a_n}(m,n)^{2\alpha}}{(mn)^\alpha}. \tag{0.2}$$

But the right-hand side above is also close to $\frac{1}{\zeta(2\alpha)}\sum_{n\le N}|b_n|^2$. We show that (0.2) also holds for N as large as T^λ for some $\lambda > 0$ if $(a_n) \in l^2$. Results of this type (with a larger

[1]Acta Arithmetica **139** (2009) 341-367

range of λ) are well-known (see for example [1] and [8]) especially for $\alpha = \frac{1}{2}$, but under the condition that $a_n = O(n^{-\alpha})$. As a result we find that $B_\alpha(N)$ provides a lower bound for $\max_{t\le T} |\zeta(\alpha+it)|$: for every $\varepsilon > 0$,

$$\max_{t\le T} |\zeta(\alpha+it)| \ge B_\alpha(T^{\frac{2}{3}(\alpha-\frac{1}{2})-\varepsilon})$$

for all T sufficiently large.

Using the lower bounds obtained in §2, one has

$$\max_{0\le t\le T} |\zeta(\alpha+it)| \ge \exp\left\{c\frac{(\log T)^{1-\alpha}}{\log\log T}\right\}, \qquad (\tfrac{1}{2} < \alpha < 1)$$

for some $c > 0$, and $\max_{1\le t\le T} |\zeta(1+it)| \ge e^\gamma \log\log T + O(1)$. The result for $\alpha = 1$ is close to best known, but for $\frac{1}{2} < \alpha < 1$, the better bound with $\log\log T$ replaced by $(\log\log T)^\alpha$ is available (see [9]). However, with little extra effort, we show in Theorem 3.5 that $|\zeta(\alpha+it)|$ is this large for a fairly large set of values from $[0,T]$ by showing that for all $c > 0$ sufficiently small, the measure of the set

$$\left\{t \in [0,T] : |\zeta(\alpha+it)| > \exp\left\{c\frac{(\log T)^{1-\alpha}}{\log\log T}\right\}\right\}$$

is at least $T^{(1+2\alpha)/3}$ for $\frac{1}{2} < \alpha < 1$, while for A sufficiently large, the measure of the set

$$\left\{t \in [1,T] : |\zeta(1+it)| > e^\gamma \log\log T - A\right\}$$

is at least $T\exp\{-a\frac{\log T}{\log\log T}\}$ for some $a > 0$. (By quite different methods a similar (but superior) result was obtained recently in [2].)

Of interest here is that these bounds are found by (almost) purely arithmetical means, involving neither detailed estimates of $\zeta(s)$ in and near the critical strip nor the Dirichlet or Kronecker theorems. Indeed, they basically involve estimating the maximum order of the function

$$\frac{1}{d(n)}\sum_{d|n} \sigma_{-\alpha}(d)^2.$$

The size of $B_\alpha(N)$ for large N is also closely connected to the largest eigenvalue $\Lambda_N(\alpha)$ of the $N\times N$-matrix with entries $\frac{(i,j)^{2\alpha}}{(ij)^\alpha}$, which was discussed recently in [7]. The approximate formulas obtained for $B_\alpha(N)$ then imply similar formulas for $\Lambda_N(\alpha)$; for example, we show

$$\Lambda_N(1) = \frac{6}{\pi^2}(e^\gamma \log\log N + O(1))^2.$$

Acknowledgement. The author is grateful to the referee for pointing out the recent paper by Soundararajan [11], in which a "resonator" method was developed and used to find Ω-results for $\zeta(\frac{1}{2}+it)$. The method employed in this paper regarding Ω-results for $\zeta(\alpha+it)$ is similar in nature. Indeed, we subsequently used Soundararajan's method to obtain the upper bounds for $B_{p,p,\alpha}(N)$ for $\alpha > \frac{1}{p}$ and the approximate formula for the case $\alpha = \frac{1}{p}$.

1.1 Some preliminaries

(a) For $a \in l^p$ $(1 \le p \le \infty)$, let $\|a\|_p$ denote the usual l^p-norm:

$$\|a\|_p = \|(a_n)\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} \qquad \text{for } p < \infty, \qquad \|a\|_\infty = \sup_{n \ge 1} |a_n|.$$

A linear operator $f : l^p \to l^q$ is *bounded* if there exists A such that $\|f(x)\|_q \le A\|x\|_p$ for all $x \in l^p$. As such, the *operator norm* is defined by $\|f\| = \sup_{\|x\|_p=1} \|f(x)\|_q$.

(b) **Maximal order of some arithmetical functions**
(1) $\sigma_{-\alpha}(n) = \sum_{d|n} d^{-\alpha}$. We have the well-known results (see for example [3]):

$$(i) \qquad \limsup_{n\to\infty} \sigma_{-\alpha}(n) = \zeta(\alpha) \qquad \textit{for } \alpha > 1,$$

$$(ii) \qquad \limsup_{n\to\infty} \frac{\sigma_{-1}(n)}{e^\gamma \log\log n} = 1 \qquad \textit{where } \gamma \textit{ is Euler's constant,}$$

$$(iii) \qquad \max_{r\le n} \sigma_{-\alpha}(r) = \exp\left\{\frac{(1+o(1))(\log n)^{1-\alpha}}{(1-\alpha)\log\log n}\right\} \qquad \textit{for } 0 < \alpha < 1.$$

(2) For $\alpha, \beta > 0$, let $\eta_{\alpha,\beta}(n)$ denote the multiplicative function

$$\eta_{\alpha,\beta}(n) = \frac{1}{d(n)} \sum_{d|n} \sigma_{-\alpha}(d)^\beta.$$

We have

$$(i) \qquad \limsup_{n\to\infty} \eta_{\alpha,\beta}(n) = \zeta(\alpha)^\beta \qquad \textit{for } \alpha > 1,$$

$$(ii) \qquad \max_{r\le n} \eta_{1,\beta}(r) = (e^\gamma \log\log n + O(1))^\beta,$$

$$(iii) \qquad \exp\left\{\frac{(\beta+o(1))(\log n)^{1-\alpha}}{2(1-\alpha)\log\log n}\right\} \le \max_{r\le n} \eta_{\alpha,\beta}(r) \le \exp\left\{\frac{(\beta+o(1))(\log n)^{1-\alpha}}{(1-\alpha)\log\log n}\right\},$$

for $0 < \alpha < 1$.

Proof. Note that for $\beta > 0$, $\eta_{\alpha,\beta}(n) \le \sigma_{-\alpha}(n)^\beta$. Hence, $\eta_{\alpha,\beta}(n) < \zeta(\alpha)^\beta$ for $\alpha > 1$, $\eta_{1,\beta}(n) \le (e^\gamma \log\log n + O(1))^\beta$ (see [10]), and the upper bound in (iii) holds. We need therefore only consider lower bounds.

As $\eta_{\alpha,\beta}$ is multiplicative, consider the behaviour at powers of a prime. We have for p prime and $k \in \mathbb{N}$

$$\eta_{\alpha,\beta}(p^k) = \frac{1}{k+1} \sum_{r=0}^{k} \sigma_{-\alpha}(p^r)^\beta = \frac{1}{k+1} \sum_{r=0}^{k} \Big(\frac{1-p^{-(r+1)\alpha}}{1-p^{-\alpha}}\Big)^\beta$$
$$= \Big(1 - \frac{1}{p^\alpha}\Big)^{-\beta} \cdot \frac{1}{k+1} \sum_{r=0}^{k} \Big\{1 + O\Big(\frac{1}{p^{(r+1)\alpha}}\Big)\Big\} = \Big(1 - \frac{1}{p^\alpha}\Big)^{-\beta} \Big(1 + O\Big(\frac{1}{p^\alpha k}\Big)\Big).$$

(i) Suppose now $\alpha > 1$. Let n be of the form $2^{a_2} 3^{a_3} \dots P^{a_P}$ where $a_p = [\frac{\log P}{\log p}]$. Note that $\log n = \psi(P)$, where ψ is the usual Chebyshev function. By the Prime Number

Theorem, $\log n \sim P$ as $n \to \infty$ through such values. Putting $k = [\frac{\log P}{\log p}]$ in the above gives

$$\eta_{\alpha,\beta}(n) = \prod_{p \le P} \eta_{\alpha,\beta}(p^{a_p}) = \prod_{p \le P} \Big(1 - \frac{1}{p^\alpha}\Big)^{-\beta} \cdot \exp\Big\{O\Big(\frac{1}{\log P} \sum_{p \le P} \frac{\log p}{p^\alpha}\Big)\Big\}.$$

As $n \to \infty$, the RHS tends to $\zeta(\alpha)^\beta$, proving the result.

(ii) Now we consider the case $\alpha = 1$. If we take n as in (i), we only obtain[2] $\eta_{1,\beta}(n) \asymp (\log\log n)^\beta$. Instead we take $n = \prod_{p \le P} p^{b_p}$ with $b_p = [\sqrt{P/p}]$. Then

$$\begin{aligned}\eta_{1,\beta}(n) &= \prod_{p \le P} \eta_{1,\beta}(p^{b_p}) = \prod_{p \le P} \Big(1 - \frac{1}{p}\Big)^{-\beta} \cdot \exp\Big\{O\Big(\frac{1}{\sqrt{P}} \sum_{p \le P} \frac{1}{\sqrt{p}}\Big)\Big\} \\ &= (e^\gamma \log P)^\beta \Big(1 + O\Big(\frac{1}{\log P}\Big)\Big) = (e^\gamma \log P + O(1))^\beta,\end{aligned}$$

by Merten's Theorem and the Prime Number Theorem. But $\log n = \sum_{p \le P} [\sqrt{P/p}] \log p \asymp P$, so that $\log P = \log\log n + O(1)$. Now, if s_k is the k^{th} number of this form (i.e. $s_k = \prod_{p \le p_k} p^{b_p}$ where p_k is the k^{th} prime), then $\log s_k \asymp p_k \asymp \log s_{k+1}$. Hence for $s_k \le n < s_{k+1}$, $\log n \asymp \log s_k$ and $\log\log n = \log\log s_k + O(1)$. It follows that

$$\max_{r \le n} \eta_{1,\beta}(r) \ge \eta_{1,\beta}(s_k) = (e^\gamma \log\log s_k + O(1))^\beta = (e^\gamma \log\log n + O(1))^\beta.$$

For (iii), we have for n squarefree

$$\eta_{\alpha,\beta}(n) = \prod_{p|n} \eta_{\alpha,\beta}(p) = \prod_{p|n} \frac{1}{2}\Big(1 + \Big(1 + \frac{1}{p^\alpha}\Big)^\beta\Big) = \prod_{p|n}\Big(1 + \frac{\beta}{2p^\alpha} + O\Big(\frac{1}{p^{2\alpha}}\Big)\Big).$$

In particular, for $n = 2.3 \ldots P$ (so that $\log n \sim P$), we have

$$\begin{aligned}\eta_{\alpha,\beta}(n) &= \prod_{p \le P}\Big(1 + \frac{\beta}{2p^\alpha} + O\Big(\frac{1}{p^{2\alpha}}\Big)\Big) = \exp\Big\{\frac{\beta}{2}(1 + o(1)) \sum_{p \le P} \frac{1}{p^\alpha}\Big\} \\ &= \exp\left\{\frac{(\beta + o(1))P^{1-\alpha}}{2(1-\alpha)\log P}\right\} = \exp\left\{\frac{(\beta + o(1))(\log n)^{1-\alpha}}{2(1-\alpha)\log\log n}\right\}.\end{aligned}$$

Now, if t_k is the k^{th} number of the form $2.3 \ldots P$ (i.e. $t_k = p_1 \ldots p_k$), then $\log t_k \sim k \log k \sim \log t_{k+1}$. Hence for $t_k \le n < t_{k+1}$, $\log n \sim k \log k$. It follows that

$$\max_{r \le n} \eta_{\alpha,\beta}(r) \ge \eta_{\alpha,\beta}(t_k) \ge \exp\left\{\frac{(\beta + o(1))(\log t_k)^{1-\alpha}}{2(1-\alpha)\log\log t_k}\right\} = \exp\left\{\frac{(\beta + o(1))(\log n)^{1-\alpha}}{2(1-\alpha)\log\log n}\right\}.$$

□

[2]Here $F(n) \asymp G(n)$ means there exist $a, A > 0$ such that $a < F(n)/G(n) < A$ for all n under consideration.

1.2 General considerations

For $\alpha \in \mathbb{R}$, let φ_α be the operator defined by (0.1). We wish to investigate when φ_α is a bounded mapping from l^p to l^q (for given $1 \le p, q, \le \infty$). Note that φ_α is a linear bijection on the space of all sequences. Linearity is trivial, and if $\varphi_\alpha(a) = 0$ then, by Möbius inversion, $a = 0$, showing that φ_α is injective. Finally, given $b = (b_n)$, we can define $a = (a_n)$ by

$$a_n = \frac{1}{n^\alpha} \sum_{d|n} \mu\Big(\frac{n}{d}\Big) d^\alpha b_d,$$

where $\mu(\cdot)$ is the Möbius function. Then $\varphi_\alpha(a) = b$, showing surjectivity, and hence, bijectivity.

First some general necessary conditions:

$$\varphi_\alpha(l^p) \subset l^q \Longrightarrow q \ge p; \tag{1.1}$$

$$\varphi_\alpha(a) \in l^q \Longrightarrow \alpha > \frac{1}{q}; \tag{1.2}$$

$$\varphi_\alpha(a) \in l^\infty \Longrightarrow \alpha \ge 0. \tag{1.2$'$}$$

These follow from the elementary inequalities $b_n \ge a_n$ and $b_n \ge a_1 n^{-\alpha}$, which hold if $a_n \ge 0$ for all n. (For (1.2), we have $b_n^q \ge a_1^q n^{-q\alpha}$, so that $q\alpha > 1$ is necessary for the convergence of $\sum |b_n|^q$.)

For $1 \le p \le q \le \infty$, let $r \in [1, \infty]$ be defined by $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$, where we use the convention that $\frac{1}{\infty} = 0$. Note that $r = 1$ if and only if $p = q$, and $r = \infty$ if and only if $p = 1$ and $q = \infty$.

Theorem 1.1

Let $1 \le p \le q \le \infty$ and let r be as defined above. If $r < \infty$, then $\varphi_\alpha : l^p \to l^q$ is bounded if $\alpha > \frac{1}{r}$, with $\|\varphi_\alpha\| \le \sqrt[r]{\zeta(\alpha r)}$. If $r = \infty$ (i.e. $(p, q) = (1, \infty)$), then $\varphi_\alpha : l^1 \to l^\infty$ is bounded if and only if $\alpha \ge 0$, with $\|\varphi_\alpha\| = 1$.

Furthermore, for the cases $p = 1$ (any q), $q = \infty$ (any p), and $p = q$, we have $\|\varphi_\alpha\| = \sqrt[r]{\zeta(\alpha r)}$.

Proof. First we consider the case where $1 < p \le q < \infty$, so that $r < \infty$. Let $\alpha > \frac{1}{r}$.

Let $\kappa = (1 - \frac{1}{p})\alpha r$ and $\lambda = \frac{1}{q}\alpha r$, so that $\kappa > 1 - \frac{1}{p}$, $\lambda > \frac{1}{q}$ and $\kappa + \lambda = \alpha$.

By Hölder's inequality,

$$\begin{aligned}
\sum_{d|n} d^\alpha |a_d| &= \sum_{d|n} d^\kappa \cdot d^\lambda |a_d|^{\frac{p}{q}} \cdot |a_d|^{1-\frac{p}{q}} \\
&\le \Big(\sum_{d|n} d^{\frac{\kappa}{1-1/p}}\Big)^{1-\frac{1}{p}} \Big(\sum_{d|n} d^{\lambda q} |a_d|^p\Big)^{\frac{1}{q}} \Big(\sum_{d|n} |a_d|^p\Big)^{\frac{1}{p}-\frac{1}{q}} \\
&= n^\kappa \Big(\sum_{d|n} d^{-\alpha r}\Big)^{1-\frac{1}{p}} \Big(\sum_{d|n} d^{\alpha r} |a_d|^p\Big)^{\frac{1}{q}} \Big(\sum_{d|n} |a_d|^p\Big)^{\frac{1}{p}-\frac{1}{q}} \quad \text{(using } \tfrac{\kappa}{1-1/p} = \lambda q = \alpha r\text{)} \\
&\le n^\kappa \zeta(\alpha r)^{1-\frac{1}{p}} \|a\|_p^{1-\frac{p}{q}} \Big(\sum_{d|n} d^{\alpha r} |a_d|^p\Big)^{\frac{1}{q}}.
\end{aligned}$$

Hence

$$|b_n|^q \le \frac{1}{n^{\alpha q}} \Big(\sum_{d|n} d^\alpha |a_d| \Big)^q \le \frac{\zeta(\alpha r)^{q(1-1/p)} \|a\|_p^{q-p}}{n^{\lambda q}} \sum_{d|n} d^{\alpha r} |a_d|^p,$$

and so

$$\begin{aligned} \sum_{n\le x} |b_n|^q &\le \zeta(\alpha r)^{q(1-1/p)} \|a\|_p^{q-p} \sum_{n\le x} \frac{1}{n^{\alpha r}} \sum_{d|n} d^{\alpha r} |a_d|^p \\ &= \zeta(\alpha r)^{q(1-1/p)} \|a\|_p^{q-p} \sum_{n\le x} |a_n|^p \sum_{d\le x/n} \frac{1}{d^{\alpha r}} \\ &\le \zeta(\alpha r)^{q(1-1/p)+1} \|a\|_p^q. \end{aligned}$$

Hence $b \in l^q$ and $\|b\|_q \le \zeta(\alpha r)^{1-1/p+1/q} \|a\|_p = \sqrt[r]{\zeta(\alpha r)} \|a\|_p$. Thus $\varphi_\alpha : l^p \to l^q$ is bounded and $\|\varphi_\alpha\| \le \sqrt[r]{\zeta(\alpha r)}$.

For $1 = p \le q < \infty$ (so that $r = q$), we take $\kappa = 0$ and $\lambda = \alpha$ in the above. Then

$$\sum_{d|n} d^\alpha |a_d| = \sum_{d|n} d^\alpha |a_d|^{\frac{1}{q}} \cdot |a_d|^{1-\frac{1}{q}} \le \Big(\sum_{d|n} d^{\alpha q} |a_d| \Big)^{\frac{1}{q}} \Big(\sum_{d|n} |a_d| \Big)^{1-\frac{1}{q}} \le \|a\|_1^{1-\frac{1}{q}} \Big(\sum_{d|n} d^{\alpha q} |a_d| \Big)^{\frac{1}{q}},$$

and we proceed as before.

For $1 < p < q = \infty$ (so that $r = \frac{p}{p-1}$), we take $\kappa = \alpha$ and $\lambda = 0$ in the above. Then

$$\sum_{d|n} d^\alpha |a_d| \le \Big(\sum_{d|n} d^{\alpha r} \Big)^{1-\frac{1}{p}} \Big(\sum_{d|n} |a_d|^p \Big)^{\frac{1}{p}} \le n^\alpha \zeta(\alpha r)^{1-\frac{1}{p}} \|a\|_p,$$

which implies $|b_n| \le \zeta(\alpha r)^{1-\frac{1}{p}} \|a\|_p$, so b_n is bounded and this case follows.

For $p = q = \infty$ (so that $r = 1$), we have $|b_n| \le \|a\|_\infty \sum_{d|n} d^{-\alpha} \le \zeta(\alpha) \|a\|_\infty$.

Finally, for $p = 1, q = \infty$ (so that $r = \infty$), we see that from condition (1.2′), $\alpha \ge 0$ is necessary, in which case we have

$$|b_n| \le \sum_{d|n} \frac{|a_{n/d}|}{d^\alpha} \le \sum_{d|n} |a_{n/d}| \le \|a\|_1,$$

showing that $\|\varphi_\alpha\| \le 1$.

Now we show that the bound $\sqrt[r]{\zeta(\alpha r)}$ is sharp if either $p = 1$, $q = \infty$, or $p = q$.

(i) For $p = 1$ (which implies $r = q$), let $(a_n) = (1, 0, 0, \dots)$ (so that $\|a\|_1 = 1$). Then $b_n = n^{-\alpha}$, so that for $q < \infty$, $\sum_{n=1}^\infty b_n^q = \zeta(\alpha q)$, and the bound is attained. If $q = \infty$, we have $\max_{n\ge 1} |b_n| = 1$. Hence $\|\varphi_\alpha\| = 1$.

(ii) For the case $q = \infty$, consider $1 < p < \infty$ and $p = \infty$ separately, the $p = 1$ case having been dealt with. Here, $r = \frac{p}{p-1}$. In the former case, define $a = (a_n)$ as follows: for fixed $N \in \mathbb{N}$, let

$$a_n = n^{\frac{\alpha}{p-1}} \sigma_{\alpha r}(N)^{-1/p} \quad \text{if } n|N, \text{ and zero otherwise.}$$

Then $\|a\|_p = 1$ since

$$\sum_{n=1}^{\infty} |a_n|^p = \sum_{n|N} |a_n|^p = \frac{1}{\sigma_{\alpha r}(N)} \sum_{n|N} n^{\alpha r} = 1.$$

But for this choice of a, we have

$$b_N = \frac{1}{N^\alpha} \sum_{d|N} d^\alpha a_d = \frac{1}{N^\alpha \sigma_{\alpha r}(N)^{1/p}} \sum_{d|N} d^{\alpha r} = \sigma_{-\alpha r}(N)^{1-1/p},$$

and $\sigma_{-\alpha r}(N)$ can be made arbitrarily close to $\zeta(\alpha r)$ by choosing N appropriately. Thus $\|\varphi_\alpha\| = \zeta(\alpha r)^{1-\frac{1}{p}} = \zeta(\alpha r)^{\frac{1}{r}}$.

For the $p = q = \infty$ case, take $a_n = 1$ for all n, then $b_n = \sum_{d|n} d^{-\alpha} = \sigma_{-\alpha}(n)$, which can be made arbitrarily close to $\zeta(\alpha)$, so that $\|\varphi_\alpha\| = \zeta(\alpha)$.

(iii) For the case $p = q \in (1, \infty)$, define $a = (a_n)$ as follows: for fixed $N \in \mathbb{N}$, let

$$a_n = \frac{1}{d(N)^{1/p}} \quad \text{if } n|N, \text{ and zero otherwise.}$$

Then $\|a\|_p = 1$ and, for $n|N$ (so that $d|N$ whenever $d|n$), $b_n = \frac{\sigma_{-\alpha}(n)}{d(N)^{1/p}}$. Hence

$$\sum_{n=1}^{\infty} |b_n|^p \geq \sum_{n|N} |b_n|^p = \frac{1}{d(N)} \sum_{n|N} \sigma_{-\alpha}(n)^p = \eta_{\alpha,p}(N).$$

As shown in the preliminaries, $\eta_{\alpha,p}(N)$ can be made arbitrarily close to $\zeta(\alpha)^p$.

□

Remark. For each of the cases in which $\|\varphi_\alpha\| = \sqrt[r]{\zeta(\alpha r)}$, the condition $\alpha > \frac{1}{r}$ is also necessary for the boundedness of φ_α. To see this, note that $\zeta(\alpha r)$ becomes arbitrarily large as α tends to $\frac{1}{r}$. Since b_n increases as α decreases whenever $a_m \geq 0$ $(\forall m)$ it follows that for $\alpha \leq \frac{1}{r}$, $\sum_{n=1}^{\infty} |b_n|^q$ (or $\max_{n\geq 1} |b_n|$) can be made arbitrarily large (with $\|a\|_p = 1$), and so φ_α is unbounded for such α.

Let us call the cases where $p = 1$, $q = \infty$, or $p = q$, the *boundary cases*, since in the $p - q$ plane, they form the sides of a triangle. For these cases we therefore know precisely when φ_α is a bounded mapping from l^p to l^q, as well as knowing the operator norm. What happens for the remaining cases ($1 < p < q < \infty$) inside the triangle is not very clear. Theorem 1.1 gives only a partial answer.

One could perhaps conjecture that the conclusions of Theorem 1.1 are true for these cases as well.

2. Unbounded operators

For the boundary cases (at least) we know that for $\alpha \leq \frac{1}{r}$, φ_α fails to be a bounded mapping from l^p to l^q. In these cases it is of interest to investigate how large $\sqrt[q]{\sum_{n\leq N} |b_n|^q}$ (and $\max_{n\leq N} |b_n|$ if $q = \infty$) can become. With this in mind, define the following functions: with b_n defined from $a = (a_n)$ by (0.1), let

$$B_{p,q,\alpha}(N) = \sup_{\|a\|_p=1} \left(\sum_{n=1}^{N} |b_n|^q \right)^{1/q} \quad (q < \infty), \qquad B_{p,\infty,\alpha}(N) = \sup_{\|a\|_p=1} \max_{n\leq N} |b_n| \quad (q = \infty).$$

We shall consider the three 'boundary' cases; $p=1$, $q=\infty$, and $p=q$ in turn.

2.1 The case $p=1$
This is the simplest case and is summed up in the following:

Theorem 2.1
For $1 \le q < \infty$

$$B_{1,q,\alpha}(N) = \left(\sum_{n=1}^{N} \frac{1}{n^{\alpha q}}\right)^{1/q}, \quad \textit{while} \quad B_{1,\infty,\alpha}(N) = \begin{cases} 1 & \textit{if } \alpha \ge 0 \\ N^{-\alpha} & \textit{if } \alpha < 0 \end{cases}.$$

Proof. Let $a \in l^1$ with $\|a\|_1 = 1$, and suppose $q < \infty$. From the proof of Theorem 1.1, we have

$$|b_n|^q \le \frac{1}{n^{\alpha q}} \sum_{d|n} d^{\alpha q} |a_d|.$$

Hence

$$\sum_{n \le N} |b_n|^q \le \sum_{n \le N} \frac{1}{n^{\alpha q}} \sum_{d|n} d^{\alpha q} |a_d| = \sum_{n \le N} |a_n| \sum_{d \le N/n} \frac{1}{d^{\alpha q}} \le \sum_{n \le N} \frac{1}{n^{\alpha q}}.$$

On the other hand, putting $a_1 = 1$ and $a_n = 0$ otherwise (so that $\|a\|_1 = 1$), then $b_n = \frac{1}{n^\alpha}$, which gives

$$\sum_{n \le N} |b_n|^q = \sum_{n \le N} \frac{1}{n^{\alpha q}}.$$

It follows that the maximum is achieved with this choice of a and the result follows.

For $q = \infty$ (with $\|a\|_1 = 1$), $|b_n| \le \sum_{d|n} d^{-\alpha} |a_d| \le \min\{1, n^{-\alpha}\}$, and the choice $a = (1, 0, 0, \ldots)$ shows this maximum is achieved. Thus $\max_{n \le N} |b_n| = \min\{1, N^{-\alpha}\}$, as required.

□

2.2 The case $q=\infty$
Next we consider the case when $q = \infty$. We shall take $p > 1$, the case $p = 1$ having been dealt with.

Theorem 2.2
Let $1 < p \le \infty$*. Then, with* $r = \frac{p}{p-1}$ $(= 1$ *if* $p = \infty)$*, we have*

$$B_{p,\infty,\alpha}(N) = \max_{n \le N} \sigma_{-\alpha r}(n)^{\frac{1}{r}}.$$

Proof. Suppose first that $p < \infty$. Let $a \in l^p$ with $\|a\|_p = 1$. From the proof of Theorem 1.1, we have

$$|b_n| \le \frac{1}{n^\alpha} \left(\sum_{d|n} d^{\alpha r}\right)^{1-\frac{1}{p}} \left(\sum_{d|n} |a_d|^p\right)^{\frac{1}{p}} \le \sigma_{-\alpha r}(n)^{1-\frac{1}{p}}.$$

Thus $B_{p,\infty,\alpha}(N) \le \max_{n \le N} \sigma_{-\alpha r}(n)^{1-\frac{1}{p}}$.

For a lower bound, let $a = (a_k)$ be the following sequence: fix $n \in \mathbb{N}$, and let

$$a_{n/d} = \frac{d^{-\frac{\alpha}{p-1}}}{\sqrt[p]{\sigma_{-\alpha r}(n)}} \quad \text{if } d|n, \text{ and zero otherwise.}$$

Then $\|a\|_p = 1$ and

$$b_n = \sum_{d|n} \frac{a_{n/d}}{d^\alpha} = \frac{1}{\sqrt[p]{\sigma_{-\alpha r}(n)}} \sum_{d|n} \frac{1}{d^{\alpha r}} = \sigma_{-\alpha r}(n)^{1-\frac{1}{p}}.$$

Thus, given $N \geq 1$ and $n \leq N$, we can find a such that $b_n = \sigma_{-\alpha r}(n)^{1-\frac{1}{p}}$. It follows that $B_{p,\infty,\alpha}(N) = \sup_{\|a\|_p=1} \max_{n \leq N} |b_n| \geq \max_{n\leq N} \sigma_{-\alpha r}(n)^{1-\frac{1}{p}}$, and hence we have equality.

For $p = \infty$, let $a \in l^\infty$ with $\|a\|_\infty = 1$. Then $|b_n| \leq \sigma_{-\alpha}(n)$, with equality if $a_n \equiv 1$. Hence

$$B_{\infty,\infty,\alpha}(N) = \max_{n \leq N} \sigma_{-\alpha}(n).$$

□

Remark. In Theorem 2.2 we see that although $B_{p,\infty,\alpha}(N)$ tends to infinity as $N \to \infty$ for $\alpha \leq 1 - \frac{1}{p}$, it does not give an example of an $a \in l^p$ for which $\max_{n \leq N} |b_n| \to \infty$. In the appendix, we give such an example.

2.3 The case $p = q$

This case is much more tricky and interesting. We cannot obtain an exact formula as for the previous two cases, but only an approximate formula.

Theorem 2.3

Let $1 < p < \infty$. Then

$$B_{p,p,1}(N) = e^\gamma \log\log N + O(1) \qquad (\alpha = 1)$$

$$\log B_{p,p,\alpha}(N) \asymp \frac{(\log N)^{1-\alpha}}{\log\log N} \qquad (\tfrac{1}{p} < \alpha < 1)$$

$$\log B_{p,p,\frac{1}{p}}(N) \sim (p-1)^{-\frac{1}{p}} \left(\frac{\log N}{\log\log N}\right)^{1-\frac{1}{p}}. \qquad (\alpha = \tfrac{1}{p})$$

Proof. We start with upper bounds. For these we use the methods of [11].

First we note that for any positive arithmetical function $g(n)$,

$$B_{p,p,\alpha}(N) \leq \left(\sum_{n \leq N} \frac{g(n)}{n^\alpha}\right)^{\frac{1}{p}} \cdot \left(\max_{n \leq N} \sum_{d|n} \frac{1}{g(d)^{\frac{1}{p-1}} d^\alpha}\right)^{1-\frac{1}{p}}. \tag{2.1}$$

This is because

$$|b_n| = \left|\sum_{d|n} \frac{1}{g(d)^{\frac{1}{p}} d^{\alpha(1-\frac{1}{p})}} \cdot \frac{g(d)^{\frac{1}{p}} a_{n/d}}{d^{\frac{\alpha}{p}}}\right| \leq \left(\sum_{d|n} \frac{1}{g(d)^{\frac{1}{p-1}} d^\alpha}\right)^{1-\frac{1}{p}} \left(\sum_{d|n} \frac{g(d)|a_{n/d}|^p}{d^\alpha}\right)^{\frac{1}{p}},$$

using Hölder's inequality. Writing $G(n) = \sum_{d|n} g(d)^{-\frac{1}{p-1}} d^{-\alpha}$, we have

$$\sum_{n \leq N} |b_n|^p \leq \sum_{n \leq N} G(n)^{p-1} \sum_{d|n} \frac{g(d)|a_{n/d}|^p}{d^\alpha} \leq \max_{n \leq N} G(n)^{p-1} \sum_{d \leq N} \frac{g(d)}{d^\alpha} \sum_{n \leq N/d} |a_n|^p.$$

Taking $\|a\|_p = 1$, we see that (2.1) follows.

We choose g appropriately, so that the RHS of (2.1) is small.

For $\frac{1}{p} < \alpha \leq 1$, choose $g(n)$ to be the following multiplicative function: for a prime power[3] p_1^k let

$$g(p_1^k) = \begin{cases} 1 & \text{if } p_1^k \leq M \\ (\frac{M}{p_1^k})^\beta & \text{if } p_1^k > M \end{cases}.$$

Here $M, \beta > 0$ are constants to be determined later. They may depend on N and α. In fact, we shall require $1 - \alpha < \beta < (p-1)\alpha$. Note that $g(p_1^k) \leq g(p_1)$ for every $k \in \mathbb{N}$ and p_1 prime.

We estimate the expressions in (2.1) separately. First

$$\sum_{n \leq N} \frac{g(n)}{n^\alpha} \leq \prod_{p_1} \Big(1 + \sum_{k=1}^{\infty} \frac{g(p_1^k)}{p_1^{k\alpha}}\Big) \leq \prod_{p_1} \Big(1 + \frac{g(p_1)}{p_1^\alpha - 1}\Big) \leq \exp\Big\{\sum_{p_1} \frac{g(p_1)}{p_1^\alpha - 1}\Big\}. \tag{2.2}$$

Thus for $\alpha < 1$ (for $\alpha = 1$ we argue slightly differently)

$$\log \sum_{n \leq N} \frac{g(n)}{n^\alpha} \leq \sum_{p_1 \leq M} \frac{1}{p_1^\alpha - 1} + M^\beta \sum_{p_1 > M} \frac{1}{p_1^\beta (p_1^\alpha - 1)}.$$

(Here we require $\beta > 1 - \alpha$.) By the prime number theorem, the RHS above is asymptotic to

$$\frac{M^{1-\alpha}}{(1-\alpha)\log M} + \frac{M^{1-\alpha}}{(\alpha + \beta - 1)\log M} = \frac{\beta M^{1-\alpha}}{(1-\alpha)(\alpha+\beta-1)\log M}.$$

Hence

$$\log \sum_{n \leq N} \frac{g(n)}{n^\alpha} \lesssim \frac{\beta M^{1-\alpha}}{(1-\alpha)(\alpha+\beta-1)\log M}. \tag{2.3}$$

Now consider $G(n)$, which is multiplicative as g is. At the prime powers we have

$$\begin{aligned} G(p_1^k) = \sum_{r=0}^{k} \frac{1}{p_1^{\alpha r} g(p_1^r)^{\frac{1}{p-1}}} &= \sum_{\substack{r \geq 0 \\ p_1^r \leq M}} \frac{1}{p_1^{\alpha r}} + \frac{1}{M^{\beta/(p-1)}} \sum_{\substack{r \leq k \\ p_1^r > M}} \frac{1}{p_1^{(\alpha - \frac{\beta}{p-1})r}} \\ &\leq 1 + \frac{1}{p_1^\alpha - 1} + \frac{1}{M^\alpha (1 - p_1^{\frac{\beta}{p-1} - \alpha})}. \end{aligned}$$

(Here we require $\beta < (p-1)\alpha$.) Note that this is independent of k. It follows that

$$G(n) \leq \exp\Big\{\sum_{p_1 | n} \frac{1}{p_1^\alpha - 1} + \frac{1}{M^\alpha} \sum_{p_1 | n} \frac{1}{1 - p_1^{\frac{\beta}{p-1} - \alpha}}\Big\}.$$

The RHS is maximised when n is as large as possible (i.e. N) and N is of the form $N = 2.3 \ldots P$. For such a choice, $\log N = \theta(P) \sim P$, so that (using the prime number theorem)

$$\log \max_{n \leq N} G(n) \lesssim \sum_{p_1 \leq P} \frac{1}{p_1^\alpha - 1} + \frac{1}{M^\alpha} \sum_{p_1 \leq P} 1 \sim \frac{(\log N)^{1-\alpha}}{(1-\alpha)\log\log N} + \frac{\log N}{M^\alpha \log\log N}. \tag{2.4}$$

Now choose $M = \lambda \log N$ for $\lambda > 0$. (2.1), (2.3) and (2.4) then imply

$$\log B_{p,p,\alpha}(N) \lesssim \Big(\frac{\beta \lambda^{1-\alpha}}{p(1-\alpha)(\alpha+\beta-1)} + \frac{1 - 1/p}{(1-\alpha)} + \frac{1-1/p}{\lambda^\alpha}\Big) \frac{(\log N)^{1-\alpha}}{\log\log N}$$

[3]Since p is already used, we denote primes by p_1 in this proof.

for every $\beta \in (1-\alpha, (p-1)\alpha)$ and $\lambda > 0$. Since $\frac{\beta}{\alpha+\beta-1}$ decreases with β, the optimal choice is to take β arbitrarily close to $(p-1)\alpha$. Hence we require $\inf_{\lambda>0} h(\lambda)$, where

$$h(\lambda) = \frac{\alpha\lambda^{1-\alpha}}{(1-\alpha)(p\alpha-1)} + \frac{1}{(1-\alpha)} + \frac{1}{\lambda^\alpha}.$$

Since $h'(\lambda) = \frac{\alpha}{\lambda^{\alpha+1}}(\frac{\lambda}{p\alpha-1} - 1)$, we see that the optimal choice is $\lambda = p\alpha - 1$. Substituting this value of λ gives

$$\log B_{p,p,\alpha}(N) \lesssim \Big(1 - \frac{1}{p}\Big) \frac{(1 + (p\alpha-1)^{-\alpha})}{(1-\alpha)} \frac{(\log N)^{1-\alpha}}{\log\log N}.$$

For $\alpha = 1$, we use the same function $g(n)$ as before (though with possibly different values of M and β). From (2.2)

$$\sum_{n\le N} \frac{g(n)}{n} \le \prod_{p_1\le M}\bigg(\frac{1}{1-\frac{1}{p_1}}\bigg) \cdot \prod_{p_1>M}\bigg(1 + \frac{M^\beta}{p_1^\beta(p_1-1)}\bigg).$$

By Merten's Theorem, the first product is $e^\gamma \log M + O(1)$ while $M^\beta \sum_{p_1>M} p_1^{-1-\beta} = O(1/\log M)$, so this implies

$$\sum_{n\le N} \frac{g(n)}{n} \le \Big(e^\gamma \log M + O(1)\Big) \exp\{O(1/\log M)\} = e^\gamma \log M + O(1). \tag{2.5}$$

For the $G(n)$ term we have, as for the $\alpha < 1$ case,

$$G(p_1^k) \le \frac{1}{1-\frac{1}{p_1}} + \frac{1}{M(1 - p_1^{\frac{\beta}{p-1}-1})}.$$

Thus, with $N = 2.3.\dots P$,

$$G(N) \le \prod_{p_1\le P}\bigg(\frac{1}{1-\frac{1}{p_1}}\bigg)\bigg(1 + \frac{1-1/p_1}{M(1-p_1^{\beta-1})}\bigg) = \Big(e^\gamma \log P + O(1)\Big)\Big(1 + O\Big(\frac{P}{M\log P}\Big)\Big).$$

Taking $M = \log N$ and noting that $P \sim \log N$, the RHS is $e^\gamma \log\log N + O(1)$. Combining with (2.5) shows that

$$B_{p,p,1}(N) \le e^\gamma \log\log N + O(1).$$

The case $\alpha = \frac{1}{p}$. The function g as chosen for $\alpha \in (\frac{1}{p}, 1]$ is not suitable for an upper bound as we would require $1 - \frac{1}{p} < \beta < 1 - \frac{1}{p}$! Instead we take g to be the multiplicative function as follows: for a prime power p_1^k let

$$g(p_1^k) = \min\Big\{1, \Big(\frac{M}{p_1^k(\log p_1)^p}\Big)^{1-\frac{1}{p}}\Big\}.$$

Here $M > 0$ is independent of p_1 and k and will be determined later. Thus $g(p_1^k) = 1$ if and only if $p_1^k(\log p_1)^p \le M$. Note that $g(p_1^k) \le g(p_1) \le 1$ for all $k \ge 1$ and primes p_1.

Thus (2.2) holds with $\alpha = \frac{1}{p}$ and (using the prime number theorem)

$$\log \sum_{n \le N} \frac{g(n)}{n^{1/p}} \lesssim \sum_{p_1 \lesssim \frac{M}{(\log M)^p}} \frac{1}{\sqrt[p]{p_1} - 1} + M^{1-\frac{1}{p}} \sum_{p_1 \gtrsim \frac{M}{(\log M)^p}} \frac{1}{p_1 (\log p_1)^{p-1}}$$
$$\sim \frac{M^{1-\frac{1}{p}}}{(p-1)(\log M)^{p-1}}. \tag{2.6}$$

(The first sum is of order $M^{1-\frac{1}{p}}/(\log M)^p$ and the main contribution comes from the second term.)

Regarding $G(n)$, this time we have

$$G(n) = \prod_{p_1^k \| n} G(p_1^k) \le \prod_{p_1^k \| n} \left(1 + \sum_{r=1}^{k} \frac{1}{p_1^{r/p}} + M^{-\frac{1}{p}} \sum_{r=1}^{k} \log p_1\right),$$

so that

$$\log G(n) \le \sum_{p_1 | n} \frac{1}{\sqrt[p]{p_1} - 1} + M^{-\frac{1}{p}} \sum_{p_1^k \| n} k \log p_1 \le M^{-\frac{1}{p}} \log n + \sum_{p_1 | n} \frac{1}{\sqrt[p]{p_1} - 1}.$$

The right hand side above is maximal when $n = N = 2.3 \dots P$, hence

$$\log \max_{n \le N} G(n) \lesssim M^{-\frac{1}{p}} \log N + \sum_{p_1 \le P} \frac{1}{\sqrt[p]{p_1}} \sim M^{-\frac{1}{p}} \log N + \frac{(\log N)^{1-\frac{1}{p}}}{(1 - 1/p) \log \log N}.$$

Combining with (2.6), then (2.1) gives

$$\log B_{p,p,\frac{1}{p}} \lesssim \frac{M^{1-\frac{1}{p}}}{p(p-1)(\log M)^{p-1}} + (1 - 1/p) M^{-\frac{1}{p}} \log N + \frac{(\log N)^{1-\frac{1}{p}}}{\log \log N}.$$

The optimal choice for M is easily seen to be $M = (p-1) \log N (\log \log N)^{p-1}$, and this gives the upper bound in (iii).

Now we proceed to give lower bounds.

For a fixed $n \in \mathbb{N}$, let

$$a_d = \frac{1}{\sqrt[p]{d(n)}} \quad \text{if } d|n \text{, and zero otherwise.}$$

Then $\|a\|_p = 1$ while

$$b_d = \frac{1}{\sqrt[p]{d(n)}} \sum_{c|d} \frac{1}{c^\alpha} = \frac{\sigma_{-\alpha}(d)}{\sqrt[p]{d(n)}}.$$

Hence for $N \ge n$,

$$\sum_{k \le N} |b_k|^p \ge \sum_{d|n} b_d^p = \frac{1}{d(n)} \sum_{d|n} \sigma_{-\alpha}(d)^p = \eta_{\alpha,p}(n).$$

Thus $B_{p,p,\alpha}(N) \ge \max_{n \le N} \sqrt[p]{\eta_{\alpha,p}(n)}$.

Hence for $\frac{1}{p} < \alpha \le 1$, the lower bounds follow from the maximal order of $\eta_{\alpha,p}(n)$.

For the case $\alpha = \frac{1}{p}$, the above choice doesn't give the correct order and we lose a power of $\log\log N$. Instead we follow an idea of Soundararajan [11]. Let f be the multiplicative function supported on the squarefree numbers whose values at primes p_1 is

$$f(p_1) = \begin{cases} (\frac{M}{p_1})^{1/p}\frac{1}{\log p_1} & \text{for } M \le p_1 \le R \\ 0 & \text{otherwise} \end{cases}.$$

Here $M = (p-1)\log N(\log\log N)^{p-1}$ as before and $\log R = (\log M)^2$.

Now take $a_n = f(n)F(N)^{-1/p}$ where $F(N) = \sum_{n\le N} f(n)^p$ so that $\sum_{n\le N} a_n^p = 1$. Then by Hölder's inequality

$$\begin{aligned}\Big(\sum_{n=1}^{N} b_n^p\Big)^{1/p} &\ge \sum_{n=1}^{N} a_n^{p-1}b_n = \frac{1}{F(N)}\sum_{n=1}^{N}\frac{f(n)^{p-1}}{n^{1/p}}\sum_{d|n} d^{1/p}f(d) \\ &= \frac{1}{F(N)}\sum_{n\le N}\frac{f(n)^{p-1}}{n^{1/p}}\sum_{\substack{d\le N/n\\(n,d)=1}} f(d)^p. \end{aligned}\tag{2.7}$$

Now using 'Rankin's trick'[4] we have, for any $\beta > 0$

$$\begin{aligned}\sum_{n\le N}\frac{f(n)^{p-1}}{n^{1/p}}\sum_{\substack{d\le N/n\\(n,d)=1}} f(d)^p &= \sum_{n\le N}\frac{f(n)^{p-1}}{n^{1/p}}\Big(\sum_{\substack{d\ge 1\\(n,d)=1}} f(d)^p - \sum_{\substack{d\ge N/n\\(n,d)=1}} f(d)^p\Big) \\ &= \sum_{n\le N}\frac{f(n)^{p-1}}{n^{1/p}}\Big(\prod_{p_1\nmid n}\Big(1+f(p_1)^p\Big) + O\Big(\Big(\frac{n}{N}\Big)^{\beta}\prod_{p_1\nmid n}\Big(1+p_1^{\beta}f(p_1)^p\Big)\Big)\Big).\end{aligned}\tag{2.8}$$

The O-term in (2.8) is at most a constant times

$$\frac{1}{N^{\beta}}\sum_{n\le N} f(n)^{p-1}n^{\beta-1/p}\prod_{p_1\nmid n}\Big(1+p_1^{\beta}f(p_1)^p\Big) \le \frac{1}{N^{\beta}}\prod_{p_1}\Big(1+p_1^{\beta}f(p_1)^p + p_1^{\beta-1/p}f(p_1)^{p-1}\Big),$$

while the main term in (2.8) is (using Rankin's trick again)

$$\prod_{p_1}\Big(1+f(p_1)^p+\frac{f(p_1)^{p-1}}{p_1^{1/p}}\Big) + O\Big(\frac{1}{N^{\beta}}\prod_{p_1}\Big(1+f(p_1)^p+p_1^{\beta-1/p}f(p_1)^{p-1}\Big)\Big).$$

Hence (2.7) implies

$$\Big(\sum_{n=1}^{N} b_n^p\Big)^{1/p} \ge \frac{1}{F(N)}\Big(\prod_{p_1}\Big(1+f(p_1)^p+\frac{f(p_1)^{p-1}}{p_1^{1/p}}\Big)+O\Big(\frac{1}{N^{\beta}}\prod_{p_1}\Big(1+p_1^{\beta}f(p_1)^p+p_1^{\beta-1/p}f(p_1)^{p-1}\Big)\Big)\Big).$$

The ratio of the O-term to the main term on the right is less than

$$\exp\Big\{-\beta\log N + \sum_{M\le p_1\le R}(p_1^{\beta}-1)\Big(f(p_1)^p+\frac{f(p_1)^{p-1}}{p_1^{1/p}}\Big)\Big\}$$

[4] If $c_n > 0$, then for any $\beta > 0$, $\sum_{n>x} c_n \le x^{-\beta}\sum_{n=1}^{\infty} n^{\beta}c_n$.

which equals

$$\exp\left\{-\beta\log N+\sum_{M\le p_1\le R}(p_1^\beta-1)\left(\frac{M}{p_1(\log p_1)^p}+\frac{M^{1-1/p}}{p_1(\log p_1)^{p-1}}\right)\right\}.$$

Take $\beta=(\log M)^{-3}$. The term involving $M^{1-1/p}$ is at most $(\log N)^{1-1/p+\varepsilon}$ for every $\varepsilon>0$, while the remaining terms in the exponent are (on the prime number theorem in the form $\pi(x)=\mathrm{li}(x)+O(x(\log x)^{-A})$ for all A)

$$\begin{aligned}&-\beta\log N+M\int_M^R\frac{t^\beta-1}{t(\log t)^{p+1}}\,dt+O\left(\frac{\log N}{(\log\log N)^A}\right)\\&=-\beta\log N+\beta M\int_M^R\frac{dt}{t(\log t)^p}+O\left(\beta^2M\int_M^R\frac{dt}{t(\log t)^{p-1}}\right)\\&\sim-\beta(p-1)^2\frac{\log N\log\log\log N}{\log\log N},\end{aligned}$$

after some calculations.

Finally, since $F(N)\le\prod_{p_1}(1+f(p_1)^p)$, this implies

$$B_{p,p,1/p}(N)\ge\frac{1}{2}\prod_{M\le p_1\le R}\left(1+\frac{f(p_1)^{p-1}}{p_1^{1/p}(1+f(p_1)^p)}\right),$$

for all N sufficiently large. Hence

$$\log B_{p,p,1/p}(N)\gtrsim M^{1-1/p}\sum_{M\le p_1\le R}\frac{1}{p_1(\log p_1)^{p-1}}\sim\frac{1}{(p-1)^{1/p}}\left(\frac{\log N}{\log\log N}\right)^{1-1/p}.$$

as required.

□

Remark. The result for $\frac{1}{p}<\alpha<1$ is

$$\frac{(\log N)^{1-\alpha}}{2(1-\alpha)\log\log N}\lesssim\log B_{p,p,\alpha}(N)\lesssim\left(1-\frac{1}{p}\right)\frac{(1+(p\alpha-1)^{-\alpha})}{(1-\alpha)}\frac{(\log N)^{1-\alpha}}{\log\log N}.$$

It would be nice to obtain an asymptotic formula for $\log B_{p,p,\alpha}(N)$. Indeed, it is possible to improve the lower bound at the cost of more work by using the method for the case $\alpha=\frac{1}{p}$, but we have not been able to obtain the same upper and lower limits.

3. Connections with $\zeta(s)$ and the eigenvalues of certain arithmetical matrices. Now we restrict ourselves to the case $p=q=2$, this being perhaps the most interesting case. We shall show that the bounds obtained for[5] $B_\alpha(N)$ in Theorem 2.3 for $\frac{1}{2}<\alpha\le 1$ can be used to obtain information regarding the maximum order of $\zeta(s)$ on the line $\Re s=\alpha$.

Proposition 3.1
We have, for any α,

$$\sum_{n\le N}|b_n|^2=\sum_{m,n\le N}\frac{a_m\overline{a_n}(m,n)^{2\alpha}}{m^\alpha n^\alpha}\sum_{k\le\frac{N}{[m,n]}}\frac{1}{k^{2\alpha}}.$$

[5] In this section we write $B_\alpha(N)$ for $B_{2,2,\alpha}(N)$

Proof. We have

$$|b_n|^2 = b_n\overline{b_n} = \frac{1}{n^{2\alpha}} \sum_{c|n,d|n} c^\alpha d^\alpha a_c \overline{a_d} = \frac{1}{n^{2\alpha}} \sum_{[c,d]|n} c^\alpha d^\alpha a_c \overline{a_d},$$

since $c|n, d|n$ if and only if $[c,d]|n$. Hence

$$\sum_{n\le N} |b_n|^2 = \sum_{c,d\le N} c^\alpha d^\alpha a_c \overline{a_d} \sum_{n\le N, [c,d]|n} \frac{1}{n^{2\alpha}} = \sum_{c,d\le N} \frac{c^\alpha d^\alpha a_c \overline{a_d}}{[c,d]^{2\alpha}} \sum_{k\le \frac{N}{[c,d]}} \frac{1}{k^{2\alpha}},$$

by writing $n = [c,d]k$. Since $(c,d)[c,d] = cd$, the result follows.

□

Remark. We can use this to show that $B_\alpha(N) \asymp N^{\frac{1}{2}-\alpha}$ for $\alpha < \frac{1}{2}$. For such α and $\|a\|_2 = 1$,

$$\begin{aligned}
\sum_{n\le N} |b_n|^2 &\le AN^{1-2\alpha} \sum_{m,n\le N} \frac{|a_m a_n|(m,n)^{2\alpha}}{(mn)^\alpha [m,n]^{1-2\alpha}} = AN^{1-2\alpha} \sum_{m,n\le N} \frac{|a_m a_n|(m,n)}{(mn)^{1-\alpha}} \\
&= AN^{1-2\alpha} \sum_{d\le N} d \sum_{\substack{m,\, n\, \le\, N \\ (m,\, n)\, =\, d}} \frac{|a_m a_n|}{(mn)^{1-\alpha}} \le AN^{1-2\alpha} \sum_{d\le N} \frac{1}{d^{1-2\alpha}} \left(\sum_{m\le N/d} \frac{|a_{md}|}{m^{1-\alpha}} \right)^2 \\
&\le A'N^{1-2\alpha} \sum_{d\le N} \frac{1}{d^{1-2\alpha}} \sum_{m\le N/d} \frac{|a_{md}|^2 (\log m + 1)^2}{m^{1-2\alpha}} \qquad \text{(by Cauchy-Schwarz)} \\
&= A'N^{1-2\alpha} \sum_{n\le N} \frac{|a_n|^2}{n^{1-2\alpha}} \sum_{d|n} (\log d + 1)^2 \le A'N^{\frac{1}{2}-\alpha} \sum_{n\le N} \frac{|a_n|^2 d(n) (\log n + 1)^2}{n^{1-2\alpha}}.
\end{aligned}$$

But since $d(n)(\log n + 1)^2 = O(n^\varepsilon)$ and $\alpha < \frac{1}{2}$, the sum on the right is $O(1)$.

On the other hand, if we take $a_1 = 1$ and $a_n = 0$ otherwise, then $b_n = n^{-\alpha}$ and

$$\sum_{n\le N} |b_n|^2 = \sum_{n\le N} \frac{1}{n^{2\alpha}} \sim \frac{N^{1-2\alpha}}{1-2\alpha}.$$

Letting $N \to \infty$ in Proposition 3.1 gives:

Corollary 3.2
Let $\alpha > \frac{1}{2}$ and let $a \in l^2$. Then $\varphi_\alpha(a) \in l^2$ if and only if the series

$$\sum_{m,n\ge 1} \frac{a_m \overline{a_n} (m,n)^{2\alpha}}{m^\alpha n^\alpha}$$

converges. In which case, we have $\|\varphi_\alpha(a)\|_2^2 = \zeta(2\alpha) \sum_{m,n\ge 1} \frac{a_m \overline{a_n}(m,n)^{2\alpha}}{m^\alpha n^\alpha}$.

Also, it follows from Proposition 3.1 that if $a_n \ge 0$ for all n and $\alpha > \frac{1}{2}$, then

$$\sum_{n\le N} |b_n|^2 \le \zeta(2\alpha) \sum_{m,n\le N} \frac{a_m \overline{a_n} (m,n)^{2\alpha}}{(mn)^\alpha} \le (1+\varepsilon) \sum_{n\le N^3} |b_n|^2 \tag{3.2}$$

for every $\varepsilon > 0$, whenever $N \geq N_0$, some $N_0 = N_0(\varepsilon)$. The left-hand inequality is immediate while the right-hand sum (without the $(1+\varepsilon)$) is greater than

$$\sum_{m,n\leq N} \frac{a_m\overline{a_n}(m,n)^{2\alpha}}{m^\alpha n^\alpha} \sum_{k\leq \frac{N^3}{[m,n]}} \frac{1}{k^{2\alpha}} > (\zeta(2\alpha)-\varepsilon) \sum_{m,n\leq N} \frac{a_m\overline{a_n}(m,n)^{2\alpha}}{m^\alpha n^\alpha}$$

since $\frac{N^3}{[m,n]} \geq N$ for $m,n \leq N$.

Theorem 3.3
Let $\frac{1}{2} < \alpha \leq 1$ and let $a \in l^2$ with $\|a\|_2 = 1$. Let $A_N(t) = \sum_{n=1}^{N} a_n n^{it}$. Let $N \leq T^\lambda$ where $0 < \lambda < \frac{2}{3}(\alpha - \frac{1}{2})$. Then for some $\eta > 0$,

$$\frac{1}{T}\int_1^T |\zeta(\alpha+it)|^2 |A_N(t)|^2\, dt = \zeta(2\alpha) \sum_{m,n\leq N} \frac{a_m\overline{a_n}(m,n)^{2\alpha}}{(mn)^\alpha} + O(T^{-\eta}). \qquad (3.3)$$

Proof. We shall assume $\frac{1}{2} < \alpha < 1$, adjusting the proof for the case $\alpha = 1$ afterwards. For $\alpha \neq 1$, we can integrate from 0 to T since the error involved is at most $O(N/T) = O(T^{-\eta})$.

Starting from the approximation $\zeta(\alpha+it) = \sum_{n\leq t} n^{-\alpha-it} + O(t^{-\alpha})$, we have

$$|\zeta(\alpha+it)|^2 = \left|\sum_{n\leq t} \frac{1}{n^{\alpha+it}}\right|^2 + O(t^{1-2\alpha}).$$

Let $k, l \in \mathbb{N}$ such that $(k,l) = 1$. Let $M = \max\{k,l\} < T$. The above gives

$$\int_0^T |\zeta(\alpha+it)|^2 \Big(\frac{k}{l}\Big)^{it} dt = \int_0^T \left|\sum_{n\leq t} \frac{1}{n^{\alpha+it}}\right|^2 \Big(\frac{k}{l}\Big)^{it} dt + O(T^{2-2\alpha}).$$

The integral on the right is

$$\int_0^T \sum_{m,n\leq t} \frac{1}{(mn)^\alpha}\Big(\frac{km}{ln}\Big)^{it} dt = \sum_{m,n\leq T} \frac{1}{(mn)^\alpha} \int_{\max\{m,n\}}^T \Big(\frac{km}{ln}\Big)^{it} dt.$$

The terms with $km = ln$ (which implies $m = rl, n = rk$ with r integral) contribute

$$\frac{1}{(kl)^\alpha} \sum_{r\leq T/M} \frac{T - rM}{r^{2\alpha}} = \frac{\zeta(2\alpha)}{(kl)^\alpha} T + O\Big(\frac{M^{2\alpha-1}T^{2-2\alpha}}{(kl)^\alpha}\Big).$$

The remaining terms contribute at most

$$\begin{aligned}
&2 \sum_{\substack{m,\,n\,\leq\,T\\ km\,\neq\, ln}} \frac{1}{(mn)^\alpha |\log\frac{km}{ln}|} \leq 2M^{2\alpha} \sum_{\substack{m,\,n\,\leq\,T\\ km\,\neq\, ln}} \frac{1}{(kmln)^\alpha|\log\frac{km}{ln}|} \\
&\leq 2M^{2\alpha} \sum_{\substack{m_1\,\leq\,kT,\, n_1\,\leq\, lT\\ m_1\,\neq\, n_1}} \frac{1}{(m_1n_1)^\alpha|\log\frac{m_1}{n_1}|} \leq 2M^{2\alpha} \sum_{\substack{m_1,\, n_1\,\leq\, MT\\ m_1\,\neq\, n_1}} \frac{1}{(m_1n_1)^\alpha|\log\frac{m_1}{n_1}|} \\
&= O(M^{2\alpha}(MT)^{2-2\alpha}\log(MT)) = O(M^2T^{2-2\alpha}\log T),
\end{aligned}$$

using Lemma 7.2 from [13]. Hence

$$\int_0^T |\zeta(\alpha+it)|^2\Big(\frac{k}{l}\Big)^{it} dt = \frac{\zeta(2\alpha)}{(kl)^\alpha}T + O(M^2T^{2-2\alpha}\log T).$$

It follows that for any positive integers $m, n < T$,

$$\int_0^T |\zeta(\alpha+it)|^2 \Big(\frac{m}{n}\Big)^{it} dt = \frac{\zeta(2\alpha)(m,n)^{2\alpha}}{(mn)^\alpha} T + O(\max\{m,n\}^2 T^{2-2\alpha}\log T).$$

Thus, with $A_N(t) = \sum_{n=1}^N a_n n^{it}$,

$$\begin{aligned}\int_0^T |\zeta(\alpha+it)|^2 |A_N(t)|^2\, dt &= \sum_{m,n\le N} a_m \overline{a_n} \int_0^T |\zeta(\alpha+it)|^2 \Big(\frac{m}{n}\Big)^{it} dt \\ &= \zeta(2\alpha) T \sum_{m,n\le N} \frac{a_m\overline{a_n}(m,n)^{2\alpha}}{(mn)^\alpha} + O\Big(T^{2-2\alpha}\log T \sum_{m,n\le N} \max\{m,n\}^2 |a_m a_n|\Big).\end{aligned}$$

The sum in the O-term is at most $N^2(\sum_{n\le N}|a_n|)^2 \le N^3$, using Cauchy-Schwarz. Hence

$$\frac{1}{T}\int_0^T |\zeta(\alpha+it)|^2|A_N(t)|^2\, dt = \zeta(2\alpha)\sum_{m,n\le N}\frac{a_m\overline{a_n}(m,n)^{2\alpha}}{(mn)^\alpha} + O\Big(\frac{N^3\log T}{T^{2\alpha-1}}\Big).$$

Since $N^3 \le T^{3\lambda}$ and $3\lambda < 2\alpha - 1$, the error term is $O(T^{-\eta})$ for some $\eta > 0$.

If $\alpha = 1$ we integrate from 1 to T instead and the O-term above will contain an extra $\log T$ factor, but this is still $O(T^{-\eta})$.

□

We note that with more care, the N^3 could be turned into an N^2, so that we can take $\lambda < \alpha - \frac{1}{2}$ in the theorem. This is however not too important for us.

Corollary 3.4
Let $\frac{1}{2} < \alpha \le 1$. Then for every $\varepsilon > 0$ and N sufficiently large

$$\max_{t\le N} |\zeta(\alpha+it)| \ge B_\alpha(N^{\frac{2}{3}(\alpha-\frac{1}{2})-\varepsilon}) + O(N^{-\eta}) \tag{3.4}$$

for some $\eta > 0$.

Proof. Let $a_n \ge 0$ be such that $\|a\|_2 = 1$, and take $N = T^\lambda$ with $\lambda < \frac{2}{3}(\alpha - \frac{1}{2})$. By (3.2) and (3.3)

$$\begin{aligned}\sum_{n\le N}|b_n|^2 &\le \frac{1}{T}\int_0^T |\zeta(\alpha+it)|^2|A_N(t)|^2\, dt + O(T^{-\eta}) \\ &\le \max_{t\le T}|\zeta(\alpha+it)|^2 \frac{1}{T}\int_0^T |A_N(t)|^2\, dt + O(T^{-\eta}) \\ &= \max_{t\le T}|\zeta(\alpha+it)|^2 \sum_{n\le N}|a_n|^2(1+O(N/T)) + O(T^{-\eta})\end{aligned}$$

using the Montgomery and Vaughan mean value theorem. The implied constants in the O-terms depend only on T and not on the sequence $\{a_n\}$. Taking the supremum over all such a, this gives

$$B_\alpha(N)^2 = \sup_{\|a\|_2=1}\sum_{n\le N}|b_n|^2 \le \max_{t\le T}|\zeta(\alpha+it)|^2 + O(T^{-\eta}),$$

for some $\eta > 0$, and (3.4) follows.

$\square$

In particular, this gives the (known) lower bounds

$$\max_{t \le T} |\zeta(\alpha + it)| \ge \exp\Big\{ \frac{c(\log T)^{1-\alpha}}{\log\log T} \Big\}$$

for $\frac{1}{2} < \alpha < 1$ and $\max_{t \le T} |\zeta(1+it)| \ge e^\gamma \log\log T + O(1)$ (obtained by Levinson in [6]).

Morever, we can say more about how often $|\zeta(\alpha+it)|$ is as large as this. For $A \in \mathbb{R}$ and $c > 0$, let

$$F_A(T) = \Big\{ t \in [1,T] : |\zeta(1+it)| \ge e^\gamma \log\log T - A \Big\}. \tag{3.5}$$

$$F_{\alpha,c}(T) = \Big\{ t \in [0,T] : |\zeta(\alpha+it)| \ge \exp\Big\{ \frac{c(\log T)^{1-\alpha}}{\log\log T} \Big\} \Big\}. \tag{3.5$'$}$$

Consider first the $\alpha = 1$ case. We have, for $N \le T^\lambda$ with $0 < \lambda < \frac{1}{3}$,

$$\sum_{n \le N} |b_n|^2 \le \frac{1}{T} \Big(\int_{F_A(T)} + \int_{[1,T] \setminus F_A(T)} \Big) |\zeta(1+it)|^2 |A_N(t)|^2 \, dt + O(T^{-\eta}). \tag{3.6}$$

The second integral on the right is at most

$$(e^\gamma \log\log T - A)^2 \cdot \frac{1}{T} \int_0^T |A_N(t)|^2 \, dt = (e^\gamma \log\log T - A)^2 (1 + O(N/T)),$$

while, by choosing $a_n = d(N)^{-1/2}$ for $n|N$ and zero otherwise, the LHS of (3.6) is at least $\eta_{1,2}(N)$. Now, every interval $[T^{\lambda/3}, T^\lambda]$ contains an N of the form[6] s_k. For such an N, $\eta_{1,2}(N) \ge (e^\gamma \log\log N - a)^2 \ge (e^\gamma \log\log T - a')^2$ for some $a, a' > 0$. Hence for $A > a'$,

$$\frac{1}{T} \int_{F_A(T)} |\zeta(1+it)|^2 |A_N(t)|^2 \, dt \ge (e^\gamma \log\log T - a')^2 - (e^\gamma \log\log T - A)^2 + O(T^{-\eta}) \ge 1$$

for T sufficiently large. But $|\zeta(1+it)| = O(\log T)$ and $|A_N(t)|^2 \le d(N)$, so

$$1 \le \frac{1}{T} \int_{F_A(T)} |\zeta(1+it)|^2 |A_N(t)|^2 \, dt \le \frac{(\log T)^2 d(N) \mu(F_A(T))}{T},$$

where $\mu(\cdot)$ is Lebesque measure. Thus $\mu(F_A(T)) \ge T/d(N)(\log T)^2 \ge T \exp\{-\frac{a \log T}{\log\log T}\}$ for some $a > 0$ (which depends on A only).

Now consider $\frac{1}{2} < \alpha < 1$. Again

$$\sum_{n \le N} |b_n|^2 \le \frac{1}{T} \Big(\int_{F_{\alpha,c}(T)} + \int_{[0,T] \setminus F_{\alpha,c}(T)} \Big) |\zeta(\alpha+it)|^2 |A_N(t)|^2 \, dt + O(T^{-\eta}). \tag{3.6$'$}$$

The second integral on the right is at most

$$\exp\Big\{ \frac{2c(\log T)^{1-\alpha}}{\log\log T} \Big\} \cdot \frac{1}{T} \int_0^T |A_N(t)|^2 \, dt = O\Big(\exp\Big\{ \frac{2c(\log T)^{1-\alpha}}{\log\log T} \Big\} \Big),$$

[6] Recall $s_k = \prod_{p \le p_k} p^{[\sqrt{\frac{p_k}{p}}]}$. From (1.1), it is easy to see that $s_k < s_{k+1} \le s_k^{2+o(1)}$.

while, by choosing a_n and N as before, the LHS of (3.6′) is at least $\exp\left\{\frac{c'(\log T)^{1-\alpha}}{\log\log T}\right\}$ for some $c' > 0$. Hence for $2c < c'$,

$$\frac{1}{T}\int_{F_{\alpha,c}(T)} |\zeta(\alpha+it)|^2|A_N(t)|^2\,dt \geq \exp\left\{\frac{c'(\log T)^{1-\alpha}}{2\log\log T}\right\}.$$

We have $|\zeta(\alpha+it)| = O(T^\nu)$ for some ν and $|A_N(t)|^2 \leq d(N) = O(T^\varepsilon)$, so

$$\frac{1}{T}\int_{F_{\alpha,c}(T)} |\zeta(\alpha+it)|^2|A_N(t)|^2\,dt \leq T^{2\nu-1+\varepsilon}\mu(F_{\alpha,c}(T)).$$

Thus $\mu(F_{\alpha,c}(T)) \geq T^{1-2\nu-\varepsilon}$ for all c sufficiently small.

In particular, since $\nu < \frac{1-\alpha}{3}$, we have:

Theorem 3.5
Let $F_A(T)$ and $F_{\alpha,c}(T)$ denote the sets in (3.5) and (3.5′) respectively. Then for all A sufficiently large (and positive) $\mu(F_A(T)) \geq T\exp\{-a\frac{\log T}{\log\log T}\}$ for some $a > 0$, and for all c sufficiently small, $\mu(F_{\alpha,c}(T)) \geq T^{(1+2\alpha)/3}$ for all T sufficiently large. Furthermore, on the Lindelöf Hypothesis, the exponent can be replaced by $1-\varepsilon$.

Connection with the largest eigenvalue of certain arithmetical matrices
In [7], the eigenvalues of the $N \times N$ matrix with entries $\frac{(i,j)^{2\alpha}}{i^\alpha j^\alpha}$ was discussed. Denote the largest eigenvalue by $\Lambda_N(\alpha)$. Using deep properties of Dirichlet series (see [4]), it was shown that for $\alpha > 1$, $\Lambda_N(\alpha)$, though never larger than $\frac{\zeta(\alpha)^2}{\zeta(2\alpha)}$, can be made arbitrarily close to this; i.e. $\limsup_{N\to\infty}\Lambda_N(\alpha) = \frac{\zeta(\alpha)^2}{\zeta(2\alpha)}$, while the lim sup is infinite for $\alpha \leq 1$. It was further suggested that an 'arithmetical proof' of this would be unlikely. However, since

$$\Lambda_N(\alpha) = \sup_{\|a\|_2=1}\sum_{m,n\leq N}\frac{a_m\overline{a_n}(m,n)^{2\alpha}}{m^\alpha n^\alpha},$$

and the supremum (actually maximum) occurs when $a_n \geq 0$, we see from (3.2) that

$$\frac{B_\alpha(N)^2}{\zeta(2\alpha)} \leq \Lambda_N(\alpha) \leq (1+o(1))\frac{B_\alpha(N^3)^2}{\zeta(2\alpha)} \qquad \text{for } \alpha > \tfrac{1}{2}.$$

But by purely arithmetical means we showed in Theorem 1.1 that, for $\alpha > 1$, $\|\varphi_\alpha\| = \zeta(\alpha)$; i.e. $B_\alpha(N) \to \zeta(\alpha)$. (Indeed, this depended on the fact that $\limsup_{n\to\infty}\eta_{\alpha,2}(n) = \zeta(\alpha)^2$).

Furthermore, from the bounds on $B_\alpha(N)$ obtained in Theorem 2.3, we have corresponding bounds for $\Lambda_N(\alpha)$ for $\frac{1}{2} \leq \alpha \leq 1$ for large N, namely:

$$\Lambda_N(1) = \frac{1}{\zeta(2)}(e^\gamma\log\log N + O(1))^2,$$

and

$$\log\Lambda_N(\alpha) \asymp \frac{(\log N)^{1-\alpha}}{\log\log N} \qquad \text{for } \tfrac{1}{2} < \alpha < 1.$$

Adjusting (3.2) for $\alpha = \frac{1}{2}$ ($\zeta(2\alpha)$ gets replaced by $\sum_{n\leq N}\frac{1}{n}$) gives

$$\log\Lambda_N(1/2) \asymp \Big(\frac{\log N}{\log\log N}\Big)^{1/2}.$$

References

[1] R. Balasubramanian, J. B. Conrey, and D. R. Heath-Brown, Asymptotic mean square of the product of the Riemann zeta-function and a Dirichlet polynomial, *J. Reine Angew. Math.* **357** (1985) 161-181.

[2] A. Granville and K. Soundararajan, Extreme values of $|\zeta(1+it)|$, *Ramanujan Math. Soc. Lect. Notes Ser 2, Ramanujan Math. Soc., Mysore* (2006) 65-80.

[3] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, (fifth edition) Oxford University Press, 1979.

[4] H. Hedenmalm, P. Lindqvist and K. Seip, A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0,1)$, *Duke Math. J.* **86** (1997) 1-37.

[5] T. W. Hilberdink, Determinants of Multiplicative Toeplitz matrices, *Acta Arith.* **125** (2006) 265-284.

[6] N. Levinson, Ω-theorems for the Riemann zeta function, *Acta Arith.* **20** (1972) 319-332.

[7] P. Lindqvist and K. Seip, Note on some greatest common divisor matrices, *Acta Arith.* **84** (1998) 149-154.

[8] K. Matsumoto, On the mean square of the product of $\zeta(s)$ and a Dirichlet polynomial, *Comment. Math. Univ. St. Pauli* **53** (2004) 1-21.

[9] H. L. Montgomery, Extreme values of the Riemann zeta-function, *Comment. Math. Helv.* **52** (1977) 511-518.

[10] G. Robin, Grandes valeurs de la fonction somme des diviseurs et l'hypothèse de Riemann, *J. Math. Pures Appl.* **63** (1984) 187-213.

[11] K. Soundararajan, Extreme values of zeta and L-functions, *Math. Ann.* **342** (2008) 467-486.

[12] A. E. Taylor, *Introduction to Functional Analysis*, Wiley and Sons, 1958.

[13] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Second edition, Oxford University Press, 1986.

[14] O. Toeplitz, Zur Theorie der quadratischen und bilinearen Formen von unendlichvielen Veränderlichen, *Math. Ann.* **70** (1911) 351-376.

[15] O. Toeplitz, Zur Theorie der Dirichletschen Reihen, *Amer. J. Math.* **60** (1938) 880-888.

[16] A. Wintner, Diophantine approximations and Hilbert's space, *Amer. J. Math.* **66** (1944) 564-578.

APPENDIX

Examples where $\max_{n\le N}|b_n|\to\infty$ and $\sum_{n\le N}|b_n|^p\to\infty$ for a given $a\in l^p$

For $\alpha\le\frac{1}{r}$, φ_α fails to be a bounded operator (for the boundary cases at least). As φ_α is a 'matrix' mapping, it is closed ([12], p.183). For such mappings, if $\varphi_\alpha(l^p)\subset l^q$, then $\varphi_\alpha: l^p\to l^q$ is necessarily bounded. Since we know this is false for $\alpha\le\frac{1}{r}$, it follows that for such α, $\varphi_\alpha(l^p)\not\subset l^q$; i.e. $\exists\, a\in l^p$ such that $\varphi_\alpha(a)\notin l^q$.

In Theorem 2.2, we see that although $B_{p,\infty,\alpha}(N)$ tends to infinity as $N\to\infty$ for $\alpha\le 1-\frac{1}{p}$, it does not give an example of an $a\in l^p$ for which $\max_{n\ge1}|b_n|=\infty$. Similarly, in Theorem 2.3, $B_{p,p,\alpha}(N)\to\infty$ for $\alpha\le1$ but this does not provide an example of an $a\in l^p$ for which $\sum_{n\ge1}|b_n|^p=\infty$. For $\alpha\le0$, it is easy to construct such examples but for $\alpha>0$ this is not obvious. Below, we provide examples for both cases. For simplicity, we take $p=2$, as both examples can easily be adjusted for general p.

1. Example for which $\max_{n\le N}|b_n|\to\infty$ **as** $N\to\infty$.
First note that although $B_{2,\infty,\alpha}(N)\to\infty$ for $\alpha\le\frac{1}{2}$, b_n is usually quite small (at least if $\alpha>0$). For example, we know from the Remark following Proposition 3.1 that for $\alpha<\frac{1}{2}$, $\sum_{n\le N}|b_n|^2=O(N^{1-2\alpha})$. Thus the set $S_N=\{n\le N:|b_n|\ge N^{-\alpha+\varepsilon}\}$ satisfies $|S_N|=o(N)$ for every $\varepsilon>0$, since

$$cN^{1-2\alpha}\ge\sum_{n\le N}|b_n|^2\ge\sum_{n\in S_N}N^{-2\alpha+2\varepsilon}=|S_N|N^{-2\alpha+2\varepsilon}.$$

Let R be an infinite subset of numbers of the form $2.3\ldots P$. Let $\sigma'_{-\alpha}(n)=\sum_{d|n,d\le\sqrt{n}}d^{-\alpha}$. Now for $n\in R$ and $d|n$ with $d\le\sqrt{n}$, define

$$a_{n/d}=\frac{\varepsilon_n}{d^\alpha\sqrt{\sigma'_{-2\alpha}(n)}}\qquad\text{and zero otherwise.}$$

Here $\varepsilon_n>0$ is to be determined later. For this to be well-defined we need $n'>n^2$ for consecutive elements n,n' of R. Hence

$$\sum_{k=1}^\infty a_k^2=\sum_{n\in R}\frac{\varepsilon_n^2}{\sigma'_{-2\alpha}(n)}\sum_{d|n,d\le\sqrt{n}}\frac{1}{d^{2\alpha}}=\sum_{n\in R}\varepsilon_n^2.\tag{A1}$$

Thus $a\in l^2$ with $\|a\|_2=1$ if $\sum_{n\in R}\varepsilon_n^2=1$, which we shall now assume.

Now, for $n\in R$, we have

$$b_n=\sum_{d|n}\frac{a_{n/d}}{d^\alpha}=\frac{\varepsilon_n}{\sqrt{\sigma'_{-2\alpha}(n)}}\sum_{d|n,d\le\sqrt{n}}\frac{1}{d^{2\alpha}}=\varepsilon_n\sqrt{\sigma'_{-2\alpha}(n)}.$$

But for any given $\varepsilon>0$, $\sigma'_{-2\alpha}(n)\ge(1-\varepsilon)\sigma_{-2\alpha}(n)$ for n sufficiently large (as long as $\alpha>0$). Thus $b_n\ge(1-\varepsilon)\varepsilon_n\sqrt{\sigma_{-2\alpha}(n)}$.

Thus, in order to have an example, we need to choose R and ε_n in such a way that the sum in (A1) converges but $\varepsilon_n\sqrt{\sigma_{-2\alpha}(n)}$ is unbounded. This is easily done; for example, by taking $\varepsilon_n=\sigma_{-2\alpha}(n)^{-\beta}$ with $0<\beta<\frac{1}{2}$ and making R sufficiently 'thin'. Indeed, by choosing R sufficiently thin, ε_n can be chosen to tend to zero as slowly as we please. Using the bounds on $\sigma_{-2\alpha}(n)$, this proves:

Theorem A
(i) Given any function $\phi(n)$ increasing to infinity, however slowly, there exists $a\in l^2$ such that $\varphi_{\frac{1}{2}}(a)=(b_n)$ satisfies

$$b_n=\Omega\Big(\frac{\sqrt{\log\log n}}{\phi(n)}\Big).$$

(ii) For $0<\alpha<\frac{1}{2}$, there exists $a\in l^2$ such that $\varphi_\alpha(a)=(b_n)$ satisfies

$$b_n=\Omega\bigg(\exp\Big\{\frac{c(\log n)^{1-2\alpha}}{\log\log n}\Big\}\bigg)$$

for some $c>0$.

Part (i) is best possible, for, writing $b_n=b'_n+b''_n$ where

$$b'_n=\frac{1}{\sqrt{n}}\sum_{d|n,d\le\sqrt{n}}\sqrt{d}a_d\quad\text{and } b''_n=\frac{1}{\sqrt{n}}\sum_{d|n,d>\sqrt{n}}\sqrt{d}a_d.$$

We have

$$|b'_n| \le \frac{1}{\sqrt{n}}\Big(\sum_{d|n, d\le\sqrt{n}} d\Big)^{\frac12}\Big(\sum_{d|n}|a_d|^2\Big)^{\frac12} \le \Big(\frac{d(n)}{\sqrt{n}}\Big)^{\frac12} \to 0,$$

while

$$\begin{aligned}|b''_n| = \left|\sum_{d|n, d<\sqrt{n}} \frac{a_{n/d}}{\sqrt{d}}\right| &\le \Big(\sum_{d|n, d<\sqrt{n}} \frac{1}{d}\Big)^{\frac12}\Big(\sum_{d|n, d<\sqrt{n}} |a_{n/d}|^2\Big)^{\frac12} \\ &\le \sqrt{\sigma_{-1}(n)}\Big(\sum_{\sqrt{n}\le m\le n} |a_m|^2\Big)^{\frac12} = o(\sqrt{\sigma_{-1}(n)}).\end{aligned}$$

Hence $b_n = o(\sqrt{\log\log n})$ in any case.

2. Example for which $\sum_{n\le N}|b_n|^2 \to \infty$

Again, let R be an infinite subset of numbers of the form $2.3\ldots P$. Define a_n as follows: for $n \in R$ and $d|n$ such that $d \le n^{3/4}$, let

$$a_{n/d} = \frac{\varepsilon_n}{\sqrt{d(n)}}, \qquad \text{and zero otherwise.}$$

This is well-defined if, for consecutive terms n, n' of R, we have $n' > n^4$, which we shall now assume. Then $a \in l^2$ if and only if

$$\sum_{n\in R}\sum_{\substack{d|n \\ d\le n^{3/4}}} a_{n/d}^2 = \sum_{n\in R}\frac{\varepsilon_n^2}{d(n)}\sum_{\substack{d|n \\ d\le n^{3/4}}} 1 < \infty.$$

But the inner sum on the right is at least $\frac12 d(n)$, so $a \in l^2$ if $\sum_{n\in R}\varepsilon_n^2$ converges.

Now, for $n \in R$ and $d|n$ such that $d \le n^{3/4}$, we have

$$b_{n/d} = \sum_{c|\frac{n}{d}} \frac{a_{n/cd}}{c^\alpha} = \frac{\varepsilon_n}{\sqrt{d(n)}}\sum_{c|\frac{n}{d}, c\le\frac{n^{3/4}}{d}} \frac{1}{c^\alpha}.$$

Hence

$$\begin{aligned}\sum_{\substack{d|n \\ d\le n^{3/4}}} b_{n/d}^2 = \frac{\varepsilon_n^2}{d(n)}\sum_{\substack{d|n \\ d\le n^{3/4}}}\Big(\sum_{\substack{c|\frac{n}{d} \\ c\le\frac{n^{3/4}}{d}}}\frac{1}{c^\alpha}\Big)^2 &\ge \frac{\varepsilon_n^2}{d(n)}\sum_{\substack{d|n \\ d\le\sqrt{n}}}\Big(\sum_{\substack{c|\frac{n}{d} \\ c\le\frac{n^{1/4}}{d}}}\frac{1}{c^\alpha}\Big)^2 \\ \ge \frac{(1-\varepsilon)\varepsilon_n^2}{d(n)}\sum_{\substack{d|n \\ d\le\sqrt{n}}}\sigma_{-\alpha}(n/d)^2 &= \frac{(1-\varepsilon)\varepsilon_n^2\sigma_{-\alpha}(n)^2}{d(n)}\sum_{\substack{d|n \\ d\le\sqrt{n}}}\frac{1}{\sigma_{-\alpha}(d)^2},\end{aligned}$$

using the fact that n is squarefree, so that $(\frac{n}{d}, d) = 1$ and $\sigma_{-\alpha}(n) = \sigma_{-\alpha}(n/d)\sigma_{-\alpha}(d)$. Now, without the restriction $d \le \sqrt{n}$, the sum on the far right is of order $d(n)/\sigma_{-\alpha}(n)$ (by using the results in 1.1 on $\eta_{\alpha,\beta}(n)$). But

$$\Big(\sum_{\substack{d|n \\ d\le\sqrt{n}}} 1\Big)^2 \le \sum_{d|n}\sigma_{-\alpha}(d)^2\sum_{\substack{d|n \\ d\le\sqrt{n}}}\frac{1}{\sigma_{-\alpha}(d)^2} \asymp d(n)\sigma_{-\alpha}(n)\sum_{\substack{d|n \\ d\le\sqrt{n}}}\frac{1}{\sigma_{-\alpha}(d)^2}.$$

Since the LHS is just $\frac{1}{4}d(n)^2$, it follows that $\sum_{d|n, d\le\sqrt{n}} \frac{1}{\sigma_{-\alpha}(d)^2} \asymp \frac{d(n)}{\sigma_{-\alpha}(n)}$ also. In particular, if $N \in R$

$$\sum_{k\le N} b_k^2 \ge \sum_{n\le N, n\in R} \sum_{\substack{d|n \\ d\,\le\, n^{3/4}}} b_{n/d}^2 \ge A \sum_{n\le N, n\in R} \varepsilon_n^2 \sigma_{-\alpha}(n) \ge A\varepsilon_N^2 \sigma_{-\alpha}(N)$$

for some $A > 0$. This can be made to tend to infinity as $N \to \infty$. Indeed, by choosing R as 'thin' as we please, we can make ε_n tend to zero as slowly as we please. Thus:

Theorem B
(i) Given any function $\phi(n)$ increasing to infinity, however slowly, there exists $a \in l^2$ such that $\varphi_1(a) = (b_n)$ satisfies

$$\sum_{n\le N} b_n^2 = \Omega\Big(\frac{\log\log N}{\phi(N)}\Big).$$

(ii) For $\frac{1}{2} \le \alpha < 1$ there exists $a \in l^2$ such that $\varphi_\alpha(a) = (b_n)$ satisfies

$$\sum_{n\le N} b_n^2 = \Omega\bigg(\exp\Big\{\frac{c(\log N)^{1-\alpha}}{\log\log N}\Big\}\bigg)$$

for some $c > 0$.

Probably with some more effort, the $\log\log N$ in (i) can be turned into a $(\log\log N)^2$.

Abstract

In this paper we study the linear mapping that sends a sequence (a_n) to (b_n) where $b_n = \sum_{d|n} d^{-\alpha} a_{n/d}$. We investigate for which values of α this is a bounded operator from l^p to l^q and show the operator norm is closely connected to the Riemann zeta function.

We consider the unbounded case, in particular on l^2, giving formulas (exact and asymptotic) for the maximal behaviour of the norm $\sqrt{\sum_{n\le N} |b_n|^2}$. We show that these provide lower bounds for $\max_{t\le N} |\zeta(\alpha+it)|$, giving a new proof that $\max_{t\le T} |\zeta(\alpha + it)| \ge \exp\{c\frac{(\log T)^{1-\alpha}}{\log\log T}\}$ for some $c > 0$ for $\frac{1}{2} < \alpha < 1$. Further we show that this lower bound holds in $[0, T]$ on a set of measure at least $T^{(1+2\alpha)/3}$. Analogously, we show $|\zeta(1+it)| \ge e^\gamma \log\log T + O(1)$ in $[1, T]$ on a set of measure at least $T\exp\{-\frac{a\log T}{\log\log T}\}$.

We also find connections to the largest eigenvalues of certain arithmetical matrices.

2010 AMS Mathematics Subject Classification: 11N56, 11M06.
Keywords and phrases: Bounded linear mappings, maximal growth of arithmetical functions, Ω-theorems.