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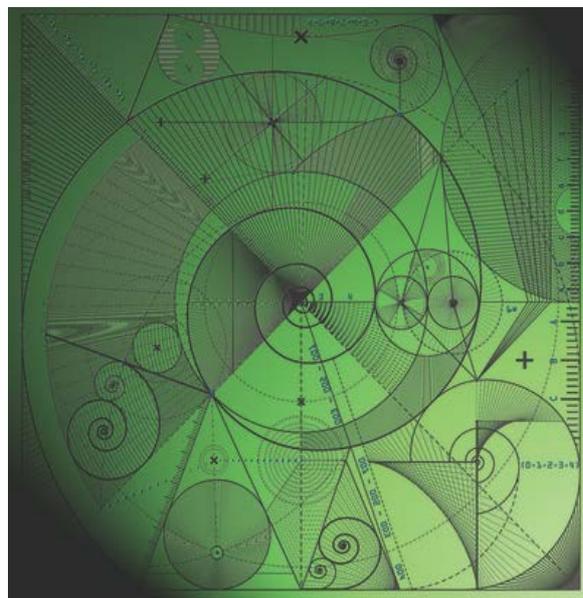
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Rigidity and flatness of the image of certain classes of mappings having tangential Laplacian

by

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RIGIDITY AND FLATNESS OF THE IMAGE OF CERTAIN CLASSES OF MAPPINGS HAVING TANGENTIAL LAPLACIAN

HUSSIEN ABUGIRDA, BIRZHAN AYANBAYEV AND NIKOS KATZOURAKIS

This paper is dedicated to Gunnar Aronsson with the utmost esteem for his pioneering work.

ABSTRACT. In this paper we consider the PDE system of vanishing normal projection of the Laplacian for C^2 maps $u : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}^N$:

$$\llbracket Du \rrbracket^\perp \Delta u = 0 \quad \text{in } \Omega.$$

This system has discontinuous coefficients and geometrically expresses the fact that the Laplacian is a vector field tangential to the image of the mapping. It arises as a constituent component of the p -Laplace system for all $p \in [2, \infty]$. For $p = \infty$, the ∞ -Laplace system is the archetypal equation describing extrema of supremal functionals in vectorial Calculus of Variations in L^∞ . Herein we show that the image of a solution u is piecewise affine if either the rank of Du is equal to one or $n = 2$ and u has additively separated form. As a consequence we obtain corresponding flatness results for p -Harmonic maps, $p \in [2, \infty]$.

1. INTRODUCTION

Suppose that n, N are integers and Ω an open subset of \mathbb{R}^n . In this paper we study geometric aspects of the image $u(\Omega) \subseteq \mathbb{R}^N$ of certain classes of C^2 vectorial solutions $u : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}^N$ to the following nonlinear degenerate elliptic PDE system:

$$(1.1) \quad \llbracket Du \rrbracket^\perp \Delta u = 0 \quad \text{in } \Omega.$$

Here, for the map u with components $(u_1, \dots, u_N)^\top$ the notation Du symbolises the gradient matrix

$$Du(x) = (D_i u_\alpha(x))_{\substack{\alpha=1 \dots N \\ i=1 \dots n}} \in \mathbb{R}^{N \times n}, \quad D_i \equiv \partial / \partial x_i,$$

Δu stands for the Laplacian

$$\Delta u(x) = \sum_{i=1}^n D_{ii}^2 u(x) \in \mathbb{R}^N$$

and for any $X \in \mathbb{R}^{N \times n}$, $\llbracket X \rrbracket^\perp$ denotes the orthogonal projection on the orthogonal complement of the range of linear map $X : \mathbb{R}^n \rightarrow \mathbb{R}^N$:

$$(1.2) \quad \llbracket X \rrbracket^\perp := \text{Proj}_{\mathbb{R}(X)^\perp}.$$

Our general notation will be either self-explanatory, or otherwise standard as e.g. in [E, D]. Note that, since the rank is a discontinuous function, the map $\llbracket \cdot \rrbracket^\perp$ is

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discontinuous on $\mathbb{R}^{N \times n}$; therefore, the PDE system (1.1) has *discontinuous coefficients*. The geometric meaning of (1.1) is that *the Laplacian vector field Δu is tangential to the image $u(\Omega)$* and hence (1.1) is equivalent to the next statement: there exists a vector field

$$A : \mathbb{R}^n \supseteq \Omega \longrightarrow \mathbb{R}^n$$

such that

$$\Delta u = Du A \quad \text{in } \Omega.$$

As we show later, the vector field is generally discontinuous (Lemma 7).

Our interest in (1.1) stems from the fact that it is a constituent component of the p -Laplace PDE system for all $p \in [2, \infty]$. Further, contrary perhaps to appearances, (1.1) is in itself a *variational PDE system* but in a non-obvious way. Deferring temporarily the specifics of how exactly (1.1) arises and what is the variational principle associated with it, let us recall that, for $p \in [2, \infty)$, the celebrated p -Laplacian is the divergence system

$$(1.3) \quad \Delta_p u := \text{Div}(|Du|^{p-2} Du) = 0 \quad \text{in } \Omega$$

and comprises the Euler-Lagrange equation which describes extrema of the model p -Dirichlet integral functional

$$(1.4) \quad E_p(u) := \int_{\Omega} |Du|^p, \quad u \in W^{1,p}(\Omega, \mathbb{R}^N),$$

in conventional vectorial Calculus of Variations. Above and subsequently, for any $X \in \mathbb{R}^{N \times n}$, the notation $|X|$ symbolises its Euclidean (Frobenious) norm:

$$|X| = \left(\sum_{\alpha=1}^N \sum_{i=1}^n (\mathbf{X}_{\alpha i})^2 \right)^{1/2}.$$

The pair (1.3)-(1.4) is of paramount important in applications and has been studied exhaustively. The extremal case of $p \rightarrow \infty$ in (1.3)-(1.4) is much more modern and intriguing. It turns out that one then obtains the following nondivergence PDE system

$$(1.5) \quad \Delta_{\infty} u := \left(Du \otimes Du + |Du|^2 [[Du]]^{\perp} \otimes I \right) : D^2 u = 0 \quad \text{in } \Omega,$$

which is known as the ∞ -Laplacian. In index from, (1.5) reads

$$\sum_{\beta=1}^N \sum_{i,j=1}^n \left(D_i u_{\alpha} D_j u_{\beta} + |Du|^2 [[Du]]_{\alpha\beta}^{\perp} \delta_{ij} \right) D_{ij}^2 u_{\beta} = 0, \quad \alpha = 1, \dots, N.$$

The system (1.5) plays the role of the Euler-Lagrange equation and arises in connexion with variational problems for the supremal functional

$$(1.6) \quad E_{\infty}(u, \mathcal{O}) := \|Du\|_{L^{\infty}(\mathcal{O})}, \quad u \in W^{1,\infty}(\Omega, \mathbb{R}^N), \quad \mathcal{O} \Subset \Omega.$$

The scalar case of $N = 1$ in (1.5)-(1.6) was pioneered by G. Aronsson in the 1960s [A1]-[A5] who initiated the field of Calculus of Variations in L^{∞} , namely the study of supremal functionals and of their associated equations describing critical points. Since then, the field has developed tremendously and there is an extensive relevant literature (for a pedagogical introduction see e.g. [K7] and [C]). The vectorial case $N \geq 2$ began much more recently and first arose in work of the third author in the early 2010s [K1]. The area is developing very rapidly due to both the mathematical significance as well as the importance for applications, particularly

in Data Assimilation (see [K2]-[K6], [K8]-[K11] and the joint contributions with the first and second author, Croce, Manfredi, Moser, Parini, Pisante and Pryer [AK, AyK, KM, KMo, CKP, KPa, KP, KP2, KP3]).

In this paper we focus on the C^2 case and establish the geometric rigidity and flatness of the images of solutions $u : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}^N$ to the nonlinear system (1.1), under the assumption that either Du has rank at most 1, or that $n = 2$ and u has an additively separated form. As a consequence, we obtain corresponding flatness results for the images of solutions to (1.3) and (1.5). Both aforementioned classes of solutions furnish very important particular examples which provide substantial intuition for the behaviour or general extremal maps in Calculus of Variations in L^∞ , see [A6, A7, K3, K4, C, K7, KP, KP2]. Obtaining further information for the still largely mysterious behaviour of ∞ -Harmonic maps is perhaps the greatest driving force to isolate and study the particular nonlinear system (1.1).

Let us note that the rank-one case includes the scalar and the one-dimensional case (i.e. when $\min\{n, N\} = 1$), although in the case of $N = 1$ (in which the single ∞ -Laplacian reduces to $Du \otimes Du : D^2u = 0$) (1.1) has no bearing since it vanishes identically at any non-critical point.

The effect of (1.1) to the flatness of the image can be seen through the L^∞ variational principle introduced in [K2], wherein it was shown that solutions to (1.1) of constant rank can be characterised as those having minimal area with respect to (1.6)-(1.4). More precisely, therein the following result was proved:

Theorem 1 (cf. [K2, Theorem 2.7, Lemma 2.2]). *Given $N \geq n \geq 1$, let $u : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}^N$ be a C^2 immersion defined on the open set Ω (more generally u can be a map with constant rank of its gradient on Ω). Then, the following statements are equivalent:*

- (1) *The map u solves the PDE system (1.1) on Ω .*
- (2) *For all $p \in [2, \infty]$, for all compactly supported domains $\mathcal{O} \Subset \Omega$ and all C^1 vector fields $\nu : \mathcal{O} \rightarrow \mathbb{R}^N$ which are normal to the image $u(\mathcal{O}) \subseteq \mathbb{R}^N$ (without requiring to vanish on $\partial\mathcal{O}$), namely those for which $\nu = \llbracket Du \rrbracket^\perp \nu$ in \mathcal{O} , we have*

$$\|Du\|_{L^p(\mathcal{O})} \leq \|Du + D\nu\|_{L^p(\mathcal{O})}.$$

- (3) *The same statement as in item (2) holds, but only for some $p \in [2, \infty]$.*

If in addition $p < \infty$ in (2)-(3), then we may further restrict the class of normal vector fields to those satisfying $\nu|_{\partial\mathcal{O}} = 0$.

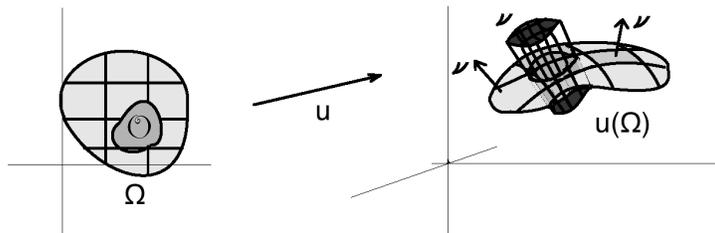


Figure 1. Illustration of the variational principle characterising (1.1).

In the paper [K2], it was also shown that in the conformal class, (1.1) expresses the vanishing of the mean curvature vector of $u(\Omega)$.

The effect of (1.1) to the flatness of the image can be easily seen in the case of $n = 1 \leq N$ as follows: since

$$[[u']]^\perp u'' = 0 \quad \text{in } \Omega \subseteq \mathbb{R}$$

and in one dimension we have

$$[[u']]^\perp = \begin{cases} \text{I} - \frac{u' \otimes u'}{|u'|^2}, & \text{on } \{u' \neq 0\}, \\ \text{I}, & \text{on } \{u' = 0\}, \end{cases}$$

we therefore infer that $u'' = fu'$ on the open set $\{u' \neq 0\} \subseteq \mathbb{R}$ for some function f , readily yielding after an integration that $u(\Omega)$ is necessarily contained in a piecewise polygonal line of \mathbb{R}^N . As a generalisation of this fact, our first main result herein is the following:

Theorem 2 (Rigidity and flatness of rank-one maps with tangential Laplacian). *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $n, N \geq 1$. Let $u \in C^2(\Omega, \mathbb{R}^N)$ be a solution to the nonlinear system (1.1) in Ω , satisfying that the rank of its gradient matrix is at most one:*

$$\text{rk}(Du) \leq 1 \quad \text{in } \Omega.$$

Then, its image $u(\Omega)$ is contained in a polygonal line in \mathbb{R}^N , consisting of an at most countable union of affine straight line segments (possibly with self-intersections).

Let us note that the rank-one assumption for Du is equivalent to the existence of two vector fields $\xi : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}^N$ and $a : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}^n$ such that $Du = \xi \otimes a$ in Ω .

Example 3 below shows that Theorem 2 is optimal and in general rank-one solutions to the system (1.1) can not have affine image but only piecewise affine.

Example 3. *Consider the C^2 rank-one map $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by*

$$u(x, y) = \begin{cases} (-x^4, x^4), & x \leq 0, y \in \mathbb{R}, \\ (+x^4, x^4), & x > 0, y \in \mathbb{R}. \end{cases}$$

Then, $u = \nu \circ f$ with $\nu : \mathbb{R} \rightarrow \mathbb{R}^2$ given by $\nu(t) = (t, |t|)$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x, y) = \text{sgn}(x)x^4$. It follows that u solves (1.1) on \mathbb{R}^2 : indeed, Δu is a non-vanishing vector field on $\{x \neq 0\}$, being tangential to the image thereon since it is parallel to the derivative $\nu'(t) = (1, \pm 1)$ for $t \neq 0$. On the other hand, on $\{x = 0\}$ we have that $\Delta u = 0$. However, the image of u is piecewise affine but not affine (see Figure 2 below).

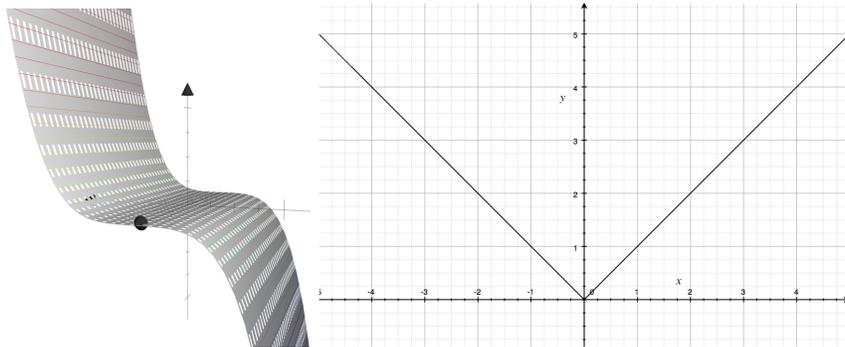


Figure 2. The function f and the image of the curve ν comprising the function u .

As a consequence of Theorem 2, we obtain the next result regarding the rigidity of p -Harmonic maps for $p \in [2, \infty)$ which complements one of the results in the paper [K3] wherein the case $p = \infty$ was considered.

Corollary 4 (Rigidity of p -Harmonic maps, cf. [K3]). *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $n, N \geq 1$. Let $u \in C^2(\Omega, \mathbb{R}^N)$ be a p -Harmonic map in Ω for some $p \in [2, \infty)$, that is u solves (1.3). Suppose that the rank of its gradient matrix is at most one:*

$$\text{rk}(Du) \leq 1 \quad \text{in } \Omega.$$

Then, the same result as in Theorem 2 is true.

In addition, there exists a partition of Ω to at most countably many Borel sets, where each set of the partition is a non-empty open set with a (perhaps empty) boundary portion, such that, on each of these, u can be represented as

$$u = \nu \circ f.$$

Here, f is a scalar C^2 p -Harmonic function (for the respective $p \in [2, \infty)$), defined on an open neighbourhood of the Borel set, whilst $\nu : \mathbb{R} \rightarrow \mathbb{R}^N$ is a Lipschitz curve which is twice differentiable and with unit speed on the image of f .

Now we move on to discuss our second main result which concerns the rigidity of solutions $u : \mathbb{R}^2 \supseteq \Omega \rightarrow \mathbb{R}^N$ to (1.1) for $N \geq 2$, having the additively separated form

$$u(x, y) = f(x) + g(y)$$

for some curves $f, g : \mathbb{R} \rightarrow \mathbb{R}^N$. Solutions of this form are very important in relation to the ∞ -Laplacian. If $N = 1$, all ∞ -Harmonic functions of this form after a normalisation reduce to the so-called Aronsson solution on \mathbb{R}^2

$$u(x, y) = |x|^{4/3} - |y|^{4/3}$$

which is the standard explicit example of a non- C^2 ∞ -Harmonic function with conjectured optimal regularity. In the vectorial case, the family of separated solutions is very large. For $N = 2$, a large class of such vectorial solutions was constructed in [K4] and is given by

$$u(x, y) = \int_x^y (\cos(K(t)), \sin(K(t))) dt$$

with K a function in $C^1(\mathbb{R})$ satisfying certain general conditions. The simplest non-trivial example of an ∞ -Harmonic map with this form (defined on the strip $\{|x - y| < \pi/4\} \subseteq \mathbb{R}^2$) is given by the choice $K(t) = t$. Our second main result asserts that solutions of separated form to (1.1) have images which are piecewise affine, contained in a union of intersecting planes of \mathbb{R}^N . More precisely, we have:

Theorem 5 (Rigidity and flatness of maps with tangential Laplacian in separated form). *Let $\Omega \subseteq \mathbb{R}^2$ be an open set and $N \geq 2$. Let $u \in C^2(\Omega, \mathbb{R}^N)$ be a solution to the nonlinear system (1.1) in Ω , having the separated form $u(x, y) = f(x) + g(y)$, for some C^2 curves $f, g : \mathbb{R} \rightarrow \mathbb{R}^N$.*

Then, the image $u(\Omega)$ of the solution is contained in an at most countable union of affine planes in \mathbb{R}^N .

In addition, every connected component of the set $\{\text{rk}(Du) = 2\}$ is contained entirely in an affine plane and every connected component of the set $\{\text{rk}(Du) \leq 1\}$ is contained entirely in an affine line.

Note that our result is trivial in the case that $N = n = 2$ since the codimension $N - n$ vanishes. Further, one might also restrict their attention to domains of rectangular shape, since any map with separated form can be automatically extended to the smallest rectangle containing the domain.

Also, herein we consider only the illustrative case of $n = 2 < N$ and do not discuss more general situations, since numerical evidence obtained in [KP] suggests that Theorem 5 does not hold in general for solutions in non-separated form. However, as a consequence of Theorem 5 we have the next particular result:

Corollary 6 (Rigidity and flatness of p -Harmonic maps in separated form). *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $n, N \geq 1$. Suppose that $u \in C^2(\Omega, \mathbb{R}^N)$ is a p -Harmonic map in Ω for some $p \in [2, \infty]$, that is u solves (1.3) if $p < \infty$ and (1.5) if $p = \infty$.*

Then, if u has the separated form $u(x, y) = f(x) + g(y)$ for some C^2 curves $f, g : \mathbb{R} \rightarrow \mathbb{R}^N$, the same conclusion as in Theorem 5 holds true.

In this paper we try to keep the exposition as simple as possible and therefore we refrain from discussing generalised solutions to (1.1) and (1.5) (or (1.3)). We confine ourselves to merely mentioning that in the scalar case, ∞ -Harmonic functions are understood in the viscosity sense of Crandall-Ishii-Lions (see e.g. [C, K7]), whilst in the vectorial case a new candidate theory for systems has been proposed in [K8] which has already borne significant fruit in [K8, K9, K10, CKP, AyK, KMo], [KP]-[KP3].

We now expound on how exactly the nonlinear system (1.1) arises from (1.3) and (1.5). By expanding the derivatives in (1.3) and normalising, we arrive at

$$(1.7) \quad Du \otimes Du : D^2u + \frac{|Du|^2}{p-2} \Delta u = 0.$$

For any $X \in \mathbb{R}^{N \times n}$, let $\llbracket X \rrbracket^\parallel$ denote the orthogonal projection on the range of the linear map $X : \mathbb{R}^n \rightarrow \mathbb{R}^N$:

$$(1.8) \quad \llbracket X \rrbracket^\parallel := \text{Proj}_{\mathbb{R}(X)}.$$

Since the identity of \mathbb{R}^N splits as $I = \llbracket Du \rrbracket^\parallel + \llbracket Du \rrbracket^\perp$, by expanding Δu with respect to these projections,

$$Du \otimes Du : D^2u + \frac{|Du|^2}{p-2} \llbracket Du \rrbracket^\parallel \Delta u = -\frac{|Du|^2}{p-2} \llbracket Du \rrbracket^\perp \Delta u.$$

The mutual perpendicularity of the vector fields of the left and right hand side leads via a renormalisation argument (see e.g. [K1, K2, K3]) to the equivalence of the p -Laplacian with the pair of systems

$$(1.9) \quad Du \otimes Du : D^2u + \frac{|Du|^2}{p-2} \llbracket Du \rrbracket^\parallel \Delta u = 0, \quad |Du|^2 \llbracket Du \rrbracket^\perp \Delta u = 0.$$

The ∞ -Laplacian corresponds to the limiting case of (1.9) as $p \rightarrow \infty$, which takes the form

$$(1.10) \quad Du \otimes Du : D^2u = 0, \quad |Du|^2 \llbracket Du \rrbracket^\perp \Delta u = 0.$$

Hence, the ∞ -Laplacian (1.5) actually consists of the two independent systems in (1.10) above. The second system in (1.9)-(1.10) is, at least on $\{Du \neq 0\}$, equivalent to our PDE system (1.1). Note that in the scalar case of $N = 1$ as well as in the case of submersion solutions (for $N \leq n$), the second system trivialises.

We conclude the introduction with a geometric interpretation of the nonlinear system (1.1), which can be expressed in a more geometric language as follows:¹ Suppose that $u(\Omega)$ is a C^2 manifold and let $\mathbf{A}(u)$ denote its second fundamental form. Then

$$[[Du]]^\perp \Delta u = -\operatorname{tr} \mathbf{A}(u)(Du, Du).$$

The tangential part $[[Du]]^\parallel \Delta u$ of the Laplacian is commonly called the *tension field* in the theory of *Harmonic maps* and is symbolised by $\tau(u)$ (see e.g. [M]). Hence, we have the orthogonal decomposition

$$\Delta u = \tau(u) - \operatorname{tr} \mathbf{A}(u)(Du, Du).$$

Therefore, in the case of higher regularity of the image of u , we obtain that the nonlinear system

$$(1.11) \quad \Delta u = \tau(u) \quad \text{in } \Omega,$$

is a further geometric reformulation of our PDE system (1.1).

2. PROOFS

In this section we prove the results of this paper. Before delving into that, we present a result of independent interest in which we represent explicitly the vector field A arising in the parametric system $\Delta u = Du A$, in the illustrative case of $n = 2$.

We will be using the symbolisations “cof”, “det” and “rk” to denote the cofactor matrix, the determinant function and the rank of a matrix, respectively.

Lemma 7 (Representation of A). *Let $u \in C^2(\Omega, \mathbb{R}^N)$ be given, $\Omega \subseteq \mathbb{R}^2$ open, $N \geq 2$. The following are equivalent:*

- (1) *The map u is a solution to the PDE system (1.1).*
- (2) *There exists a vector field $A : \mathbb{R}^2 \supseteq \Omega \rightarrow \mathbb{R}^N$ such that*

$$\Delta u = Du A \quad \text{in } \Omega.$$

In (2), as A one might choose

$$\bar{A} := \begin{cases} \frac{\operatorname{cof}(Du^\top Du)^\top}{\det(Du^\top Du)} (Du)^\top \Delta u, & \text{on } \{\operatorname{rk}(Du) = 2\}, \\ (\Delta u)^\top \frac{Du Du^\top}{|Du Du^\top|^2} Du, & \text{on } \{\operatorname{rk}(Du) = 1\}, \\ 0, & \text{on } \{\operatorname{rk}(Du) = 0\}. \end{cases}$$

A is uniquely determined on $\{\operatorname{rk}(Du) = 2\}$ but not on $\{\operatorname{rk}(Du) < 2\}$ and any other A has the form $\bar{A} + V$, where $V(x)$ lies in the nullspace of $Du(x)$, $x \in \Omega$.

Proof of Lemma 7. The equivalence between (1)-(2) is immediate, therefore it suffices to show that \bar{A} satisfies $\Delta u = Du \bar{A}$ and is unique on $\{\operatorname{rk}(Du) = 2\}$. Let A be as in (2). On $\{\operatorname{rk}(Du) = 2\}$, the 2×2 matrix-valued map $Du^\top Du$ is invertible and

$$(Du^\top Du)^{-1} = \frac{\operatorname{cof}(Du^\top Du)^\top}{\det(Du^\top Du)}.$$

Since $Du^\top \Delta u = Du^\top Du A$, we obtain that $A = \bar{A}$.

¹This fact has been brought to our attention by Roger Moser.

The claim being obvious for $\{\text{rk}(Du) = 0\} = \{Du = 0\}$, it suffices to consider only the set $\{\text{rk}(Du) = 1\}$ in order to conclude. Thereon we have that Du can be written as

$$Du = \xi \otimes a, \quad \text{in } \{\text{rk}(Du) = 1\},$$

for some non-vanishing vector fields ξ and a . By replacing ξ with $\xi|a|$ and a with $a/|a|$, we may assume $|a| \equiv 1$ throughout $\{\text{rk}(Du) = 1\}$. If $\Delta u = Du A$, we have $\Delta u = (\xi \otimes a)A$ and since any component of A which is orthogonal to a is annihilated, we may replace A by λa for some function λ . Therefore,

$$\Delta u = (\xi \otimes a) A = (\xi \otimes a)(\lambda a) = \xi \lambda |a|^2 = \lambda \xi$$

and hence $\xi \cdot \Delta u = \lambda |\xi|^2$ and also $\xi^\top Du = a |\xi|^2$. On the other hand, since

$$Du Du^\top = (\xi \otimes a)(a \otimes \xi) = \xi \otimes \xi, \quad |Du Du^\top| = |\xi|^2$$

we infer that

$$A = \lambda a = \left(\frac{\Delta u \cdot \xi}{|\xi|^2} \right) \left(\frac{\xi^\top Du}{|\xi|^2} \right) = \frac{\Delta u^\top (\xi \otimes \xi) Du}{|\xi \otimes \xi|^2} = (\Delta u)^\top \frac{Du Du^\top}{|Du Du^\top|^2} Du,$$

as claimed. \square

We now continue with the proof of the main results.

The main analytical tool needed in the proof of Theorem 2 is the next rigidity theorem for maps whose gradient has rank at most one. It was established in [K3] and we recall it below for the convenience of the reader and only in the case needed in this paper.

Theorem 8 (Rigidity of Rank-One maps, cf. [K3]). *Suppose $\Omega \subseteq \mathbb{R}^n$ is an open set and u is in $C^2(\Omega, \mathbb{R}^N)$. Then, the following are equivalent:*

(i) *The map u satisfies that $\text{rk}(Du) \leq 1$ on Ω . Equivalently, there exist vector fields $\xi : \Omega \rightarrow \mathbb{R}^N$ and $a : \Omega \rightarrow \mathbb{R}^n$ with $a \in C^1(\Omega, \mathbb{R}^n)$ and $\xi \in C^1(\Omega \setminus \{a = 0\}, \mathbb{R}^N)$ such that*

$$Du = \xi \otimes a, \quad \text{on } \Omega.$$

(ii) *There exists Borel subset $\{B_i\}_{i \in \mathbb{N}}$ of Ω such that*

$$\Omega = \bigcup_{i=1}^{\infty} B_i$$

and each B_i equals a non-empty connected open set with a (possibly empty) boundary portion, functions $\{f_i\}_{i \in \mathbb{N}} \in C^2(\Omega)$ and curves $\{\nu_i\}_{i \in \mathbb{N}} \subseteq W_{\text{loc}}^{1,\infty}(\mathbb{R}, \mathbb{R}^N)$ such that, on each B_i the map u has the form

$$(2.1) \quad u = \nu_i \circ f_i, \quad \text{on } B_i.$$

Moreover, $|\nu_i'| \equiv 1$ on the interval $f_i(B_i)$, $\nu_i' \equiv 0$ on $\mathbb{R} \setminus f_i(B_i)$ and ν_i'' exists everywhere on $f_i(B_i)$, interpreted as 1-sided derivative on $\partial f_i(B_i)$ (if $f_i(B_i)$ is not open). Also,

$$(2.2) \quad \begin{cases} Du = (\nu_i' \circ f_i) \otimes Df_i, & \text{on } B_i, \\ D^2u = (\nu_i'' \circ f_i) \otimes Df_i \otimes Df_i + (\nu_i' \circ f_i) \otimes D^2f_i, & \text{on } B_i. \end{cases}$$

In addition, the local functions $(f_i)_1^\infty$ extend to a global function $f \in C^2(\Omega)$ with the same properties as above if Ω is contractible (namely, homotopically equivalent to a point).

We may now prove our first main result.

Proof of Theorem 2. Suppose that $u : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}^N$ is a solution to the nonlinear system (1.1) in $C^2(\Omega, \mathbb{R}^N)$ which in addition satisfies that $\text{rk}(Du) \leq 1$ in Ω . Since $\{Du = 0\}$ is closed, necessarily its complement in Ω which is $\{\text{rk}(Du) = 1\}$ is open.

By invoking Theorem 8, we have that there exists a partition of the open subset $\{\text{rk}(Du) = 1\}$ to countably many Borel sets $(B_i)_{i \in \mathbb{N}}$ with respective functions $(f_i)_{i \in \mathbb{N}}$ and curves $(\nu_i)_{i \in \mathbb{N}}$ as in the statement such that (2.1)-(2.2) hold true and in addition

$$Df_i \neq 0 \quad \text{on } B_i, \quad i \in \mathbb{N}.$$

Consequently, on each B_i we have

$$\begin{aligned} \llbracket Du \rrbracket^\perp &= \llbracket (\nu'_i \circ f_i) \otimes Df_i \rrbracket^\perp = \mathbf{I} - \frac{(\nu'_i \circ f_i) \otimes (\nu'_i \circ f_i)}{|\nu'_i \circ f_i|^2}, \\ \Delta u &= (\nu''_i \circ f_i)|Df_i|^2 + (\nu'_i \circ f_i)\Delta f_i. \end{aligned}$$

Hence, (1.1) becomes

$$\left[\mathbf{I} - \frac{(\nu'_i \circ f_i) \otimes (\nu'_i \circ f_i)}{|\nu'_i \circ f_i|^2} \right] \left((\nu''_i \circ f_i)|Df_i|^2 + (\nu'_i \circ f_i)\Delta f_i \right) = 0,$$

on B_i . Since $|\nu_i|^2 \equiv 1$ on $f_i(B_i)$, we have that ν''_i is orthogonal to ν'_i thereon and therefore the above equation reduces to

$$(\nu''_i \circ f_i)|Df_i|^2 = 0 \quad \text{on } B_i, \quad i \in \mathbb{N}.$$

Therefore, ν_i is affine on the interval $f_i(B_i) \subseteq \mathbb{R}$ and as a result $u(B_i) = \nu_i(f_i(B_i))$ is contained in an affine line of \mathbb{R}^N , for each $i \in \mathbb{N}$. On the other hand, since

$$u(\Omega) = u(\{Du = 0\}) \bigcup_{i \in \mathbb{N}} u(B_i)$$

and u is constant on each connected component of the interior of $\{Du = 0\}$, the conclusion ensues by the regularity of u because $u(\{Du = 0\})$ is also contained in the previous union of affine lines. The result ensues. \square

Now we establish Corollary 4 by following similar lines to those of the respective result in [K3].

Proof of Corollary 4. Suppose u is as in the statement of the corollary. By Theorem 8, there exists, a partition of Ω to Borel sets $\{B_i\}_{i \in \mathbb{N}}$, functions $f_i \in C^2(\Omega)$ and Lipschitz curves $\{\nu_i\}_{i \in \mathbb{N}} : \mathbb{R} \rightarrow \mathbb{R}^N$ with $|\nu'_i| \equiv 1$ on $f_i(B_i)$, $|\nu'_i| \equiv 0$ on $\mathbb{R} \setminus f_i(B_i)$ and twice differentiable on $f_i(B_i)$, such that $u|_{B_i} = \nu_i \circ f_i$ and (2.2) holds as well. Since on each B_i we have

$$|Du| = |(\nu'_i \circ f_i) \otimes Df_i| = |Df_i|,$$

by (1.7) and the above, we obtain

$$\begin{aligned} &((\nu'_i \circ f_i) \otimes Df_i) \otimes ((\nu'_i \circ f_i) \otimes Df_i) : \left[(\nu''_i \circ f_i) \otimes Df_i \otimes Df_i + (\nu'_i \circ f_i) \otimes D^2 f_i \right] \\ &+ \frac{|Df_i|^2}{p-2} \left\{ (\nu'_i \circ f_i) \Delta f_i + (\nu''_i \circ f_i) |Df_i|^2 \right\} = 0, \end{aligned}$$

on B_i . Since ν_i'' is orthogonal to ν_i' and also ν_i' has unit length, the above reduces to

$$(\nu_i' \circ f_i) \left[Df_i \otimes Df_i : D^2 f_i + \frac{|Df_i|^2}{p-2} \Delta f_i \right] + \frac{1}{p-2} (\nu_i'' \circ f_i) |Df_i|^4 = 0,$$

on B_i . Again by orthogonality, the above is equivalent to the pair of independent systems

$$(\nu_i' \circ f_i) \left[Df_i \otimes Df_i : D^2 f_i + \frac{|Df_i|^2}{p-2} \Delta f_i \right] = 0, \quad (\nu_i'' \circ f_i) |Df_i|^4 = 0,$$

on B_i . Since $|\nu_i'| \equiv 1$ of $f_i(B_i)$, it follows that $\Delta_p f_i = 0$ on B_i and since $(B_i)_{\mathbb{I}}^\infty$ is a partition of Ω of the form described in the statement, the result ensues by invoking Theorem 2. \square

We may now prove our second main result.

Proof of Theorem 5. We begin by temporarily assuming that Ω is a rectangle of the form

$$Q = (a, b) \times (c, d) \subseteq \mathbb{R}^2$$

and we fix $(x_0, y_0) \in Q = \Omega$. Let us also assume that the rank of the gradient is full throughout:

$$\text{rk}(Du) \equiv 2 \quad \text{in } Q.$$

Later we will remove both these extra assumptions. Let $f : (a, b) \rightarrow \mathbb{R}^N$ and $g : (c, d) \rightarrow \mathbb{R}^N$ be such that $u(x, y) = f(x) + g(y)$. Then, the gradient matrix then has the form

$$Du(x, y) = (f'(x), g'(y)) \in \mathbb{R}^{N \times 2}.$$

By assumption, we have that $[Du]^\perp \Delta u = 0$ in Ω . By Lemma 7, there exists a vector field $A : \mathbb{R}^2 \supseteq \Omega \rightarrow \mathbb{R}^2$ such that $\Delta u = Du A$. If A has components (a, b) , this means that the functions f and g satisfy

$$(2.3) \quad f''(x) + g''(y) = a(x, y) f'(x) + b(x, y) g'(y).$$

Although we will not utilise this in the sequel, it is instructive and quite possible to express the coefficients a, b in terms of f, g, f', g' along the lines of Lemma 7 but more concretely, as follows. By applying the $\mathbb{R}^{N \times N}$ matrix

$$|g'(y)|^2 \mathbf{I} - g'(y) \otimes g'(y)$$

to (2.3), the summand $b(x, y) g'(y)$ on the right hand side is annihilated and we obtain

$$\left[|g'(y)|^2 \mathbf{I} - g'(y) \otimes g'(y) \right] (f''(x) + g''(y)) = a(x, y) \left[|g'(y)|^2 \mathbf{I} - g'(y) \otimes g'(y) \right] f'(x).$$

Hence, on $\{\text{rk}(Du) = 2\}$ we have

$$a(x, y) = \frac{|g'(y)|^2 \mathbf{I} - g'(y) \otimes g'(y)}{\left[|g'(y)|^2 \mathbf{I} - g'(y) \otimes g'(y) \right] f'(x)} (f''(x) + g''(y)).$$

Arguing symmetrically, we may obtain

$$b(x, y) = \frac{|f'(x)|^2 \mathbf{I} - f'(x) \otimes f'(x)}{\left[|f'(x)|^2 \mathbf{I} - f'(x) \otimes f'(x) \right] g'(y)} (f''(x) + g''(y)).$$

Similar expression can be obtained on $\{\text{rk}(Du) \leq 1\}$ as well.

Let us consider (2.3) as a first order ODE with unknown function f' in the variable x . By integrating the equation in x , we view its first integral as an first

order ODE with solution the function g' in the variable y , which can be integrated again. Therefore, by sparing the reader the tedious computations, we arrive at the following integral identity

$$(2.4) \quad \left\{ \begin{aligned} g'(y) e^{-\int_{y_0}^y \frac{B(x,s)}{A(x,s)} ds} + f'(x) \int_{y_0}^y \frac{e^{-\int_{x_0}^x a(s,t) ds}}{A(x,t)} e^{-\int_{y_0}^t \frac{B(x,\tau)}{A(x,\tau)} d\tau} dt \\ = g'(y_0) + f'(x_0) \int_{y_0}^y \frac{1}{A(x,t)} e^{-\int_{y_0}^t \frac{B(x,\tau)}{A(x,\tau)} d\tau} dt, \end{aligned} \right.$$

where

$$\left\{ \begin{aligned} A(x,y) &:= \int_{x_0}^x e^{-\int_{x_0}^s a(\tau,y) d\tau} ds, \\ B(x,y) &:= \int_{x_0}^x b(s,y) e^{-\int_{x_0}^s a(\tau,y) d\tau} ds. \end{aligned} \right.$$

Note that

$$A(x,y) \leq 0 \quad \text{if and only if} \quad x - x_0 \leq 0, \quad \text{for all } y \in \mathbb{R}.$$

Let us now define for convenience the functions

$$\left\{ \begin{aligned} C(x,y) &:= e^{-\int_{y_0}^y \frac{B(x,s)}{A(x,s)} ds}, \\ D(x,y) &:= \int_{y_0}^y \frac{e^{-\int_{x_0}^x a(s,t) ds}}{A(x,t)} e^{-\int_{y_0}^t \frac{B(x,\tau)}{A(x,\tau)} d\tau} dt, \\ E(x,y) &:= \int_{y_0}^y \frac{1}{A(x,t)} e^{-\int_{y_0}^t \frac{B(x,\tau)}{A(x,\tau)} d\tau} dt. \end{aligned} \right.$$

For the sake of brevity, we will be suppressing the dependence on the variables (x,y) . In view of these definitions, (2.4) can be re-written in the abbreviated form

$$(2.5) \quad g'(y)C + f'(x)D = g'(y_0) + f'(x_0)E.$$

By the symmetry of the equations, the substitutions

$$g \longleftrightarrow f, \quad x \longleftrightarrow y, \quad x_0 \longleftrightarrow y_0, \quad a \longleftrightarrow b$$

lead to the additional equation

$$(2.6) \quad \left\{ \begin{aligned} f'(x) e^{-\int_{x_0}^x \frac{G(s,y)}{F(s,y)} ds} + g'(y) \int_{x_0}^x \frac{e^{-\int_{y_0}^y b(t,s) ds}}{F(t,y)} e^{-\int_{x_0}^t \frac{G(\tau,y)}{F(\tau,y)} d\tau} dt \\ = f'(x_0) + g'(y_0) \int_{x_0}^x \frac{1}{F(t,y)} e^{-\int_{x_0}^t \frac{G(\tau,y)}{F(\tau,y)} d\tau} dt, \end{aligned} \right.$$

where

$$\left\{ \begin{aligned} F(x,y) &:= \int_{y_0}^y e^{-\int_{y_0}^s b(x,\tau) d\tau} ds, \\ G(x,y) &:= \int_{y_0}^y a(x,s) e^{-\int_{y_0}^s b(x,\tau) d\tau} ds. \end{aligned} \right.$$

Note that

$$F(x,y) \leq 0 \quad \text{if and only if} \quad y - y_0 \leq 0 \quad \text{for all } x \in \mathbb{R}.$$

For brevity, we set

$$\begin{cases} H(x, y) := e^{-\int_{x_0}^x \frac{G(s,y)}{F(s,y)} ds}, \\ I(x, y) := \int_{x_0}^x \frac{e^{-\int_{y_0}^y b(t,s) ds}}{F(t, y)} e^{-\int_{x_0}^t \frac{G(\tau,y)}{F(\tau,y)} d\tau} dt, \\ J(x, y) := \int_{x_0}^x \frac{1}{F(t, y)} e^{-\int_{x_0}^t \frac{G(\tau,y)}{F(\tau,y)} d\tau} dt. \end{cases}$$

Then (2.6) can be re-written in the simpler form

$$f'(x) = H^{-1} \left(g'(y_0)J + f'(x_0) - g'(y)I \right).$$

Substituting the above into (2.5), after some elementary calculations we obtain the equation

$$(2.7) \quad (C - IH^{-1}) g'(y) = (E - DH^{-1})f'(x_0) + (1 + JH^{-1})g'(y_0),$$

for all $(x, y) \in Q$. Note that

$$I < 0 \quad \text{if} \quad (x - x_0)(y - y_0) < 0, \quad (x, y) \in Q.$$

Moreover, we have that $C > 0$ and also $H > 0$, for all $(x, y) \in Q$. Hence, if $y < y_0$, we may choose $x > x_0$ and if $y > y_0$, we may choose $x < x_0$. In either case, we can arrange

$$C - IH^{-1} > 0$$

for all $y \neq y_0$ such that $(x - x_0)(y - y_0) < 0$. Therefore, from (2.7) we deduce that

$$g'(y) = \left(\frac{E - DH^{-1}}{C - IH^{-1}} \right) f'(x_0) + \left(\frac{1 + JH^{-1}}{C - IH^{-1}} \right) g'(y_0),$$

which yields

$$g'(y) \in \text{span}[\{f'(x_0), g'(y_0)\}].$$

By an integration, the above inclusion implies that the curve g is valued in an affine plane of the form

$$g(y) \in g(y_0) + \text{span}[\{f'(x_0), g'(y_0)\}].$$

By arguing similarly for f , we also infer that

$$f'(x) \in \text{span}[\{f'(x_0), g'(y_0)\}]$$

and hence

$$f(x) \in f(x_0) + \text{span}[\{f'(x_0), g'(y_0)\}].$$

Conclusively, by putting the above together we have obtained

$$f(x) + g(y) \in (f(x_0) + g(y_0)) + \text{span}[\{f'(x_0), g'(y_0)\}],$$

that is,

$$u(x, y) \in u(x_0, y_0) + \text{R}(Du(x_0, y_0)),$$

for all (x, y) running in the rectangle Q . Hence, $u(Q)$ is contained in an affine plane of \mathbb{R}^N , as claimed.

Finally, we remove the additional assumptions that Ω is a rectangle and that the rank of Du is full. Let us consider separately each connected component of the

open subset of Ω given by $\{\text{rk}(Du) = 2\}$. We cover each component with countably many (overlapping) rectangles $\{Q_i : i \in \mathbb{N}\}$ where

$$Q_i = (a_i, b_i) \times (c_i, d_i), \quad i \in \mathbb{N},$$

and respective points inside the rectangles

$$\{(x_{0i}, y_{0i}) \in Q_i : i \in \mathbb{N}\}.$$

On each rectangle, by the previous reasoning the solution will be contained in an affine plane $\Pi_i \subseteq \mathbb{R}^N$. By choosing (x_{0i}, y_{0i}) on the overlaps of each Q_i with all its neighbouring rectangles, connectedness of the component allows us to conclude that all the planes coincide.

It remains to consider the complement $\Omega \setminus \{\text{rk}(Du) = 2\}$, which we decompose to the sets

$$\text{int}\{\text{rk}(Du) \leq 1\} \cup \partial\{\text{rk}(Du) \leq 1\},$$

where “int” denotes topological interior. If the interior is non-empty, on each connected component of it we apply Theorem 2 to infer that $u(\text{int}\{\text{rk}(Du) \leq 1\})$ is contained in a polygonal curve of \mathbb{R}^N , given by an at most countable union of affine straight line segments. Finally, by continuity of u , we have that $u(\partial\{\text{rk}(Du) \leq 1\})$ is also contained in the union of planes. The result ensues. \square

We conclude by establishing the remaining corollary.

Proof of Corollary 6. Suppose that u is a C^2 p -Harmonic mapping as in the hypotheses of the corollary for some $p \in [2, \infty]$, namely that it has additively separated form and solves either (1.3) if $p < \infty$ or (1.5) if $p = \infty$. By (1.9)-(1.10), we deduce that u solves the PDE system

$$|Du|^2 \llbracket Du \rrbracket^\perp \Delta u = 0 \quad \text{in } \Omega.$$

On the open set $\{Du \neq 0\}$, we readily have that u solve the system (1.1). On the other hand, we decompose its complement to

$$\text{int}\{Du = 0\} \cup \partial\{Du = 0\},$$

If $\text{int}\{Du = 0\}$ is non-empty, then u is constant on each connected component of it. Finally, again by the regularity of u , we have that $u(\partial\{Du = 0\})$ is also contained in the previous union of planes. The claim has been established. \square

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