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by

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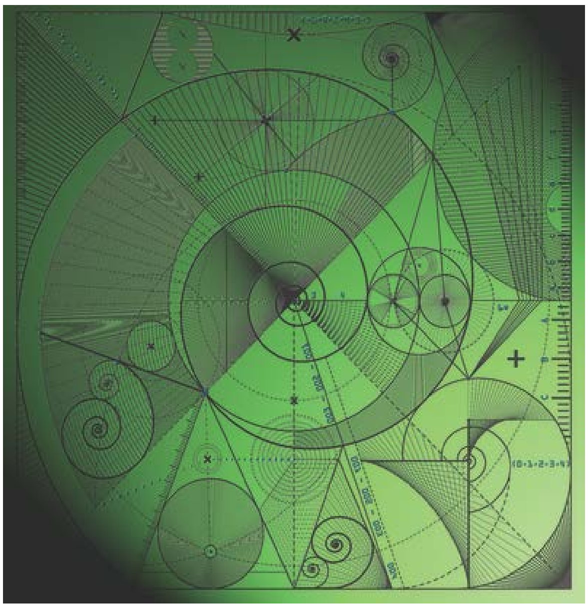

‘Quasi’-norm of an arithmetical convolution operator and the order of the Riemann zeta function[1]

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Abstract

In this paper we study Dirichlet convolution with a given arithmetical function f as a linear mapping φ_f that sends a sequence (a_n) to (b_n) where $b_n = \sum_{d|n} f(d)a_{n/d}$. We investigate when this is a bounded operator on l^2 and find the operator norm. Of particular interest is the case $f(n) = n^{-\alpha}$ for its connection to the Riemann zeta function on the line $\Re s = \alpha$. For $\alpha > 1$, φ_f is bounded with $\|\varphi_f\| = \zeta(\alpha)$.

For the unbounded case, we show that $\varphi_f : \mathcal{M}^2 \to \mathcal{M}^2$ where $\mathcal{M}^2$ is the subset of l^2 of multiplicative sequences, for many $f \in \mathcal{M}^2$. Consequently, we study the ‘quasi’-norm

$$\sup_{\substack{\|a\| = T \\ a \in \mathcal{M}^2}} \frac{\|\varphi_f a\|}{\|a\|}$$

for large T, which measures the ‘size’ of φ_f on $\mathcal{M}^2$. For the $f(n) = n^{-\alpha}$ case, we show this quasi-norm has a striking resemblance to the conjectured maximal order of $|\zeta(\alpha + iT)|$ for $\alpha > \frac{1}{2}$.

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Introduction

Given an arithmetical function $f(n)$, the mapping φ_f sends $(a_n)_{n\in\mathbb{N}}$ to $(b_n)_{n\in\mathbb{N}}$, where

$$b_n = \sum_{d|n} f(d)a_{n/d}. \tag{0.1}$$

Writing $a = (a_n)$, φ_f maps a to $f * a$ where $*$ is Dirichlet convolution. This is a ‘matrix’ mapping, where the matrix, say $M(f)$, is of ‘multiplicative Toeplitz’ type; that is,

$$M(f) = (a_{ij})_{i,j\geq 1}$$

where $a_{ij} = f(i/j)$ and f is supported on the natural numbers (see, for example, [6], [7]).

Toeplitz matrices (whose ij^{th}-entry is a function of $i - j$) are most usefully studied in terms of a ‘symbol’ (the function whose Fourier coefficients make up the matrix). Analogously, the Multiplicative Toeplitz matrix $M(f)$ has as symbol the Dirichlet series

$$\sum_{n=1}^{\infty} f(n)n^{it}.$$

Our particular interest is naturally the case $f(n) = n^{-\alpha}$ when the symbol is $\zeta(\alpha - it)$. We are especially interested how and to what extent properties of the mapping relate to properties of the symbol for $\alpha \leq 1$.

These type of mappings were considered by various authors (for example Wintner [15]) and most notably Toeplitz [13], [14] (although somewhat indirectly, through his investigations of so-called

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"D-forms"). In essence, Toeplitz proved that $\varphi_f : l^2 \to l^2$ is bounded if and only if $\sum_{n=1}^{\infty} f(n)n^{-s}$ is defined and bounded for all $\Re s > 0$. In particular, if $f(n) \geq 0$ then φ_f is bounded on l^2 if and only if $f \in l^1$; furthermore, the operator norm is $\|\varphi_f\| = \|f\|_1$. We prove this in Theorem 1.1 following Toeplitz's original idea. For example, for $f(n) = n^{-\alpha}$, φ_f is bounded on l^2 for $\alpha > 1$ with operator norm $\zeta(\alpha)$. In this special case, the mapping was studied in [7] for $\alpha \leq 1$ when it is unbounded on l^2 by estimating the behaviour of the quantity

$$\Phi_f(N) = \sup_{\|a\|_2=1} \left(\sum_{n=1}^{N} |b_n|^2 \right)^{1/2}$$

for large N. Approximate formulas for $\Phi_f(N)$ were obtained and it was shown that, for $\frac{1}{2} < \alpha \leq 1$, $\Phi_f(N)$ is a lower bound for $\max_{1 \leq t \leq T} |\zeta(\alpha + it)|$ with $N = T^\lambda$ (some $\lambda > 0$ depending on α only). In this way, it was proven that the measure of the set

$$\Big\{ t \in [1, T] : |\zeta(1 + it)| \geq e^\gamma \log\log T - A \Big\}$$

is at least $T \exp\{-a \frac{\log T}{\log\log T}\}$ (some $a > 0$) for A sufficiently large, while for $\frac{1}{2} < \alpha < 1$ one has

$$\max_{t \leq T} |\zeta(\alpha + it)| \geq \exp\left\{ \frac{c(\log T)^{1-\alpha}}{\log\log T} \right\}$$

for some $c > 0$ depending on α only, as well providing an estimate for how often $|\zeta(\alpha + iT)|$ is as large as the right-hand side above. The method is akin to Soundararajan's 'resonance' method and incidentally shows the limitation of this approach for $\alpha > \frac{1}{2}$ since $|\zeta(\alpha + iT)|$ is known to be of larger order.

In this paper we study the unbounded case in a different way, by restricting the domain. Thus in section 2, we show that for many multiplicative f, in particular for f completely multiplicative, $\varphi_f(\mathcal{M}^2) \subset \mathcal{M}^2$ even though $\varphi_f(l^2) \not\subset l^2$. Here $\mathcal{M}^2$ is the set of multiplicative functions in l^2. As a result we consider, for such f, the 'quasi'-norm

$$M_f(T) = \sup_{\substack{\|a\| = T \\ a \in \mathcal{M}^2}} \frac{\|\varphi_f a\|}{\|a\|}$$

and obtain approximate formulae for large T (here $\|\cdot\|$ is the usual l^2-norm). We find that for the particular case $f(n) = n^{-\alpha}$ ($\alpha > \frac{1}{2}$), this quasi-norm has a striking similarity to the conjectured maximal order of $|\zeta(\alpha + iT)|$. For example, with $\alpha = 1$ (i.e. $f(n) = 1/n$) we prove

$$M_f(T) = e^\gamma(\log\log T + \log\log\log T + 2\log 2 - 1) + o(1), \tag{0.2}$$

while for $\frac{1}{2} < \alpha < 1$

$$\log M_f(T) \sim \frac{B(\frac{1}{\alpha}, 1 - \frac{1}{2\alpha})^\alpha}{(1-\alpha)2^\alpha} \frac{(\log T)^{1-\alpha}}{(\log\log T)^\alpha},$$

where $B(x, y)$ is the Beta function. Writing $Z_\alpha(T) = \max_{1 \leq t \leq T} |\zeta(\alpha + it)|$, Granville and Soundararajan [3] proved that $Z_1(T)$ is at least as large as (0.2) minus a $\log\log\log\log T$ term for some arbitrarily large T and they conjectured that it equals (0.2) (possibly with a different constant term). For $\frac{1}{2} < \alpha < 1$, Montgomery [9] found

$$\log Z_\alpha(T) \geq \frac{\sqrt{\alpha - 1/2}}{20} \frac{(\log T)^{1-\alpha}}{(\log\log T)^\alpha}$$

and, using a heuristic argument, conjectured that this is (apart from the constant) the correct order of $\log Z_\alpha(T)$. Further, in a recent paper (see [8]), Lamzouri suggests $\log Z_\alpha(T) \sim C(\alpha)(\log T)^{1-\alpha}(\log\log T)^{-\alpha}$ with some specific constant $C(\alpha)$ (see also the remark after Theorem 3.1).

Similarly one can study the quantity

$$m_f(T) = \inf_{\substack{\|a\| = T \\ a \in \mathcal{M}^2}} \frac{\|\varphi_f a\|}{\|a\|}.$$

With $f(n) = n^{-\alpha}$ this is shown to behave like the known and conjectured minimal order of $|\zeta(\alpha + iT)|$ for $\alpha > \frac{1}{2}$. It should be stressed here that, unlike the case of $\Phi_f(N)$ which was shown to be a lower bound for $Z_\alpha(T)$ in [7], we have not proved any connection between $\zeta(\alpha + iT)$ and $M_f(T)$. Even to show $M_f(T)$ is a lower bound would be very interesting.

Our results, though motivated by the special case $f(n) = n^{-\alpha}$, extend naturally to completely multiplicative f for which $f|_{\mathbb{P}}$ is *regularly varying* (see section 2 for the definition).

Addendum. I would like to thank the anonymous referee for some useful comments and for pointing out a recent paper by Aistleitner and Seip [1]. They deal with an optimization problem which is different yet curiously similar. The function $\exp\{c_\alpha(\log T)^{1-\alpha}(\log\log T)^{-\alpha}\}$ appears in the same way, although their c_α is expected to remain bounded as $\alpha \to \frac{1}{2}$. It would be interesting to investigate any links further.

1. Bounded operators

Notation: Let l^1 and l^2 denote the usual spaces of sequences $(a_n)_{n\in\mathbb{N}}$, with norms $\|a\|_1 = \sum |a_n|$ and $\|a\|_2 = (\sum |a_n|^2)^{1/2}$ respectively. After section 1 we shall, for ease of notation, just write $\|\cdot\|$ for $\|\cdot\|_2$ since it is the norm we will use.

A linear mapping $\varphi : l^2 \to l^2$ is *bounded* if there exists $C > 0$ such that $\|\varphi x\|_2 \le C\|x\|_2$ for all $x \in l^2$. As such, we define the *operator norm* by

$$\|\varphi\| = \sup_{x\neq 0} \frac{\|\varphi x\|_2}{\|x\|_2} = \sup_{\|x\|_2=1} \|\varphi x\|_2.$$

We shall assume from now on that $f(n) \ge 0$ for all $n \in \mathbb{N}$. We are particularly interested in the case where φ_f acts on l^2. Define the function

$$\Phi_f(N) = \sup_{\|a\|_2=1} \sqrt{\sum_{n\le N} |b_n|^2},$$

where b_n is given in terms of a_n by (0.1). Note that the supremum will occur when $a_n \ge 0$ for all n and when $\sum_{n\le N} a_n^2 = 1$.

Suppose now that $f \in l^1$; i.e. $\|f\|_1 = \sum_{n=1}^{\infty} f(n) < \infty$. Then

$$|b_n|^2 = \left|\sum_{d|n} \sqrt{f(d)} \cdot \sqrt{f(d)} a_{n/d}\right|^2 \le \sum_{d|n} f(d) \sum_{d|n} f(d)|a_{n/d}|^2 \le \|f\|_1 \sum_{d|n} f(d)|a_{n/d}|^2.$$

Hence

$$\sum_{n\le N} |b_n|^2 \le \|f\|_1 \sum_{n\le N} \sum_{d|n} f(d)|a_{n/d}|^2 = \|f\|_1 \sum_{d\le N} f(d) \sum_{n\le N/d} |a_n|^2 \le \|f\|_1^2 \|a\|_2^2.$$

Thus

$$\Phi_f(N) \le \|f\|_1. \tag{1.1}$$

Following Toeplitz [14], we show that this inequality is sharp.

Theorem 1.1

Let f be a non-negative arithmetical function and $f \in l^1$. Then $\Phi_f(N) \to \|f\|_1$ as $N \to \infty$. Thus

$\varphi_f : l^2 \to l^2$ *is bounded if and only if* $f \in l^1$, *in which case* $\|\varphi_f\| = \|f\|_1$.

Proof. After (1.1), and since $\Phi_f(N)$ increases with N, we need only provide a lower bound for an infinite sequence of Ns. Let $a_n = d(N)^{-\frac{1}{2}}$ for $n|N$ and zero otherwise (N to be chosen later), where $d(\cdot)$ is the divisor function. Thus $a_1^2 + \ldots + a_N^2 = 1$ and

$$\Phi_f(N) \geq \sum_{n \leq N} a_n b_n = \frac{1}{d(N)} \sum_{n|N} \sum_{d|n} f(d) = \frac{1}{d(N)} \sum_{d|N} f(d) d\Big(\frac{N}{d}\Big), \tag{1.2}$$

say. We choose N such that it has all divisors d up to some (large) number, and that $\frac{d(N/d)}{d(N)}$ is close to 1 for each such divisor d of N. Take N of the form

$$N = \prod_{p \leq P} p^{\alpha_p} \qquad \text{where } \alpha_p = [\tfrac{\log P}{\log p}].$$

Thus every natural number up to P is a divisor of N. For a divisor $d = \prod_{p \leq P} p^{\beta_p}$ of N, we have

$$\frac{d(N/d)}{d(N)} = \prod_{p \leq P} \Big(1 - \frac{\beta_p}{\alpha_p + 1}\Big).$$

If we take $d \leq \sqrt{\log P}$, then $p^{\beta_p} \leq \sqrt{\log P}$ for every prime divisor p of d. Hence, for such p, $\beta_p \leq \frac{\log \log P}{2 \log p}$ and $\beta_p = 0$ if $p > \sqrt{\log P}$. Thus for $d \leq \sqrt{\log P}$,

$$\frac{d(N/d)}{d(N)} = \prod_{p \leq \sqrt{\log P}} \Big(1 - \frac{\beta_p}{\alpha_p + 1}\Big) \geq \prod_{p \leq \sqrt{\log P}} \Big(1 - \frac{\log \log P}{2 \log P}\Big) = \Big(1 - \frac{\log \log P}{2 \log P}\Big)^{\pi(\sqrt{\log P})},$$

where $\pi(x)$ is the number of primes up to x. Since $\pi(x) = O(\frac{x}{\log x})$, it follows that for all P sufficiently large, the expression in (1.2) is at least

$$\Big(1 - \frac{A}{\sqrt{\log P}}\Big) \sum_{d \leq \sqrt{\log P}} f(d)$$

for some constant A. The sum can be made as close to $\|f\|_1$ as we please by increasing P.

□

2. Unbounded operators on l^2

Now we investigate when φ_f is unbounded on l^2 (i.e. $f \notin l^1$). In a similar generalisation of Theorem 1.1 of [7], one can readily show that both $\varphi_f : l^1 \to l^2$ and $\varphi_f : l^2 \to l^\infty$ are bounded if and only if $f \in l^2$, with $\|\varphi_f\| = \|f\|_2$ in either case. So here we shall assume that $f \in l^2 \setminus l^1$. In the appendix we see that, for all cases of interest at least, if $f \notin l^2$, then $\varphi_f a \notin l^2$ for all a except $a = 0$.

For unbounded operators, there are different ways of measuring the 'unboundedness'. One way, which was done in [7] for the case $f(n) = n^{-\alpha}$, is to restrict the *range* by looking at a restricted norm; i.e. by considering $\Phi_f(N)$ for given N. Another way is to restrict the *domain* to a set S say, such that $\varphi_f(S) \subset l^2$ and to consider the size of

$$\sup_{a \in S, \|a\| = N} \frac{\|\varphi_f a\|}{\|a\|} \qquad \text{for large } N.$$

For f completely multiplicative one is naturally led to consider $S = \mathcal{M}^2$ — the set of square summable multiplicative functions. It is also natural to consider *regularly varying* functions.

Regular Variation. A function $\ell : [A, \infty) \to \mathbb{R}$ is regularly varying of index ρ if it is measurable and

$$\ell(\lambda x) \sim \lambda^\rho \ell(x) \qquad \text{as } x \to \infty \text{ for every } \lambda > 0$$

(see [2] for a detailed treatise on the subject). For example, $x^\rho(\log x)^\tau$ is regularly-varying of index ρ for any τ. The Uniform Convergence Theorem says that the above asymptotic formula is automatically *uniform* for λ in compact subsets of $(0,\infty)$. Note that every regularly varying function of non-zero index is asymptotic to one which is strictly monotonic and continuous. We shall make use of *Karamata's Theorem: for* ℓ *regularly varying of index* ρ,

$$\int^x \ell \sim \frac{x\ell(x)}{\rho+1} \quad \text{if } \rho > -1, \qquad \int_x^\infty \ell \sim -\frac{x\ell(x)}{\rho+1} \quad \text{if } \rho < -1,$$

while if $\rho = -1$, $\int^x \ell$ *is slowly varying (regularly varying with index* 0*) and* $\int^x \ell \succ x\ell(x)$.

Notation. Let $\mathcal{M}^2$ and $\mathcal{M}^2_c$ denote the subsets of l^2 of multiplicative and completely multiplicative functions respectively. Further, write $\mathcal{M}^2+$ for the non-negative members of $\mathcal{M}^2$ and similarly for $\mathcal{M}^2_c+$.

2.1 The size of $\|\varphi_f\|$ on $\mathcal{M}^2$

Now we consider φ_f on the subset $\mathcal{M}^2$ of multiplicative functions in l^2. We suppose, as in section 2, that $f \in l^2 \setminus l^1$ so that φ_f is unbounded. This implies there exist $a \in l^2$ such that $\varphi_f(a) \notin l^2$ (by the closed graph theorem). However, if f is multiplicative then, as we shall see, $\varphi_f(\mathcal{M}^2) \subset l^2$ in many cases (and hence $\varphi_f(\mathcal{M}^2) \subset \mathcal{M}^2$).

Lemma 2.1

Let $f, g \in \mathcal{M}^2$ *be non-negative. Then* $f * g \in \mathcal{M}^2$ *if and only if*

$$\sum_p \sum_{m,n\geq 1} \sum_{k=0}^{\infty} f(p^m)g(p^n)f(p^{m+k})g(p^{n+k}) \quad \text{converges.} \tag{2.1}$$

Proof. Let $h = f * g$. Since h is multiplicative,

$$\sum_{n=1}^{\infty} h(n)^2 < \infty \iff \sum_p \sum_{k\geq 1} h(p^k)^2 < \infty.$$

Let $k \geq 1$ and p prime. Then

$$h(p^k) = \sum_{r=0}^{k} f(p^r)g(p^{k-r}) = f(p^k) + g(p^k) + \sum_{r=1}^{k-1} f(p^r)g(p^{k-r}).$$

Using the inequality $a^2+b^2+c^2 \leq (a+b+c)^2 \leq 3(a^2+b^2+c^2)$ we have

$$\Big(\sum_{r=1}^{k-1} f(p^r)g(p^{k-r})\Big)^2 \leq h(p^k)^2 \leq 3f(p^k)^2 + 3g(p^k)^2 + 3\Big(\sum_{r=1}^{k-1} f(p^r)g(p^{k-r})\Big)^2.$$

Since $\sum_{p,k\geq 1} f(p^k)^2$ and $\sum_{p,k\geq 1} g(p^k)^2$ converge we find that $\sum_{p,k\geq 1} h(p^k)^2$ converges if and only if

$$\sum_p \sum_{k=2}^{\infty} \Big(\sum_{r=1}^{k-1} f(p^r)g(p^{k-r})\Big)^2 \text{ converges.}$$

But

$$\begin{aligned}\sum_{k=2}^{\infty}\Big(\sum_{r=1}^{k-1} f(p^r)g(p^{k-r})\Big)^2 &= \sum_{k=1}^{\infty} \sum_{1\leq r,s\leq k} f(p^r)f(p^s)g(p^{k-r+1})g(p^{k-s+1}) \\ &\leq 2\sum_{k=1}^{\infty}\sum_{s=1}^{k}\sum_{r=1}^{s} f(p^r)f(p^s)g(p^{k-r+1})g(p^{k-s+1}) \\ &= 2\sum_{r=1}^{\infty}\sum_{k=1}^{\infty}\sum_{s=0}^{\infty} f(p^r)f(p^{s+r})g(p^{k+s})g(p^k).\end{aligned} \tag{2.2}$$

On the other hand, the RHS of (2.2) is greater than

$$\sum_{k=1}^{\infty}\sum_{s=1}^{k}\sum_{r=1}^{s} f(p^r)f(p^s)g(p^{k-r+1})g(p^{k-s+1}).$$

Hence $h \in \mathcal{M}^2$ if and only if

$$\sum_p \sum_{m,n\geq 1}\sum_{k=0}^{\infty} f(p^m)g(p^n)f(p^{m+k})g(p^{n+k}) \quad \text{converges.}$$

□

Let $\mathcal{M}_0^2$ denote the set of $\mathcal{M}^2$ functions f for which $f * g \in \mathcal{M}^2$ whenever $g \in \mathcal{M}^2$; that is,

$$\mathcal{M}_0^2 = \{f \in \mathcal{M}^2 : g \in \mathcal{M}^2 \implies f * g \in \mathcal{M}^2\}.$$

Thus for $f \in \mathcal{M}_0^2$, $\varphi_f(\mathcal{M}^2) \subset \mathcal{M}^2$. We shall see that it may happen that $f, g \in \mathcal{M}^2$ but $f * g \notin \mathcal{M}^2$. So $\mathcal{M}_0^2 \neq \mathcal{M}^2$. The following gives a criterion for multiplicative functions to be in $\mathcal{M}_0^2$.

Proposition 2.2
Let $f \in \mathcal{M}^2$ be such that $\sum_{k=1}^{\infty} |f(p^k)|$ converges for every prime p and that $\sum_{k=1}^{\infty} |f(p^k)| \leq A$ for some constant A independent of p. Then $f \in \mathcal{M}_0^2$.

On the other hand, if $f \in \mathcal{M}^2$ with $f \geq 0$ and for some prime p_0, $f(p_0^k)$ decreases with k and $\sum_{k=1}^{\infty} f(p_0^k)$ diverges, then $f \notin \mathcal{M}_0^2$.

Proof. Without loss of generality we can take $f \geq 0$. Let $g \in \mathcal{M}^2$ (again w.l.o.g. $g \geq 0$) with $\alpha_p = \sum_{k=1}^{\infty} g(p^k)^2$. Thus $\sum_p \alpha_p$ converges. By the Cauchy-Schwarz inequality,

$$\Big(\sum_{n=1}^{\infty} g(p^n)g(p^{n+k})\Big)^2 \leq \sum_{n=1}^{\infty} g(p^n)^2 \sum_{n=1}^{\infty} g(p^{n+k})^2 \leq \alpha_p\alpha_p = \alpha_p^2.$$

Thus by Lemma 2.1, $f * g \in \mathcal{M}^2$ if

$$\sum_p \alpha_p \sum_{m=1}^{\infty} f(p^m) \sum_{k=0}^{\infty} f(p^{m+k}) \quad \text{converges.}$$

By assumption, the inner sum over k is bounded by a constant (independent of p), and hence so is the sum over m. This implies the convergence of the above. Hence $f * g \in \mathcal{M}^2$.

Now suppose $\sum_{k=1}^{\infty} f(p_0^k)$ diverges for some prime p_0. Then with $g \in \mathcal{M}^2$ and $g(p_0^k)$ decreasing (to zero) we have

$$(f * g)(p_0^k) = \sum_{r=0}^{k} f(p_0^r)g(p_0^{k-r}) \geq g(p_0^k)\sum_{r=0}^{k} f(p_0^r) = g(p_0^k)c_k,$$

where $c_k \nearrow \infty$. Thus $\sum_k (f * g)(p_0^k)^2 \geq \sum_k g(p_0^k)^2 c_k^2$. But we can always choose $g(p_0^k)$ decreasing so that $\sum_k g(p_0^k)^2$ converges while, for the given sequence c_k, $\sum_k g(p_0^k)^2 c_k^2$ diverges. (Choose $g(p_0^k)^2 = \frac{1}{c_{k-1}} - \frac{1}{c_k}$.)

Thus $f * g \notin \mathcal{M}^2$; i.e. $f \notin \mathcal{M}_0^2$.

□

Thus, in particular, $\mathcal{M}_c^2 \subset \mathcal{M}_0^2$. For $f \in \mathcal{M}_c^2$ if and only if $|f(p)| < 1$ for all primes p and $\sum_p |f(p)|^2 < \infty$. Thus

$$\sum_{k=1}^{\infty} |f(p^k)| = \frac{|f(p)|}{1 - |f(p)|} \le A,$$

independent of p (since $f(p) \to 0$).

The "quasi-norm" $M_f(T)$
Let $f \in \mathcal{M}_0^2$. From above we see that $\varphi_f(\mathcal{M}^2) \subset \mathcal{M}^2$ but, typically, φ_f is not 'bounded' on $\mathcal{M}^2$ (if $f \notin l^1$) in the sense that $\|\varphi_f a\|/\|a\|$ is not bounded by a constant for all $a \in \mathcal{M}^2$. It therefore makes sense to define, for $T \ge 1$,

$$M_f(T) = \sup_{\substack{a \in \mathcal{M}^2 \\ \|a\| = T}} \frac{\|\varphi_f a\|}{\|a\|}.$$

We aim to find the behaviour of $M_f(T)$ for large T.

We shall consider f completely multiplicative and such that $f|_{\mathbb{P}}$ is *regularly varying* of index $-\alpha$ with $\alpha > 1/2$ in the sense that there exists a regularly varying function $\tilde{f}$ (of index $-\alpha$) with $\tilde{f}(p) = f(p)$ for every prime p.

Our main result here is the following:

Theorem 2.3
Let $f \in \mathcal{M}_c^2$, such that $f \ge 0$ and $f|_{\mathbb{P}}$ is regularly varying of index $-\alpha$ where $\alpha \in (\frac{1}{2}, 1)$. Then

$$\log M_f(T) \sim c(\alpha) \tilde{f}(\log T \log\log T) \log T$$

where $\tilde{f}$ is any regularly varying extension of $f|_{\mathbb{P}}$ and

$$c(\alpha) = \frac{B(\frac{1}{\alpha}, 1 - \frac{1}{2\alpha})^{\alpha}}{(1-\alpha)2^{\alpha}}$$

and $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$ is the Beta function.

For the proof, we obtain upper and lower bounds for $\log M_f(T)$ which are asymptotic to each other. For the lower bounds, we require a formula for $\|\varphi_f a\|$ when $a \in \mathcal{M}_c^2$. This follows from the following rather elegant formula:

Lemma 2.4
For $f, g \in \mathcal{M}_c^2$,

$$\frac{\|f * g\|}{\|f\|\|g\|} = \frac{|\langle f, g\rangle|}{\|fg\|},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product for l^2.

Proof. We have

$$\begin{aligned}
\|f * g\|^2 = \sum_{n=1}^{\infty} |(f*g)(n)|^2 &= \sum_{n=1}^{\infty} \sum_{c,d|n} f(c)\overline{f(d)} g\Big(\frac{n}{c}\Big)\overline{g\Big(\frac{n}{d}\Big)} \\
&= \sum_{c,d\ge 1} f(c)\overline{f(d)} \sum_{m=1}^{\infty} g\Big(\frac{m[c,d]}{c}\Big)\overline{g\Big(\frac{m[c,d]}{d}\Big)} \\
&= \sum_{m=1}^{\infty} |g(m)|^2 \sum_{c,d\ge 1} f(c)\overline{f(d)} g\Big(\frac{d}{(c,d)}\Big)\overline{g\Big(\frac{c}{(c,d)}\Big)}.
\end{aligned}$$

Collecting those terms for which $(c,d)=k$, writing $c=km$, $d=kn$, and using complete multiplicativity of f

$$\left(\frac{\|f*g\|}{\|g\|}\right)^2=\sum_{k=1}^{\infty}|f(k)|^2\sum_{\substack{m,\,n\,\geq\,1\\(m,\,n)\,=\,1}}f(m)\overline{f(n)g(m)}g(n).$$

But

$$|\langle f,g\rangle|^2=\sum_{m,n\geq 1}f(m)\overline{f(n)g(m)}g(n)=\sum_{d=1}^{\infty}|f(d)g(d)|^2\sum_{\substack{m,\,n\,\geq\,1\\(m,\,n)\,=\,1}}f(m)\overline{f(n)g(m)}g(n),$$

so the result follows.

□

Thus for $f,a\in\mathcal{M}_c^2$,

$$\frac{\|\varphi_f a\|}{\|a\|}=\frac{\|f\|\cdot|\sum_{n=1}^{\infty}f(n)\overline{a_n}|}{(\sum_{n=1}^{\infty}|f(n)a_n|^2)^{1/2}}.$$

Since $|a_n|\leq 1$, as a corollary we have:

Corollary 2.5
For $f,a\in\mathcal{M}_c^2$,

$$\left|\sum_{n=1}^{\infty}f(n)\overline{a_n}\right|\leq\frac{\|\varphi_f a\|}{\|a\|}\leq\|f\|\left|\sum_{n=1}^{\infty}f(n)\overline{a_n}\right|.$$

Note that by complete multiplicativity,

$$\sum_{n=1}^{\infty}f(n)\overline{a_n}=\prod_p\frac{1}{1-f(p)\overline{a_p}}=\prod_p\exp\Big\{f(p)\overline{a_p}+O(|f(p)a_p|^2)\Big\},$$

and $\sum_p|f(p)a_p|^2\leq\sum_p|f(p)|^2=O(1)$, so that

$$\log\frac{\|\varphi_f a\|}{\|a\|}=\Re\sum_p f(p)\overline{a_p}+O(1).\qquad(2.3)$$

Proof of Theorem 2.3. We consider first upper bounds. The supremum occurs for $a\geq 0$ which we now assume. Write $a=(a_n)$, $\varphi_f a=b=(b_n)$. Define α_p and β_p for prime p by

$$\alpha_p=\sum_{k=1}^{\infty}a_{p^k}^2\quad\text{and}\quad\beta_p=\sum_{k=1}^{\infty}b_{p^k}^2.$$

By multiplicativity of a and b we have $T^2=\|a\|^2=\prod_p(1+\alpha_p)$ and $\|b\|^2=\prod_p(1+\beta_p)$. Thus

$$\frac{\|\varphi_f a\|}{\|a\|}=\prod_p\sqrt{\frac{1+\beta_p}{1+\alpha_p}}.$$

Now for $k\geq 1$

$$b_{p^k}=\sum_{r=0}^{k}f(p^r)a_{p^{k-r}}=a_{p^k}+f(p)b_{p^{k-1}}.$$

Thus

$$b_{p^k}^2=a_{p^k}^2+2f(p)a_{p^k}b_{p^{k-1}}+f(p)^2b_{p^{k-1}}^2.$$

Summing from $k=1$ to ∞ and adding 1 to both sides gives

$$1+\beta_p=1+\alpha_p+2f(p)\sum_{k=1}^{\infty}a_{p^k}b_{p^{k-1}}+f(p)^2(1+\beta_p).\qquad(2.4)$$

By Cauchy-Schwarz,

$$\sum_{k=1}^{\infty} a_{p^k} b_{p^{k-1}} \le \left(\sum_{k=1}^{\infty} a_{p^k}^2 \sum_{k=1}^{\infty} b_{p^{k-1}}^2 \right)^{1/2} = \sqrt{\alpha_p(1+\beta_p)},$$

so, on rearranging

$$(1+\beta_p) - \frac{2f(p)\sqrt{\alpha_p(1+\beta_p)}}{1-f(p)^2} \le \frac{1+\alpha_p}{1-f(p)^2}.$$

Completing the square we find

$$\left(\sqrt{1+\beta_p} - \frac{f(p)\sqrt{\alpha_p}}{1-f(p)^2} \right)^2 \le \frac{1+\alpha_p}{(1-f(p)^2)^2}.$$

The term on the left inside the square is non-negative for p sufficiently large since $f(p) \to 0$; in fact from (2.4), $1+\beta_p \ge \frac{1+\alpha_p}{1-f(p)^2}$ which is greater than $\frac{f(p)^2\alpha_p}{(1-f(p)^2)^2}$ if $f(p) \le 1/\sqrt{2}$. Rearranging gives

$$\sqrt{\frac{1+\beta_p}{1+\alpha_p}} \le \frac{1}{1-f(p)^2} \left(1 + f(p) \sqrt{\frac{\alpha_p}{1+\alpha_p}} \right).$$

Let $\gamma_p = \sqrt{\frac{\alpha_p}{1+\alpha_p}}$. Taking the product over all primes p gives

$$\frac{\|\varphi_f a\|}{\|a\|} \le A\|f\|^2 \prod_p (1 + f(p)\gamma_p) \le A' \exp\Big\{ \sum_p f(p)\gamma_p \Big\} \tag{2.5}$$

for some constants A, A' depending only on f. (We can take $A = 1$ if $f(p) \le 1/\sqrt{2}$.) Note that $0 \le \gamma_p < 1$ and $\prod_p \frac{1}{1-\gamma_p^2} = T^2$.

Let $\varepsilon > 0$ and put $P = \log T \log\log T$. We split up the sum on the RHS of (2.5) into $p \le aP$, $aP < p \le AP$ and $p > AP$ (for a small and A large). First

$$\sum_{p \le aP} f(p)\gamma_p \le \sum_{p \le aP} f(p) \sim \frac{a^{1-\alpha} P \tilde{f}(P)}{(1-\alpha)\log P} < \varepsilon \tilde{f}(\log T \log\log T) \log T, \tag{2.6}$$

for a sufficiently small[2]. Next, using the fact that $\log T^2 = \log \prod_p \frac{1}{1-\gamma_p^2} \ge \sum_p \gamma_p^2$, we have (since $\tilde{f}^2$ is regularly-varying of index -2α)

$$\sum_{p > AP} f(p)\gamma_p \le \left(\sum_{p > AP} f(p)^2 \sum_{p > AP} \gamma_p^2 \right)^{1/2} \lesssim \left(\frac{2A^{1-2\alpha} P \tilde{f}(P)^2 \log T}{(2\alpha - 1)\log P} \right)^{1/2}$$

$$\sim \frac{\tilde{f}(\log T \log\log T) \log T}{A^{\alpha - 1/2}\sqrt{\alpha - 1/2}} < \varepsilon \tilde{f}(\log T \log\log T) \log T \tag{2.7}$$

for A sufficiently large. This leaves the range $aP < p \le AP$.

Note that the result follows from the case $f(n) = n^{-\alpha}$. For, by the uniform convergence theorem for regularly varying functions

$$\left| f(p) - \left(\frac{P}{p} \right)^{\alpha} \tilde{f}(P) \right| < \varepsilon f(p) \tag{2.8}$$

for $aP < p \le AP$ and P sufficiently large, depending only on ε. The problem therefore reduces to maximising

$$\sum_{aP < p \le AP} \frac{\gamma_p}{p^{\alpha}}$$

[2] Using $\sum_{p \le x} f(p) \sim \int_2^x \frac{\tilde{f}(t)}{\log t}\,dt \sim \frac{x\tilde{f}(x)}{(1-\alpha)\log x}$, since $\tilde{f}$ is regularly-varying of index $-\alpha$.

subject to $0 \le \gamma_p < 1$ and $\prod_p \frac{1}{1-\gamma_p^2} = T^2$. The maximum clearly occurs for γ_p decreasing (if $\gamma_{p'} > \gamma_p$ for primes $p < p'$, then the sum increases in value if we swap γ_p and $\gamma_{p'}$). Thus we may assume that γ_p is decreasing.

By interpolation we may write $\gamma_p = g(\frac{p}{P})$ where $g : (0,\infty) \to (0,1)$ is continuously differentiable and decreasing. Of course g will depend on P. Let $h = \log \frac{1}{1-g^2}$, which is also decreasing. Note that

$$2\log T = \sum_p h\Big(\frac{p}{P}\Big) \ge \sum_{p\le aP} h\Big(\frac{p}{P}\Big) \ge h(a)\pi(aP) \ge cah(a)\log T,$$

for P sufficiently large, for some constant $c > 0$. Thus $h(a) \le C_a$ (independent of T).

Now, for $F : (0,\infty) \to [0,\infty)$ decreasing,

$$\sum_{ax<p\le bx} F\Big(\frac{p}{x}\Big) = \frac{x}{\log x}\int_a^b F + O\Big(\frac{xF(a)}{(\log x)^2}\Big), \tag{2.9}$$

where the implied constant is independent of F (and x). For, on writing $\pi(x) = \mathrm{li}(x) + e(x)$, the LHS is

$$\begin{aligned}\int_{ax}^{bx} F\Big(\frac{t}{x}\Big)\,d\pi(t) &= x\int_a^b \frac{F(t)}{\log xt}\,dt + \int_a^b F(t)\,de(xt)\\ &= \frac{x}{\log \theta x}\int_a^b F + \Big[F(t)e(xt)\Big]_a^b - \int_a^b e(xt)\,dF(t) \qquad (\text{some } \theta\in[a,b])\\ &= \frac{x}{\log x}\int_a^b F + O\Big(\frac{xF(a)}{(\log x)^2}\Big),\end{aligned}$$

on using $e(x) = O(\frac{x}{(\log x)^2})$ and the fact that F is decreasing. Thus by (2.9)

$$2\log T \ge \sum_{aP<p\le AP} h\Big(\frac{p}{P}\Big) \sim \frac{P}{\log P}\int_a^A h \sim (\log T)\int_a^A h.$$

Since a and A are arbitrary, $\int_0^\infty h$ must exist and is at most 2. Also, by (2.9)

$$\sum_{aP<p\le AP} \frac{\gamma_p}{p^\alpha} = \frac{1}{P^\alpha}\sum_{aP<p\le AP} g\Big(\frac{p}{P}\Big)\Big(\frac{p}{P}\Big)^{-\alpha} \sim \frac{P^{1-\alpha}}{\log P}\int_a^A \frac{g(u)}{u^\alpha}\,du.$$

Hence by (2.8),

$$\sum_{aP<p\le AP} f(p)\gamma_p \sim \tilde f(P)P^\alpha \sum_{aP<p\le AP}\frac{\gamma_p}{p^\alpha} \sim \frac{P\tilde f(P)}{\log P}\int_a^A \frac{g(u)}{u^\alpha}\,du.$$

As a, A are arbitrary, it follows from above and (2.5), (2.6), (2.7) that

$$\log\frac{\|\varphi_f a\|}{\|a\|} \le \Big(\int_0^\infty \frac{g(u)}{u^\alpha}\,du + o(1)\Big)\tilde f(\log T\log\log T)\log T.$$

Thus we need to maximize $\int_0^\infty g(u)u^{-\alpha}du$ subject to $\int_0^\infty h \le 2$ over all decreasing $g : (0,\infty) \to (0,1)$. Since h is decreasing,

$$\frac{1}{2}xh(x) \le \int_{x/2}^x h.$$

The RHS can be made as small as we please for x sufficiently small or large (as $\int_0^\infty h$ converges). In particular, $xh(x) \to 0$ as $x \to \infty$ and as $x \to 0^+$. In fact, for the supremum, we can consider just those g (and h) which are continuously differentiable and strictly decreasing, since we can approximate arbitrarily closely with such functions. On writing $g = s \circ h$ where $s(x) = \sqrt{1 - e^{-x}}$, we have

$$\int_0^\infty \frac{g(u)}{u^\alpha}\, du = \Big[\frac{g(u)u^{1-\alpha}}{1-\alpha}\Big]_0^\infty - \frac{1}{1-\alpha}\int_0^\infty g'(u)u^{1-\alpha}\, du$$
$$= -\frac{1}{1-\alpha}\int_0^\infty s'(h(u))h'(u)u^{1-\alpha}\, du = \frac{1}{1-\alpha}\int_0^{h(0^+)} s'(x)l(x)^{1-\alpha}\, dx,$$

where $l = h^{-1}$, since $\sqrt{u}g(u) \to 0$ as $u \to \infty$. The final integral is, by Hölder's inequality at most

$$\Big(\int_0^{h(0^+)} {s'}^{1/\alpha}\Big)^\alpha \Big(\int_0^{h(0^+)} l\Big)^{1-\alpha}. \tag{2.10}$$

But $\int_0^{h(0^+)} l = -\int_0^\infty uh'(u)du = \int_0^\infty h \le 2$, so

$$\int_0^\infty \frac{g(u)}{u^\alpha}\, du \le \frac{2^{1-\alpha}}{1-\alpha}\Big(\int_0^\infty {s'}^{1/\alpha}\Big)^\alpha.$$

A direct calculation shows that[3] $\int_0^\infty (s')^{1/\alpha} = 2^{-1/\alpha}B(\frac{1}{\alpha}, 1 - \frac{1}{2\alpha})$. This gives the upper bound.

The proof of the upper bound leads to the optimum choice for g and the lower bound. We note that we have equality in (2.10) if $l/(s')^{1/\alpha}$ is constant; i.e. $l(x) = cs'(x)^{1/\alpha}$ for some constant $c > 0$ — chosen so that $\int_0^\infty l = 2$. This means we take

$$h(x) = (s')^{-1}\Big(\Big(\frac{x}{c}\Big)^\alpha\Big) = \log\Big(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \Big(\frac{c}{x}\Big)^{2\alpha}}\Big).$$

from which we can calculate g. In fact, we show that we get the required lower bound by just considering a_n completely multiplicative. To this end we use (2.3), and define a_p by:

$$a_p = g_0\Big(\frac{p}{P}\Big),$$

where $P = \log T \log\log T$ and g_0 is the function

$$g_0(x) = \sqrt{1 - \frac{2}{1 + \sqrt{1 + (\frac{c}{x})^{2\alpha}}}},$$

with $c = 2^{1+1/\alpha}/B(\frac{1}{\alpha}, 1 - \frac{1}{2\alpha})$. As such, by the same methods as before, we have $\|a\| = T^{1+o(1)}$ and

$$\log\frac{\|\varphi_\alpha a\|}{\|a\|} = \sum_p f(p)g_0\Big(\frac{p}{P}\Big) + O(1) \sim \frac{P\tilde{f}(P)}{\log P}\int_0^\infty \frac{g_0(u)}{u^\alpha}\, du.$$

By the choice of g_0, the integral on the right is $\frac{B(\frac{1}{\alpha}, 1-\frac{1}{2\alpha})^\alpha}{(1-\alpha)2^\alpha}$, as required.

□

Remark. From the above proof, we see that the supremum (of $\|\varphi_f a\|/\|a\|$) over $\mathcal{M}_c^2$ is roughly the same size as the supremum over $\mathcal{M}^2$; i.e. they are log-asymptotic to each other. Is it true that these respective suprema are closer still; eg. are they asymptotic to each other for $\frac{1}{2} < \alpha < 1$?

3. The special case $f(n) = n^{-\alpha}$.

In this case we can take $\tilde{f}(x) = x^{-\alpha}$ which is regularly varying of index $-\alpha$. Here we shall write φ_α for φ_f and M_α for M_f.

[3]The integral is $2^{-1/\alpha}\int_0^\infty e^{-x/\alpha}(1 - e^{-x})^{-1/2\alpha}dx = 2^{-1/\alpha}\int_0^1 t^{1/\alpha - 1}(1 - t)^{-1/2\alpha}dt$.

Theorem 3.1
We have

$$M_1(T) = e^\gamma(\log\log T + \log\log\log T + 2\log 2 - 1 + o(1)), \tag{3.1}$$

while for $\frac{1}{2} < \alpha < 1$,

$$\log M_\alpha(T) = \left(\frac{B(\frac{1}{\alpha}, 1-\frac{1}{2\alpha})^\alpha}{(1-\alpha)2^\alpha} + o(1)\right)\frac{(\log T)^{1-\alpha}}{(\log\log T)^\alpha}. \tag{3.2}$$

Remark. As noted in the introduction, these asymptotic formulae bear a strong resemblance to the (conjectured) maximal order of $|\zeta(\alpha+iT)|$. It is interesting to note that the bounds found here are just larger than what is known about the lower bounds for $Z_\alpha(T) = \max_{1\le t\le T}|\zeta(\alpha+it)|$. In a recent paper (see [8]), Lamzouri suggests $\log Z_\alpha(T) \sim C(\alpha)(\log T)^{1-\alpha}(\log\log T)^{-\alpha}$ with some specific function[4] $C(\alpha)$ (for $\frac{1}{2} < \alpha < 1$). We note that the constant appearing in (3.2) is not $C(\alpha)$ since, for α near $\frac{1}{2}$, the former is roughly $\frac{1}{\sqrt{\alpha-\frac{1}{2}}}$, while $C(\alpha) \sim \frac{1}{\sqrt{2\alpha-1}}$. For $\alpha = 1$, see the comment in the introduction.

It would be very interesting to be able to extend these ideas (and results) to the $\alpha = \frac{1}{2}$ case. As we show in the appendix, we cannot do this by restricting $\varphi_{\frac{1}{2}}$ to smaller domains in l^2. Somehow the analogy — if such exists — between M_α and Z_α breaks down just here.

Proof of Theorem 3.1. For $\frac{1}{2} < \alpha < 1$ the result follows from Theorem 2.3, so we only concern ourselves with $\alpha = 1$.

For an upper bound we use (2.5) with $f(p) = 1/p$ (and $A = 1$). Thus

$$\frac{\|\varphi_1 a\|}{\|a\|} \le \zeta(2)\prod_p\left(1+\frac{\gamma_p}{p}\right).$$

Again, the maximum of the RHS (subject to $0 \le \gamma_p < 1$ and $\prod_p \frac{1}{1-\gamma_p^2} = T^2$) occurs for γ_p decreasing. Let $P = \log T\log\log T$ and a, A be arbitrary constants such that $A > 1 > a > 0$. Split the product into the ranges $p \le aP$, $aP < p \le AP$ and $p > AP$. We have

$$\zeta(2)\prod_{p\le aP}\left(1+\frac{\gamma_p}{p}\right) \le \zeta(2)\prod_{p\le aP}\left(1+\frac{1}{p}\right) = e^\gamma(\log aP + o(1))$$

by Merten's Theorem, while the product over $p > AP$ is at most

$$\exp\left\{\sum_{p>AP}\frac{\gamma_p}{p}\right\} \le \exp\left\{\left(\sum_{p>AP}\frac{1}{p^2}\sum_{p>AP}\gamma_p^2\right)^{\frac{1}{2}}\right\} \le \exp\left\{\sqrt{2\log T\sum_{p>AP}\frac{1}{p^2}}\right\}.$$

But $\sum_{p>AP} 1/p^2 \sim 1/AP\log P \sim 1/A\log T(\log\log T)^2$, so

$$\prod_{p>AP}\left(1+\frac{\gamma_p}{p}\right) \le 1 + \frac{2}{\sqrt{A}\log\log T}$$

for all large enough T. Combining the above two estimates gives

$$\zeta(2)\prod_{\substack{p\le aP\\ p>AP}}\left(1+\frac{\gamma_p}{p}\right) \le e^\gamma\left(\log_2 T + \log_3 T + \log a + \frac{2}{\sqrt{A}} + o(1)\right).$$

For the remaining range $aP < p \le AP$ we write, as before, $\gamma_p = g(\frac{p}{P})$ where $g : (0,\infty) \to (0,1)$ is decreasing. Then

$$\log\left(\prod_{aP<p\le AP}\left(1+\frac{g(p/P)}{p}\right)\right) \sim \sum_{aP<p\le AP}\frac{g(p/P)}{p} \sim \frac{1}{\log P}\int_a^A\frac{g(u)}{u}\,du,$$

[4] Lamzouri has $C(\alpha) = G_1(\alpha)^\alpha\alpha^{-2\alpha}(1-\alpha)^{\alpha-1}$, where $G_1(x) = \int_0^\infty u^{-1-1/x}\log(\sum_{n=0}^\infty\frac{(u/2)^{2n}}{(n!)^2})du$.

by (2.9). Thus

$$\frac{\|\varphi_1 a\|}{\|a\|} \le e^\gamma \Big(\log_2 T + \log_3 T + \int_a^A \frac{g(u)}{u}\, du - \int_a^1 \frac{1}{u}\, du + \frac{2}{\sqrt{A}} + o(1) \Big)$$

for all $A > 1 > a > 0$. We need to minimise the constant term. Since $g(u) < 1$, the minimum occurs for a arbitrarily small. On the other hand $\int_A^\infty \frac{g(u)}{u} du \le (\frac{1}{A} \int_A^\infty g^2)^{1/2} = o(1/\sqrt{A})$, so the constant is minimized for arbitrarily large A; i.e. it is at most $\int_1^\infty \frac{g(u)}{u} du - \int_0^1 \frac{1-g(u)}{u} du$. Thus

$$M_1(T) \le e^\gamma(\log\log T + \log\log\log T + \kappa + o(1)) \qquad \text{where} \qquad \kappa = \sup\{L(g) : g \in G\}.$$

Here $L(g) = \int_1^\infty \frac{g(u)}{u} du - \int_0^1 \frac{1-g(u)}{u} du$ and G is the set of all decreasing $g : (0, \infty) \to (0, 1)$ for which $\int_0^\infty \log \frac{1}{1-g^2} \le 2$. As in the proof of Theorem 2.3, let $h = \log \frac{1}{1-g^2}$ so that $g = s \circ h$ where $s(x) = \sqrt{1 - e^{-x}}$. Now we show $\kappa = 2 \log 2 - 1$. Trivially, by Cauchy-Schwarz, we have

$$L(g) \le \sqrt{\int_1^\infty \frac{1}{u^2}\, du \int_1^\infty g(u)^2\, du} \le \sqrt{\int_0^\infty h} \le \sqrt{2},$$

so $\kappa \le \sqrt{2}$.

Note that the supremum is achieved for $\int_0^\infty h = 2$. For if $\int_0^\infty h < 2$, then we can always increase g by a small amount while keeping it less than 1 and decreasing, while $\int h$ is increased by a prescribed amount – just take $g_1 = k \circ g$ where $k : (0, 1) \to (0, 1)$ is increasing and $k(x) > x$. With $k(x) - x$ sufficiently small, $\int h_1 \le 2$ while $L(g_1) > L(g)$.

Further, we may take the supremum over g for which g is continuously differentiable and strictly decreasing, since they can approximate functions in G arbitrarily closely.

Now, for $L(g)$ to be finite (i.e. $> -\infty$) we need $\int_0^1 \frac{1-g(u)}{u} du$ to converge. For $x \in (0, 1)$,

$$\int_x^{\sqrt{x}} \frac{1 - g(u)}{u}\, du \ge (1 - g(x)) \int_x^{\sqrt{x}} \frac{1}{u}\, du = \frac{1}{2}(1 - g(x)) \log \frac{1}{x}.$$

The LHS tends to 0 as $x \to 0^+$, so we must have

$$(1 - g(x)) \log x \to 0 \qquad \text{as } x \to 0^+.$$

In particular, $g(x) \to 1$ as $x \to 0^+$ (so $h(x) \to \infty$ as $x \to 0^+$). Also, as in Theorem 2.3, $xh(x) \to 0$ as $x \to \infty$. Now, with $g = s \circ h$,

$$\int_1^\infty \frac{g(u)}{u}\, du = [g(u) \log u]_1^\infty - \int_1^\infty s'(h(u))h'(u) \log u\, du = \int_0^{h(1)} s'(y) \log l(y)\, dy,$$

where $l = h^{-1}$ is the inverse function of h. Also,

$$\int_0^1 \frac{1 - g(u)}{u}\, du = [(1 - g(u)) \log u]_0^1 + \int_0^1 s'(h(u))h'(u) \log u\, du = -\int_{h(1)}^\infty s'(y) \log l(y)\, dy.$$

Hence $L(g) = \int_0^\infty s' \log l$ and $\int_0^\infty l = 2$.

Now, using Jensen's inequality $\int \log f d\mu \le \log(\int f d\mu)$ for μ a probability measure ([11], p.62), we have

$$\int_0^\infty s' \log(l/s') = \int_0^\infty \log(l/s')\, ds \le \log\Big(\int_0^\infty l/s'\, ds\Big) = \log\Big(\int_0^\infty l\Big) = \log 2. \tag{3.3}$$

Hence

$$\int_0^\infty s' \log l \le \log 2 + \int_0^\infty s' \log s' = \log 2 + \int_0^1 \log\Big(\frac{1 - u^2}{2u}\Big)\, du = 2 \log 2 - 1,$$

after some calculation.

The proof of the upper bound leads to the optimum choice for g and the lower bound. We note that we have equality in (3.3) if l/s' is constant; i.e. $l(x) = cs'(x)$ for some constant $c > 0$ — chosen so that $\int_0^\infty l = 2$ (i.e. we take $c = 2$). Thus, actually $\kappa = 2\log 2 - 1$ and the supremum is achieved for the function g_0, where

$$g_0(x) = \sqrt{1 - \frac{2}{1 + \sqrt{1 + (\frac{2}{x})^2}}}.$$

In fact, we show that we get the required lower bound by just considering a_n completely multiplicative. To this end we use Corollary 2.5, and define a_p by:

$$a_p = g_0\Big(\frac{p}{P}\Big),$$

where $P = \log T \log\log T$. As such, by the same methods as before, we have $\|a\| = T^{1+o(1)}$. Let $a > 0$ and $P = \log T \log\log T$. By Corollary 2.5

$$\frac{\|\varphi_1 a\|}{\|a\|} \geq \prod_p \frac{1}{1 - \frac{a_p}{p}} = \prod_{p \leq aP} \frac{1}{1 - \frac{1}{p}} \prod_{p \leq aP} \frac{1}{1 + \frac{1-a_p}{p-1}} \prod_{p > aP} \frac{1}{1 - \frac{a_p}{p}}. \tag{3.4}$$

Using Merten's Theorem, the first product on the right is $e^\gamma(\log aP + o(1))$, while the second product is greater than

$$\exp\Big\{-\sum_{p \leq aP} \frac{1 - a_p}{p - 1}\Big\} \geq 1 - 2\sum_{p \leq aP} \frac{1 - g_0(p/P)}{p}.$$

The sum is asymptotic to $\frac{a}{\log P}\int_0^a \frac{1-g_0(u)}{u}du < \frac{\varepsilon}{\log P}$, for any given $\varepsilon > 0$, for sufficiently small a. The third product in (3.4) is greater than

$$\exp\Big\{\sum_{p > aP} \frac{a_p}{p}\Big\} = \exp\Big\{\frac{(1 + o(1))}{\log P}\int_a^\infty \frac{g_0(u)}{u}\,du\Big\}$$

by (2.9). Thus

$$\frac{\|\varphi_1 a\|}{\|a\|} \geq e^\gamma\Big(\log P + \int_a^\infty \frac{g_0(u)}{u}\,du + \log a - \varepsilon\Big) \geq e^\gamma\Big(\log P + L(g_0) - \varepsilon\Big)$$

for a sufficiently small. As $L(g_0) = 2\log 2 - 1$ and ε arbitrary, this gives the required lower bound. □

Lower bounds for φ_α and some further speculations
We can study lower bounds of φ_α via the function

$$m_\alpha(T) = \inf_{\substack{a \,\in\, \mathcal{M}^2 \\ \|a\| \,=\, T}} \frac{\|\varphi_\alpha a\|}{\|a\|}.$$

Using very similar techniques, one obtains analogous results to Theorem 3.1:

$$\frac{1}{m_1(T)} = \frac{6e^\gamma}{\pi^2}(\log\log T + \log\log\log T + 2\log 2 - 1 + o(1))$$

and

$$\log\frac{1}{m_\alpha(T)} \sim \frac{B(\frac{1}{\alpha}, 1 - \frac{1}{2\alpha})^\alpha(\log T)^{1-\alpha}}{(1-\alpha)2^\alpha(\log\log T)^\alpha} \qquad \text{for } \tfrac{1}{2} < \alpha < 1.$$

We see that $m_\alpha(T)$ corresponds closely to the conjectured minimal order of $|\zeta(\alpha+iT)|$ (see [3] and [9]). We omit the proofs, but just point out that for an upper bound (for $1/m_\alpha(T)$) we use

$$\frac{\|a\|}{\|\varphi_\alpha a\|} \le \prod_p \left(1 + \frac{\gamma_p}{p^\alpha}\right),$$

which can be obtained in much the same way as (2.5). For the lower bound, we choose a_p as -1 times the choice in Theorem 3.1 and use Corollary 2.5.

The above formulae suggest that the supremum (respectively infimum) of $\|\varphi_\alpha a\|/\|a\|$ with $a \in \mathcal{M}^2$ and $\|a\| = T$ are close to the supremum (resp. infimum) of $|\zeta_\alpha|$ on $[1,T]$. One could therefore speculate further that there is a close connection between $\|\varphi_\alpha a\|/\|a\|$ (for such a) and $|\zeta(\alpha + iT)|$, and hence between $Z_\alpha(T)$ and $M_\alpha(T)$. Recent papers by Gonek [4] and Gonek and Keating [5] suggest this may be possible, or at least that M_α is a lower bound for Z_α. On the Riemann Hypothesis, it was shown in [4] (Theorem 3.5) that $\zeta(s)$ may be approximated for $\sigma > \frac{1}{2}$ up to height T by the truncated Euler product

$$\prod_{p\le P} \frac{1}{1-p^{-s}} \qquad \text{for } P \ll T.$$

Thus one might expect that, with $a \in \mathcal{M}_c^2+$ maximizing $\frac{\|\varphi_\alpha a\|}{\|a\|}$ subject to $\|a\| = T$, and $A(s) = \prod_{p\le P} \frac{1}{1-a_p p^{-s}}$ (with $P \ll T$),

$$\int_{-T}^{T} |\zeta(\alpha - it)|^2 |A(it)|^2 \, dt \sim \int_{-T}^{T} \prod_{p\le P} \Big|(1 - \frac{p^{it}}{p^\alpha})(1 - a_p p^{it})\Big|^{-2} dt = \int_{-T}^{T} \prod_{p\le P} |B_p(it)|^2 \, dt$$

where $B_p(s) = \sum_{k\ge 0} b_{p^k} p^{-ks}$. The heuristics of Gonek and Keating now suggests this is asymptotic to

$$2T \prod_{p\le P} \sum_{k\ge 0} b_{p^k}^2 \sim 2T\|\varphi_\alpha a\|^2$$

if $P \succ \log T \log\log T$ (for the last step). Thus it would follow that

$$Z_\alpha(T)^2 \ge \frac{\int_{-T}^{T} |\zeta(\alpha - it)|^2 |A(it)|^2 \, dt}{\int_{-T}^{T} |A(it)|^2 \, dt} \sim \frac{2T\|\varphi_\alpha a\|^2}{2T\|a\|^2} \sim M_\alpha(T)^2$$

and hence $Z_\alpha(T) \gtrsim M_\alpha(T)$.

As mentioned before, this would contradict Lamzouri's suggestion (that $\log Z_\alpha(T) \sim C(\alpha)(\log T)^{1-\alpha}(\log\log T)^{-\alpha}$) since $C(\alpha) < c(\alpha)$ (notation from Theorem 2.3) for α sufficiently close to $\frac{1}{2}$ at least. It is unclear to the author which possibility is more likely.

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Appendix
Here we show that if $f \notin l^2$, we cannot hope to 'capture' φ_f by considering the mapping on some non-trivial subset of l^2.

Proposition A1
Suppose $\sum_p |f(p)|^2$ diverges, where p ranges over the primes. Then $\varphi_f a \in l^2$ for $a \in l^2$ if and only if $a = 0$.

Proof. Suppose there exists $a \in l^2$ with $a \neq 0$ such that $\varphi_f a \in l^2$. Let a_m be the first non-zero coordinate for a. Let $b = (b_n) = \varphi_f a \in l^2$. Consider b_{pm} for p prime such that $p \not| m$. We have

$$b_{pm} = \sum_{d|pm} f(d) a_{pm/d} = a_m f(p) + k(p),$$

where $k(p) = \sum_{d|m} f(d) a_{pm/d}$. Since

$$\sum_p |k(p)|^2 \leq \sum_p \left(\sum_{d|m} |f(d)|^2 \sum_{d|m} |a_{pm/d}|^2 \right) \leq A \sum_{d|m} \sum_p |a_{pm/d}|^2 < \infty,$$

and $\sum_p |b_{pm}|^2$ converges, we must have

$$|a_m|^2 \sum_p |f(p)|^2 < \infty.$$

This is a contradiction.

□