‘Quasi’-norm of an arithmetical convolution operator and the order of the Riemann zeta function

by

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Abstract
In this paper we study Dirichlet convolution with a given arithmetical function $f$ as a linear mapping $\varphi_f$ that sends a sequence $(a_n)$ to $(b_n)$ where $b_n = \sum_{d|n} f(d) a_{n/d}$. We investigate when this is a bounded operator on $l^2$ and find the operator norm. Of particular interest is the case $f(n) = n^{-\alpha}$ for its connection to the Riemann zeta function on the line $\Re s = \alpha$. For $\alpha > 1$, $\varphi_f$ is bounded with $\|\varphi_f\| = \zeta(\alpha)$. For the unbounded case, we show that $\varphi_f : M^2 \to M^2$ where $M^2$ is the subset of $l^2$ of multiplicative sequences, for many $f \in M^2$. Consequently, we study the ‘quasi’-norm

$$\sup_{\|a\| \leq T} \frac{\|\varphi_f a\|}{\|a\|}$$

for large $T$, which measures the ‘size’ of $\varphi_f$ on $M^2$. For the $f(n) = n^{-\alpha}$ case, we show this quasi-norm has a striking resemblance to the conjectured maximal order of $|\zeta(\alpha + iT)|$ for $\alpha > \frac{1}{2}$.

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Introduction
Given an arithmetical function $f(n)$, the mapping $\varphi_f$ sends $(a_n)_{n \in \mathbb{N}}$ to $(b_n)_{n \in \mathbb{N}}$, where

$$b_n = \sum_{d|n} f(d) a_{n/d}.$$  \hfill (0.1)

Writing $a = (a_n)$, $\varphi_f$ maps $a$ to $f * a$ where $*$ is Dirichlet convolution. This is a ‘matrix’ mapping, where the matrix, say $M(f)$, is of ‘multiplicative Toeplitz’ type; that is,

$$M(f) = (a_{i,j})_{i,j \geq 1}$$

where $a_{i,j} = f(i/j)$ and $f$ is supported on the natural numbers (see, for example, [6], [7]).

Toeplitz matrices (whose $ij\text{th}$-entry is a function of $i - j$) are most usefully studied in terms of a ‘symbol’ (the function whose Fourier coefficients make up the matrix). Analogously, the Multiplicative Toeplitz matrix $M(f)$ has as symbol the Dirichlet series

$$\sum_{n=1}^{\infty} f(n) n^i.$$

Our particular interest is naturally the case $f(n) = n^{-\alpha}$ when the symbol is $\zeta(\alpha - it)$. We are especially interested how and to what extent properties of the mapping relate to properties of the symbol for $\alpha \leq 1$.

These type of mappings were considered by various authors (for example Wintner [15]) and most notably Toeplitz [13], [14] (although somewhat indirectly, through his investigations of so-called

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In essence, Toeplitz proved that $\varphi_f : l^2 \to l^2$ is bounded if and only if $\sum_{n=1}^{\infty} f(n)n^{-s}$ is defined and bounded for all $\Re s > 0$. In particular, if $f(n) \geq 0$, then $\varphi_f$ is bounded on $l^2$ if and only if $f \in l^1$; furthermore, the operator norm is $\|\varphi_f\| = \|f\|_1$. We prove this in Theorem 1.1 following Toeplitz’s original idea. For example, for $f(n) = n^{-\alpha}$, $\varphi_f$ is bounded on $l^2$ for $\alpha > 1$ with operator norm $\zeta(\alpha)$. In this special case, the mapping was studied in [7] for $\alpha \leq 1$ when it is unbounded on $l^2$ by estimating the behaviour of the quantity $\Phi_f(N) = \sup_{\|a\|_2=1} \left( \sum_{n=1}^{N} \|a_n\|^2 \right)^{1/2}$ for large $N$. Approximate formulas for $\Phi_f(N)$ were obtained and it was shown that, for $\frac{1}{2} < \alpha \leq 1$, $\Phi_f(N)$ is a lower bound for $\max_{1 \leq t \leq T} |\zeta(\alpha + it)|$ with $N = T^\lambda$ (some $\lambda > 0$ depending on $\alpha$ only). In this way, it was proven that the measure of the set

$$\left\{ t \in [1, T] : |\zeta(1 + it)| \geq \exp \left( -a \frac{\log T}{\log \log T} \right) \right\}$$

is at least $T \exp \left\{ -a \frac{\log T}{\log \log T} \right\}$ (some $a > 0$) for $A$ sufficiently large, while for $\frac{1}{2} < \alpha < 1$ one has

$$\max_{1 \leq t \leq T} |\zeta(\alpha + it)| \geq \exp \left\{ c \frac{(\log T)^{1-\alpha}}{\log \log T} \right\}$$

for some $c > 0$ depending on $\alpha$ only, as well providing an estimate for how often $|\zeta(\alpha + iT)|$ is as large as the right-hand side above. The method is akin to Soundararajan’s ‘resonance’ method and incidentally shows the limitation of this approach for $\alpha > \frac{1}{2}$ since $|\zeta(\alpha + iT)|$ is known to be of larger order.

In this paper we study the unbounded case in a different way, by restricting the domain. Thus in section 2, we show that for many multiplicative $f$, in particular for $f$ completely multiplicative, $\varphi_f(M^2) \subset M^2$, even though $\varphi_f(l^2) \not\subset l^2$. Here $M^2$ is the set of multiplicative functions in $l^2$. As a result we consider, for such $f$, the ‘quasi’-norm

$$M_f(T) = \sup_{a \in M^2} \frac{\|\varphi_f(a)\|}{\|a\|}$$

and obtain approximate formulae for large $T$ (here $\|\cdot\|$ is the usual $l^2$-norm). We find that for the particular case $f(n) = n^{-\alpha}$ ($\alpha > \frac{1}{2}$), this quasi-norm has a striking similarity to the conjectured maximal order of $|\zeta(\alpha + iT)|$. For example, with $\alpha = 1$ (i.e. $f(n) = 1/n$) we prove

$$M_f(T) = e^\gamma (\log \log T + \log \log \log T + 2 \log 2 - 1) + o(1), \quad (0.2)$$

while for $\frac{1}{2} < \alpha < 1$

$$\log M_f(T) \sim \frac{B(\frac{1}{2}, 1 - \frac{1}{2\alpha})}{(1 - \alpha)^2 \alpha} (\log T)^{1-\alpha} \frac{\log \log T}{(\log \log T)^{\alpha}},$$

where $B(x, y)$ is the Beta function. Writing $Z_\alpha(T) = \max_{1 \leq t \leq T} |\zeta(\alpha + iT)|$, Granville and Soundararajan [3] proved that $Z_1(T)$ is at least as large as (0.2) minus a log log log log $T$ term for some arbitrarily large $T$ and they conjectured that it equals (0.2) (possibly with a different constant term). For $\frac{1}{2} < \alpha < 1$, Montgomery [9] found

$$\log Z_\alpha(T) \geq \frac{\sqrt{\alpha - 1/2}}{20} (\log T)^{1-\alpha} \frac{(\log \log T)^{\alpha}}{(\log \log T)^{\alpha}}$$

and, using a heuristic argument, conjectured that this is (apart from the constant) the correct order of $\log Z_\alpha(T)$. Further, in a recent paper (see [8]), Lamzouri suggests $\log Z_\alpha(T) \sim C(\alpha)(\log T)^{1-\alpha}(\log \log T)^{-\alpha}$ with some specific constant $C(\alpha)$ (see also the remark after Theorem 3.1).
Similarly one can study the quantity

\[ m_f(T) = \inf\limits_{a \in M, \|a\| = 1} \frac{\|\varphi f a\|}{\|a\|}, \]

With \( f(n) = n^{-\alpha} \) this is shown to behave like the known and conjectured minimal order of \(|(\alpha + iT)|\) for \( \alpha > \frac{1}{2} \). It should be stressed here that, unlike the case of \( \Phi_f(N) \) which was shown to be a lower bound for \( Z_\alpha(T) \) in [7], we have not proved any connection between \( \zeta(\alpha + iT) \) and \( M_f(T) \). Even to show \( M_f(T) \) is a lower bound would be very interesting.

Our results, though motivated by the special case \( f(n) = n^{-\alpha} \), extend naturally to completely multiplicative \( f \) for which \( f \| \) is regularly varying (see section 2 for the definition).

**Addendum.** I would like to thank the anonymous referee for some useful comments and for pointing out a recent paper by Aistleitner and Seip [1]. They deal with an optimization problem which is different yet curiously similar. The function \( \exp\{c_n (\log T)^{1-\alpha} (\log \log T)^{-\alpha}\} \) appears in the same way, although their \( c_n \) is expected to remain bounded as \( \alpha \to \frac{1}{2} \). It would be interesting to investigate any links further.

1. **Bounded operators**

**Notation:** Let \( l^1 \) and \( l^2 \) denote the usual spaces of sequences \((a_n)_{n \in \mathbb{N}}\), with norms \( \|a\|_1 = \sum |a_n| \) and \( \|a\|_2 = (\sum |a_n|^2)^{1/2} \) respectively. After section 1 we shall, for ease of notation, just write \( \| \cdot \| \) for \( \| \cdot \|_2 \) since it is the norm we will use.

A linear mapping \( \varphi : l^2 \to l^2 \) is bounded if there exists \( C > 0 \) such that \( \|\varphi x\|_2 \leq C\|x\|_2 \) for all \( x \in l^2 \). As such, we define the operator norm by

\[ \|\varphi\| = \sup
\begin{align*}
\|x\|_2 = 1
\|\varphi x\|_2.
\end{align*} \]

We shall assume from now on that \( f(n) \geq 0 \) for all \( n \in \mathbb{N} \). We are particularly interested in the case where \( \varphi f \) acts on \( l^2 \). Define the function

\[ \Phi_f(N) = \sup_{\|a\|_2 = 1} \sqrt{\sum_{n \leq N} |b_n|^2}, \]

where \( b_n \) is given in terms of \( a_n \) by (0.1). Note that the supremum will occur when \( a_n \geq 0 \) for all \( n \) and when \( \sum_{n \leq N} a_n^2 = 1 \).

Suppose now that \( f \in l^1 \); i.e. \( \|f\|_1 = \sum_{n=1}^\infty f(n) < \infty \). Then

\[ |b_n|^2 = \left| \sum_{d|n} \sqrt{f(d)} \cdot \sqrt{f(d)} a_{n/d} \right|^2 \leq \sum_{d|n} f(d) \sum_{d|n} f(d) |a_{n/d}|^2 \leq \|f\|_1 \sum_{d|n} f(d) |a_{n/d}|^2. \]

Hence

\[ \sum_{n \leq N} |b_n|^2 \leq \|f\|_1 \sum_{n \leq N} \sum_{d|n} f(d) |a_{n/d}|^2 \leq \|f\|_1 \sum_{d \leq N} f(d) \sum_{n \leq N/d} |a_n|^2 \leq \|f\|_1 \|a\|_2^2. \]

Thus

\[ \Phi_f(N) \leq \|f\|_1. \]

(1.1)

Following Toeplitz [14], we show that this inequality is sharp.

**Theorem 1.1**

Let \( f \) be a non-negative arithmetical function and \( f \in l^1 \). Then \( \Phi_f(N) \to \|f\|_1 \) as \( N \to \infty \). Thus
\( \varphi_f : l^2 \to l^2 \) is bounded if and only if \( f \in l^1 \), in which case \( \| \varphi_f \| = \| f \|_1 \).

**Proof.** After \((1.1)\), and since \( \Phi_f(N) \) increases with \( N \), we need only provide a lower bound for an infinite sequence of \( N \)s. Let \( a_n = d(N)^{-\frac{1}{2}} \) for \( n \mid N \) and zero otherwise (\( N \) to be chosen later), where \( d(\cdot) \) is the divisor function. Thus \( a_1^2 + \ldots + a_N^2 = 1 \) and

\[
\Phi_f(N) \geq \sum_{n \leq N} a_n b_n = \frac{1}{d(N)} \sum_{n \mid N} \sum_{d \mid n} f(d) = \frac{1}{d(N)} \sum_{d \mid N} f(d) \left( \frac{N}{d} \right),
\]

say. We choose \( N \) such that it has all divisors \( d \) up to some (large) number, and that \( \frac{d(N/d)}{d(N)} \) is close to 1 for each such divisor \( d \) of \( N \). Take \( N \) of the form

\[
N = \prod_{p \leq P} p^{\alpha_p} \quad \text{where } \alpha_p = \lfloor \log P \rfloor.
\]

Thus every natural number up to \( P \) is a divisor of \( N \). For a divisor \( d = \prod_{p \leq P} p^{\beta_p} \) of \( N \), we have

\[
\frac{d(N/d)}{d(N)} = \prod_{p \leq P} \left(1 - \frac{\beta_p}{\alpha_p + 1}\right).
\]

If we take \( d \leq \sqrt{\log P} \), then \( p^{\beta_p} \leq \sqrt{\log P} \) for every prime divisor \( p \) of \( d \). Hence, for such \( p \), \( \beta_p \leq \frac{\log \log P}{2 \log p} \) and \( \beta_p = 0 \) if \( p > \sqrt{\log P} \). Thus for \( d \leq \sqrt{\log P} \),

\[
\frac{d(N/d)}{d(N)} = \prod_{p \leq \sqrt{\log P}} \left(1 - \frac{\beta_p}{\alpha_p + 1}\right) \geq \prod_{p \leq \sqrt{\log P}} \left(1 - \frac{1}{2 \log P} \log \log P \right) = \left(1 - \frac{1}{2 \log P} \right)^{\pi(\sqrt{\log P})},
\]

where \( \pi(x) \) is the number of primes up to \( x \). Since \( \pi(x) = O\left( \frac{x}{\log x} \right) \), it follows that for all \( P \) sufficiently large, the expression in \((1.2)\) is at least

\[
\left(1 - \frac{A}{\sqrt{\log P}} \right) \sum_{d \leq \sqrt{\log P}} f(d)
\]

for some constant \( A \). The sum can be made as close to \( \| f \|_1 \) as we please by increasing \( P \).

\( \square \)

2. **Unbounded operators on \( l^2 \)**

Now we investigate when \( \varphi_f \) is unbounded on \( l^2 \) (i.e. \( f \notin l^1 \)). In a similar generalisation of Theorem 1.1 of \([7]\), one can readily show that both \( \varphi_f : l^1 \to l^2 \) and \( \varphi_f : l^2 \to l^\infty \) are bounded if and only if \( f \in l^2 \), with \( \| \varphi_f \| = \| f \|_2 \) in either case. So here we shall assume that \( f \in l^2 \setminus l^1 \). In the appendix we see that, for all cases of interest at least, if \( f \notin l^2 \), then \( \varphi_f a \notin l^2 \) for all \( a \) except \( a = 0 \).

For unbounded operators, there are different ways of measuring the ‘unboundedness’. One way, which was done in \([7]\) for the case \( f(n) = \frac{n^{-\alpha}}{n} \), is to restrict the range by looking at a restricted norm; i.e. by considering \( \Phi_f(N) \) for given \( N \). Another way is to restrict the domain to a set \( S \) say, such that \( \varphi_f(S) \subset l^2 \) and to consider the size of

\[
\sup_{a \in S, \| a \| = N} \frac{\| \varphi_f a \|}{\| a \|} \quad \text{for large } N.
\]

For \( f \) completely multiplicative one is naturally led to consider \( S = M^2 \) — the set of square summable multiplicative functions. It is also natural to consider regularly varying functions.

**Regular Variation.** A function \( \ell : [A, \infty) \to \mathbb{R} \) is regularly varying of index \( \rho \) if it is measurable and

\[
\ell(\lambda x) \sim \lambda^\rho \ell(x) \quad \text{as } x \to \infty \text{ for every } \lambda > 0.
\]
(see [2] for a detailed treatise on the subject). For example, $x^\rho (\log x)^\tau$ is regularly-varying of index $\rho$ for any $\tau$. The Uniform Convergence Theorem says that the above asymptotic formula is automatically uniform for $\lambda$ in compact subsets of $[0, \infty)$. Note that every regularly varying function of non-zero index is asymptotic to one which is strictly monotonic and continuous. We shall make use of Karumata’s Theorem: for $\ell$ regularly varying of index $\rho$,

$$
\int x^\ell \sim \frac{x^\ell(x)}{\rho + 1} \quad \text{if} \ \rho > -1,
$$

$$
\int x^\ell \sim -\frac{x^\ell(x)}{\rho + 1} \quad \text{if} \ \rho < -1,
$$

while if $\rho = -1$, $\int x^\ell$ is slowly varying (regularly varying with index 0) and $\int^\ell x \sim x^\ell(x)$.

**Notation.** Let $M^2$ and $\mathcal{M}^2_2$ denote the subsets of $l^2$ of multiplicative and completely multiplicative functions respectively. Further, write $M^2_2$ for the non-negative members of $M^2$ and similarly for $M^2_2^+$.

**2.1 The size of $\|\varphi_f\|$ on $M^2$**

Now we consider $\varphi_f$ on the subset $M^2$ of multiplicative functions in $l^2$. We suppose, as in section 2, that $f \in l^2 \setminus l^1$ so that $\varphi_f$ is unbounded. This implies there exist $a \in l^2$ such that $\varphi_f(a) \notin l^2$ (by the closed graph theorem). However, if $f$ is multiplicative then, as we shall see, $\varphi_f(M^2) \subset l^2$ in many cases (and hence $\varphi_f(M^2) \subset M^2$).

**Lemma 2.1**

Let $f, g \in M^2$ be non-negative. Then $f * g \in M^2$ if and only if

$$
\sum_p \sum_{m,n \geq 1} \sum_{k=0}^{\infty} f(p^m)g(p^n)f(p^{m+k})g(p^{n+k}) \text{ converges.} \tag{2.1}
$$

**Proof.** Let $h = f * g$. Since $h$ is multiplicative,

$$
\sum_{n=1}^{\infty} h(n^2) < \infty \iff \sum_p \sum_{k \geq 1} h(p^k)^2 < \infty.
$$

Let $k \geq 1$ and $p$ prime. Then

$$
h(p^k) = \sum_{r=0}^{k-1} f(p^r)g(p^{k-r}) = f(p^k) + g(p^k) + \sum_{r=1}^{k-1} f(p^r)g(p^{k-r}).
$$

Using the inequality $a^2 + b^2 + c^2 \leq (a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ we have

$$
\left( \sum_{r=1}^{k-1} f(p^r)g(p^{k-r}) \right)^2 \leq h(p^k)^2 \leq 3 \sum_{r=1}^{k-1} f(p^r)^2 + 3g(p^k)^2 + 3 \left( \sum_{r=1}^{k-1} f(p^r)g(p^{k-r}) \right)^2.
$$

Since $\sum_{p,k \geq 1} f(p^k)^2$ and $\sum_{p,k \geq 1} g(p^k)^2$ converge we find that $\sum_{p,k \geq 1} h(p^k)^2$ converges if and only if

$$
\sum_p \sum_{k \geq 2} \left( \sum_{r=1}^{k-1} f(p^r)g(p^{k-r}) \right)^2 \text{ converges.}
$$

But

$$
\sum_{k=2}^{\infty} \left( \sum_{r=1}^{k-1} f(p^r)g(p^{k-r}) \right)^2 = \sum_{k=1}^{\infty} \sum_{1 \leq r,s \leq k} f(p^r)f(p^s)g(p^{k-r+s})g(p^{k-s+r}) \tag{2.2}
$$

$$
\leq 2 \sum_{k=1}^{\infty} \sum_{s=1}^{k} \sum_{r=1}^{s} f(p^r)f(p^s)g(p^{k-r+s})g(p^{k-s+r})
$$

$$
= 2 \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} \sum_{s=0}^{\infty} f(p^r)f(p^{r+s})g(p^{k+s})g(p^k).
$$
On the other hand, the RHS of (2.2) is greater than
\[ \sum_{k=1}^{\infty} \sum_{s=1}^{k} \sum_{r=1}^{s} f(p^r)f(p^s)g(p^{k-r+1})g(p^{k-s+1}). \]
Hence \( h \in \mathcal{M}^2 \) if and only if
\[ \sum_{p \in \mathbb{N}} \sum_{m,n \geq 1} \sum_{k=0}^{\infty} f(p^m)g(p^n)f(p^{m+k})g(p^{n+k}) \]
converges.

Let \( \mathcal{M}_0^2 \) denote the set of \( \mathcal{M}^2 \) functions \( f \) for which \( f * g \in \mathcal{M}^2 \) whenever \( g \in \mathcal{M}^2 \); that is,
\[ \mathcal{M}_0^2 = \{ f \in \mathcal{M}^2 : g \in \mathcal{M}^2 \implies f * g \in \mathcal{M}^2 \}. \]
Thus for \( f \in \mathcal{M}_0^2, \varphi_f(\mathcal{M}^2) \subseteq \mathcal{M}^2 \). We shall see that it may happen that \( f, g \in \mathcal{M}^2 \) but \( f * g \notin \mathcal{M}^2 \). So \( \mathcal{M}_0^2 \neq \mathcal{M}^2 \). The following gives a criterion for multiplicative functions to be! in \( \mathcal{M}_0^2 \).

**Proposition 2.2**
Let \( f \in \mathcal{M}^2 \) be such that \( \sum_{k=1}^{\infty} |f(p^k)| \) converges for every prime \( p \) and that \( \sum_{k=1}^{\infty} |f(p^k)| \leq A \) for some constant \( A \) independent of \( p \). Then \( f \in \mathcal{M}_0^2 \).

On the other hand, if \( f \in \mathcal{M}^2 \) with \( f \geq 0 \) and for some prime \( p_0 \), \( f(p_0^k) \) decreases with \( k \) and \( \sum_{k=1}^{\infty} f(p_0^k) \) diverges, then \( f \notin \mathcal{M}_0^2 \).

**Proof.** Without loss of generality we can take \( f \geq 0 \). Let \( g \in \mathcal{M}^2 \) (again w.l.o.g. \( g \geq 0 \)) with \( \alpha_p = \sum_{k=1}^{\infty} g(p^k)^2 \). Thus \( \sum_{p} \alpha_p \) converges. By the Cauchy-Schwarz inequality,
\[
\left( \sum_{n=1}^{\infty} g(p^n)g(p^{n+k}) \right)^2 \leq \sum_{n=1}^{\infty} g(p^n)^2 \sum_{n=1}^{\infty} g(p^{n+k})^2 \leq \alpha_p \alpha_p = \alpha_p^2.
\]
Thus by Lemma 2.1, \( f * g \in \mathcal{M}^2 \) if
\[
\sum_{p} \alpha_p \sum_{m=1}^{\infty} f(p^m) \sum_{k=0}^{\infty} f(p^{m+k}) \]
converges.

By assumption, the inner sum over \( k \) is bounded by a constant (independent of \( p \)), and hence so is the sum over \( m \). This implies the convergence of the above. Hence \( f * g \in \mathcal{M}^2 \).

Now suppose \( \sum_{k=1}^{\infty} f(p_0^k) \) diverges for some prime \( p_0 \). Then with \( g \in \mathcal{M}^2 \) and \( g(p_0^k) \) decreasing (to zero) we have
\[
(f * g)(p_0^k) = \sum_{r=0}^{k} f(p_0^r)g(p_0^{k-r}) \geq g(p_0^k) \sum_{r=0}^{k} f(p_0^r) = g(p_0^k) c_k,
\]
where \( c_k \not\to \infty \). Thus \( \sum_{k} (f * g)(p_0^k)^2 \geq \sum_{k} g(p_0^k)^2 c_k^2 \). But we can always choose \( g(p_0^k) \) decreasing so that \( \sum_k g(p_0^k)^2 \) converges while, for the given sequence \( c_k \), \( \sum_k g(p_0^k)^2 c_k^2 \) diverges. (Choose \( g(p_0^k)^2 = \frac{1}{\alpha_{p_0} - \frac{1}{c_k}} \))
Thus \( f * g \notin \mathcal{M}^2 \); i.e. \( f \notin \mathcal{M}_0^2 \).
Thus, in particular, $M^2_c \subset M^2_0$. For $f \in M^2_c$ if and only if $|f(p)| < 1$ for all primes $p$ and

$$
\sum_p |f(p)|^2 < \infty.
$$

Thus

$$
\sum_{k=1}^{\infty} |f(p^k)| = \frac{|f(p)|}{1 - |f(p)|} \leq A,
$$

independent of $p$ (since $f(p) \to 0$).

The “quasi-norm” $M_f(T)$

Let $f \in M^2_0$. From above we see that $\varphi_f(M^2) \subset M^2$ but, typically, $\varphi_f$ is not ‘bounded’ on $M^2$ (if $f \not\in l^1$) in the sense that $\|\varphi_f a\|/\|a\|$ is not bounded by a constant for all $a \in M^2$. It therefore makes sense to define, for $T \geq 1$,

$$
M_f(T) = \sup \frac{\|\varphi_f a\|}{\|a\|}.
$$

We aim to find the behaviour of $M_f(T)$ for large $T$.

We shall consider $f$ completely multiplicative and such that $f|_P$ is regularly varying of index $-\alpha$ with $\alpha > 1/2$ in the sense that there exists a regularly varying function $\tilde{f}$ (of index $-\alpha$) with $\tilde{f}(p) = f(p)$ for every prime $p$.

Our main result here is the following:

**Theorem 2.3**

Let $f \in M^2_c$, such that $f \geq 0$ and $f|_P$ is regularly varying of index $-\alpha$ where $\alpha \in (\frac{1}{2}, 1)$. Then

$$
\log M_f(T) \sim c(\alpha) \tilde{f}(\log T \log \log T) \log T
$$

where $\tilde{f}$ is any regularly varying extension of $f|_P$ and

$$
c(\alpha) = \frac{B(\frac{1}{\alpha}, 1 - \frac{1}{2\alpha})}{(1 - \alpha)^2}.
$$

and $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ is the Beta function.

For the proof, we obtain upper and lower bounds for $\log M_f(T)$ which are asymptotic to each other. For the lower bounds, we require a formula for $\|\varphi_f a\|$ when $a \in M^2_c$. This follows from the following rather elegant formula:

**Lemma 2.4**

For $f, g \in M^2_c$,

$$
\frac{\|f \ast g\|}{\|f\| \|g\|} = \frac{|\langle f, g \rangle|}{\|fg\|},
$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product for $l^2$.

**Proof.** We have

$$
\begin{align*}
\|f \ast g\|^2 &= \sum_{n=1}^{\infty} |(f \ast g)(n)|^2 = \sum_{n=1}^{\infty} \sum_{c,d|n} f(c)\overline{f(d)}g\left(\frac{n}{c}\right)\overline{g\left(\frac{n}{d}\right)} \\
&= \sum_{c,d \geq 1} f(c)\overline{f(d)} \sum_{m=1}^{\infty} g\left(\frac{m|c,d|}{c}\right)\overline{g\left(\frac{m|c,d|}{d}\right)} \\
&= \sum_{m=1}^{\infty} |g(m)|^2 \sum_{c,d \geq 1} f(c)\overline{f(d)}g\left(\frac{d}{(c,d)}\right)\overline{g\left(\frac{c}{(c,d)}\right)}.
\end{align*}
$$

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Collecting those terms for which \((c, d) = k\), writing \(c = km\), \(d = kn\), and using complete multiplicativity of \(f\)
\[
\left(\frac{\|f \ast g\|}{\|g\|}\right)^2 = \sum_{k=1}^{\infty} \left|f(k)\right|^2 \sum_{m, n \geq 1, (m, n) = 1} f(m) f(n) g(m) g(n).
\]

But
\[
|f, g\|^2 = \sum_{m, n \geq 1} f(m) f(n) g(m) g(n) = \sum_{d=1}^{\infty} \left|f(d) g(d)\right|^2 \sum_{m, n \geq 1, (m, n) = 1} f(m) f(n) g(m) g(n),
\]

so the result follows. \(\square\)

Thus for \(f, a \in M_c^2\),
\[
\frac{\|\varphi f a\|}{\|a\|} = \frac{\|f\| \cdot \sum_{n=1}^{\infty} |f(n) a_n|}{\left(\sum_{n=1}^{\infty} |f(n) a_n|^2\right)^{1/2}}.
\]
Since \(|a_n| \leq 1\), as a corollary we have:

**Corollary 2.5**
For \(f, a \in M_c^2\),
\[
\sum_{n=1}^{\infty} f(n) a_n \leq \frac{\|\varphi f a\|}{\|a\|} \leq \|f\| \sum_{n=1}^{\infty} |f(n) a_n|.
\]

Note that by complete multiplicativity,
\[
\sum_{n=1}^{\infty} f(n) a_n = \prod_p \frac{1}{1 - f(p) a_p} = \prod_p \exp\{f(p) a_p + O(|f(p) a_p|^2)\},
\]
and \(\sum_p |f(p) a_p|^2 \leq \sum_p |f(p)|^2 = O(1)\), so that
\[
\log \frac{\|\varphi f a\|}{\|a\|} = \Re \sum_p f(p) a_p + O(1). \quad (2.3)
\]

**Proof of Theorem 2.3.** We consider first upper bounds. The supremum occurs for \(a \geq 0\) which we now assume. Write \(a = (a_n)\), \(\varphi f a = b = (b_n)\). Define \(\alpha_p\) and \(\beta_p\) for prime \(p\) by
\[
\alpha_p = \sum_{k=1}^{\infty} a_{p^k}^2 \quad \text{and} \quad \beta_p = \sum_{k=1}^{\infty} b_{p^k}^2.
\]
By multiplicativity of \(a\) and \(b\) we have \(T^2 = \|a\|^2 = \prod_p (1 + \alpha_p)\) and \(\|b\|^2 = \prod_p (1 + \beta_p)\). Thus
\[
\frac{\|\varphi f a\|}{\|a\|} = \prod_p \sqrt{\frac{1 + \beta_p}{1 + \alpha_p}}.
\]

Now for \(k \geq 1\)
\[
b_{p^k} = \sum_{r=0}^{k} f(r) a_{p^{k-r}} = a_{p^k} + f(p) b_{p^{k-1}}.
\]
Thus
\[
b_{p^k}^2 = a_{p^k}^2 + 2 f(p) a_{p^k} b_{p^{k-1}} + f(p)^2 b_{p^{k-1}}^2.
\]
Summing from \(k = 1\) to \(\infty\) and adding 1 to both sides gives
\[
1 + \beta_p = 1 + \alpha_p + 2 f(p) \sum_{k=1}^{\infty} a_{p^k} b_{p^{k-1}} + f(p)^2 (1 + \beta_p). \quad (2.4)
\]
By Cauchy-Schwarz,
\[
\sum_{k=1}^{\infty} a_k b_{k-1} \leq \left( \sum_{k=1}^{\infty} a_k^2 \sum_{k=1}^{\infty} b_{k-1}^2 \right)^{1/2} = \sqrt{\alpha_p (1 + \beta_p)},
\]
so, on rearranging
\[
(1 + \beta_p) - \frac{2f(p) \sqrt{\alpha_p (1 + \beta_p)}}{1 - f(p)^2} \leq 1 + \alpha_p
\]
Completing the square we find
\[
\left( \sqrt{1 + \beta_p} - f(p)\sqrt{\alpha_p} \right)^2 \leq \frac{1 + \alpha_p}{1 - f(p)^2}.
\]
The term on the left inside the square is non-negative for \( p \) sufficiently large since \( f(p) \to 0 \); in fact from (2.4), \( 1 + \beta_p \geq \frac{1 + \alpha_p}{1 - f(p)^2} \) which is greater than \( \frac{f(p)^2 \alpha_p}{(1 - f(p)^2)} \) if \( f(p) \leq 1/\sqrt{2} \). Rearranging gives
\[
\sqrt{1 + \beta_p} - f(p)\sqrt{\alpha_p} \leq 1 - f(p)^2 \left( 1 + f(p)\sqrt{\frac{\alpha_p}{1 + \alpha_p}} \right).
\]
Let \( \gamma_p = \sqrt{\frac{\alpha_p}{1 + \alpha_p}} \). Taking the product over all primes \( p \) gives
\[
\left\| \frac{\varphi_f}{a} \right\| \leq A \| f \|^2 \prod_p (1 + f(p)\gamma_p) \leq A' \exp \left\{ \sum_p f(p)\gamma_p \right\}
\]
for some constants \( A, A' \) depending only on \( f \). (We can take \( A = 1 \) if \( f(p) \leq 1/\sqrt{2} \).) Note that \( 0 \leq \gamma_p < 1 \) and \( \prod_p \frac{1}{1 - \gamma_p} = T^2 \).

Let \( \epsilon > 0 \) and put \( P = \log T \log \log T \). We split up the sum on the RHS of (2.5) into \( p \leq aP \), \( aP < p \leq AP \) and \( p > AP \) (for a small and \( A \) large). First
\[
\sum_{p \leq aP} f(p)\gamma_p \leq \sum_{p \leq aP} f(p) \sim \frac{a^{1 - \alpha} P \hat{f}(P)}{(1 - \alpha) P} < \epsilon \hat{f}(\log T \log \log T) \log T,
\]
for \( a \) sufficiently small\(^2\). Next, using the fact that \( \log T^2 = \log \prod_p \frac{1}{1 - \gamma_p} \geq \sum_p \gamma_p^2 \), we have (since \( \hat{f}^2 \) is regularly-varying of index \( -2\alpha \))
\[
\sum_{p > AP} f(p)\gamma_p \leq \left( \sum_{p > AP} f(p)^2 \sum_{p > AP} \gamma_p^2 \right)^{1/2} \leq \left( \frac{2A^{1 - 2\alpha} P \hat{f}(P)^2 \log T}{(2\alpha - 1) \log P} \right)^{1/2}
\]
\[
\sim \frac{\hat{f}(\log T \log \log T) \log T}{A^{\alpha - 1/2} \sqrt{\alpha - 1/2}} < \epsilon \hat{f}(\log T \log \log T) \log T
\]
for \( A \) sufficiently large. This leaves the range \( aP < p \leq AP \).

Note that the result follows from the case \( f(n) = n^{-\alpha} \). For, by the uniform convergence theorem for regularly varying functions
\[
\left| f(p) - \left( \frac{P}{p} \right)^\alpha \hat{f}(P) \right| < \epsilon f(p)
\]
for \( aP < p \leq AP \) and \( P \) sufficiently large, depending only on \( \epsilon \). The problem therefore reduces to maximising
\[
\sum_{aP < p \leq AP} \frac{\gamma_p}{p^\alpha}
\]
\(^2\)Using \( \sum_{p \leq x} f(p) \sim \int_2^x \frac{\hat{f}(t)}{\log t} dt \sim \frac{\epsilon f(x)}{(1 - \alpha) \log x} \), since \( \hat{f} \) is regularly-varying of index \(-\alpha\).
subject to $0 \leq \gamma_p < 1$ and $\prod_p \frac{1}{1 - \gamma_p} = T^2$. The maximum clearly occurs for $\gamma_p$ decreasing (if $\gamma_p' > \gamma_p$ for primes $p < p'$, then the sum increases in value if we swap $\gamma_p$ and $\gamma_p'$). Thus we may assume that $\gamma_p$ is decreasing.

By interpolation we may write $\gamma_p = g\left(\frac{p}{P}\right)$ where $g : (0, \infty) \to (0, 1)$ is continuously differentiable and decreasing. Of course $g$ will depend on $P$. Let $h = \log \frac{1}{1 - \gamma^2}$, which is also decreasing. Note that

$$2 \log T = \sum_p h\left(\frac{p}{P}\right) \geq \sum_{p \leq aP} h\left(\frac{p}{P}\right) \geq h(a) \pi(aP) \geq cb(a) \log T,$$

for $P$ sufficiently large, for some constant $c > 0$. Thus $h(a) \leq C_a$ (independent of $T$).

Now, for $F : (0, \infty) \to [0, \infty)$ decreasing,

$$\sum_{aP < p \leq bP} F\left(\frac{p}{P}\right) = \frac{x}{\log x} \int_a^b F + O\left(\frac{x F(a)}{(\log x)^2}\right),$$

where the implied constant is independent of $F$ and $x$. For, on writing $\pi(x) = li(x) + e(x)$, the LHS is

$$\int_{ax}^{bx} F\left(\frac{t}{x}\right) d\pi(t) = x \int_a^b F\left(\frac{t}{\log xt}\right) dt + \int_a^b F(t) \frac{d\theta}{dt} dt$$

$$= \frac{x}{\log \theta x} \int_a^b F + \left[F(t) c(t)\right]_a^b - \int_a^b c(t) dF(t)$$

$$= \frac{x}{\log x} \int_a^b F + O\left(\frac{x F(a)}{(\log x)^2}\right),$$

on using $c(x) = O\left(\frac{x}{(\log x)^2}\right)$ and the fact that $F$ is decreasing. Thus by (2.9)

$$2 \log T \geq \sum_{aP < p \leq aP} h\left(\frac{p}{P}\right) \sim P \int_a^A \frac{h}{(\log x)^2} dt \sim (\log T) \int_a^A \frac{h}{(\log x)^2} dt.$$

Since $a$ and $A$ are arbitrary, $\int_0^\infty h$ must exist and is at most 2. Also, by (2.9)

$$\sum_{aP < p \leq aP} \frac{\gamma_p}{p^\alpha} \sim \frac{1}{P^\alpha} \sum_{aP < p \leq aP} \frac{g\left(\frac{p}{P}\right)}{\left(\frac{p}{P}\right)^\alpha} \sim \frac{P^{1-\alpha}}{\log P} \int_a^A \frac{g(u)}{u^\alpha} du.$$

Hence by (2.8),

$$\sum_{aP < p \leq aP} f(p) \gamma_p \sim \tilde{f}(P) P^\alpha \sum_{aP < p \leq aP} \frac{\gamma_p}{p^\alpha} \sim \frac{P^{1-\alpha}}{\log P} \int_a^A \frac{g(u)}{u^\alpha} du.$$

As $a$ and $A$ are arbitrary, it follows from above and (2.5), (2.6), (2.7) that

$$\log \left(\frac{\|f\|_a}{\|a\|}\right) \leq \left(\int_0^\infty \frac{g(u)}{u^\alpha} du + o(1)\right) \tilde{f}(\log T) \log(\log T) \log T.$$

Thus we need to maximize $\int_0^\infty g(u) u^{-\alpha} du$ subject to $\int_0^\infty h \leq 2$ over all decreasing $g : (0, \infty) \to (0, 1)$. Since $h$ is decreasing,

$$\frac{1}{2} x h(x) \leq \int_{x/2}^x h,$$

The RHS can be made as small as we please for $x$ sufficiently small or large (as $\int_0^\infty h$ converges). In particular, $x h(x) \to 0$ as $x \to \infty$ and as $x \to 0^+$. In fact, for the supremum, we can consider just those $g$ (and $h$) which are continuously differentiable and strictly decreasing, since we can
approximate arbitrarily closely with such functions. On writing \( g = s \circ h \) where \( s(x) = \sqrt{1 - e^{-x}} \), we have

\[
\int_0^\infty \frac{g(u)}{u^\alpha} \, du = \left[ \frac{g(u)^{1-\alpha}}{1-\alpha} \right]_0^\infty - \frac{1}{1-\alpha} \int_0^\infty g'(u)u^{1-\alpha} \, du
\]

\[
= -\frac{1}{1-\alpha} \int_0^\infty s'(h(u))h'(u)u^{1-\alpha} \, du = \frac{1}{1-\alpha} \int_0^{h(0^+)} s'(x)l(x)^{1-\alpha} \, dx,
\]

where \( l = h^{-1} \), since \( \sqrt{u}g(u) \to 0 \) as \( u \to \infty \). The final integral is, by Hölder’s inequality at most

\[
\left( \int_0^{h(0^+)} s^{1/\alpha} \right)^\alpha \left( \int_0^{h(0^+)} l \right)^{1-\alpha}.
\]

But \( \int_0^{h(0^+)} l = -\int_0^\infty uh'(u)du = \int_0^\infty h \leq 2 \), so

\[
\int_0^\infty \frac{g(u)}{u^\alpha} \, du \leq \frac{2^{1-\alpha}}{1-\alpha} \left( \int_0^\infty s^{1/\alpha} \right)^\alpha.
\]

A direct calculation shows that\(^3\) \( \int_0^\infty (s')^{1/\alpha} = 2^{1/\alpha} B(\frac{1}{\alpha}, 1 - \frac{1}{2\alpha}) \). This gives the upper bound.

The proof of the upper bound leads to the optimum choice for \( g \) and the lower bound. We note that we have equality in (2.10) if \( l/(s')^{1/\alpha} \) is constant; i.e. \( l(x) = cs'(x)^{1/\alpha} \) for some constant \( c > 0 \) — chosen so that \( \int_0^\infty l = 2 \). This means we take

\[
b(x) = (s')^{-1}\left( \left( \frac{x}{c} \right)^\alpha \right) = \log \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \left( \frac{c}{x} \right)^{2\alpha}} \right).
\]

from which we can calculate \( g \). In fact, we show that we get the required lower bound by just considering \( a_n \) completely multiplicative. To this end we use (2.3), and define \( a_p \) by:

\[
a_p = g_0\left( \frac{p}{P} \right),
\]

where \( P = \log T \log \log T \) and \( g_0 \) is the function

\[
g_0(x) = \sqrt{1 - \frac{2}{1 + \sqrt{1 + (\frac{x}{c})^{2\alpha}}}}.
\]

with \( c = 2^{1+1/\alpha}/B(\frac{1}{\alpha}, 1 - \frac{1}{2\alpha}) \). As such, by the same methods as before, we have \( \|a\| = T^{1+o(1)} \) and

\[
\log \frac{\|\varphi_a\|}{\|a\|} = \sum_p f(p)g_0\left( \frac{p}{P} \right) + O(1) \sim \frac{P\bar{f}(P)}{\log P} \int_0^\infty g_0(u) u^{\alpha} \, du.
\]

By the choice of \( g_0 \), the integral on the right is \( \frac{B(\frac{1}{\alpha}, \frac{1}{2\alpha})}{(1-\alpha)\alpha} \), as required.

\[\square\]

**Remark.** From the above proof, we see that the supremum (of \( \|\varphi_f a\|/\|a\| \)) over \( \mathcal{M}_2^2 \) is roughly the same size as the supremum over \( \mathcal{M}^2 \); i.e. they are log-asymptotic to each other. Is it true that these respective suprema are closer still; eg. are they asymptotic to each other for \( \frac{1}{2} < \alpha < 1 \)?

3. The special case \( f(n) = n^{-\alpha} \).

In this case we can take \( \bar{f}(x) = x^{-\alpha} \) which is regularly varying of index \( -\alpha \). Here we shall write \( \varphi_a \) for \( \varphi_f \) and \( M_n \) for \( M_f \).

\(\text{\footnote{3 The integral is } 2^{-1/\alpha} \int_0^\infty e^{-x/\alpha}(1 - e^{-x})^{-1/2\alpha} \, dx = 2^{-1/\alpha} \int_0^1 t^{1/\alpha - 1}(1 - t)^{-1/2\alpha} \, dt.}\)
Theorem 3.1
We have

\[ M_1(T) = e^\gamma (\log \log T + \log \log \log T + 2 \log 2 - 1 + o(1)), \]

while for \( \frac{1}{2} < \alpha < 1, \)

\[ \log M_\alpha(T) = \left( B\left(\frac{1}{\alpha}, 1 - \frac{1}{\alpha}\right)^\alpha \right) \frac{\log \log T}{\log \log T} + o(1) \]

(3.1)

Remark. As noted in the introduction, these asymptotic formulae bear a strong resemblance to the (conjectured) maximal order of \(|\zeta(\alpha + iT)|\). It is interesting to note that the bounds found here are just larger than what is known about the lower bounds for \( Z_\alpha(T) = \max_{1 \leq t \leq T} |\zeta(\alpha + it)| \). In a recent paper (see [8]), Lamzouri suggests \( \log Z_\alpha(T) \sim C(\alpha)(\log T)^{1-\alpha}(\log \log T)^{-\alpha} \) with some specific function\(^4 \) \( C(\alpha) \) (for \( \frac{1}{2} < \alpha < 1 \)). We note that the constant appearing in (3.2) is not \( C(\alpha) \) since, for \( \alpha \) near \( \frac{1}{2} \), the former is roughly \( \frac{1}{\sqrt{\alpha - \frac{1}{2}}} \), while \( C(\alpha) \sim \frac{1}{\sqrt{\alpha - \frac{1}{2}}} \). For \( \alpha = 1 \), see the comment in the introduction.

It would be very interesting to be able to extend these ideas (and results) to the \( \alpha = \frac{1}{2} \) case. As we show in the appendix, we cannot do this by restricting \( \varphi_{\frac{1}{2}} \) to smaller domains in \( l^2 \). Somewhat the analogy — if such exists — between \( M_\alpha \) and \( Z_\alpha \) breaks down just here.

Proof of Theorem 3.1. For \( \frac{1}{2} < \alpha < 1 \) the result follows from Theorem 2.3, so we only concern ourselves with \( \alpha = 1 \).

For an upper bound we use (2.5) with \( f(p) = 1/p \) (and \( A = 1 \)). Thus

\[ \frac{\|\varphi_1\|}{\|a\|} \leq \zeta(2) \prod_p \left( 1 + \frac{\gamma_p}{p} \right). \]

Again, the maximum of the RHS (subject to \( 0 \leq \gamma_p < 1 \) and \( \prod_p \frac{1}{1 - \gamma_p} = T^2 \)) occurs for \( \gamma_p \) decreasing. Let \( P = \log T \log \log T \) and \( a, A \) be arbitrary constants such that \( A > 1 > a > 0 \). Split the product into the ranges \( p \leq aP, aP < p \leq AP \) and \( p > AP \). We have

\[ \zeta(2) \prod_{p \leq aP} \left( 1 + \frac{\gamma_p}{p} \right) \leq \zeta(2) \prod_{p \leq aP} \left( 1 + \frac{1}{p} \right) = e^\gamma (\log aP + o(1)) \]

by Merten’s Theorem, while the product over \( p > AP \) is at most

\[ \exp \left\{ \sum_{p > AP} \frac{\gamma_p}{p} \right\} \leq \exp \left\{ \left( \sum_{p > AP} \frac{1}{p^2} \sum_{p > AP} \gamma_p \right)^2 \right\} \leq \exp \left\{ 2 \log T \left( \sum_{p > AP} \frac{1}{p^2} \right)^2 \right\}. \]

But \( \sum_{p > AP} 1/p^2 \sim 1/\log T \sim 1/\log \log \log T \), so

\[ \prod_{p > AP} \left( 1 + \frac{\gamma_p}{p} \right) \leq 1 + \frac{2}{\sqrt{A} \log \log T} \]

for all large enough \( T \). Combining the above two estimates gives

\[ \zeta(2) \prod_{p \leq AP} \left( 1 + \frac{\gamma_p}{p} \right) \leq e^\gamma \left( \log_2 T + \log_3 T + \log a + \frac{2}{\sqrt{A}} + o(1) \right) \]

For the remaining range \( aP < p \leq AP \) we write, as before, \( \gamma_p = g(\frac{p}{A}) \) where \( g : (0, \infty) \rightarrow (0, 1) \) is decreasing. Then

\[ \log \left( \prod_{aP < p \leq AP} \left( 1 + \frac{g(p/A)}{p} \right) \right) \sim \sum_{aP < p \leq AP} \frac{g(p/A)}{p} \sim \frac{1}{\log P} \int_a^A \frac{g(u)}{u} \, du, \]

\(^4\)Lamzouri has \( C(\alpha) = G_1(\alpha)^\alpha \alpha^{-2\alpha}(1 - \alpha)^{1-\alpha} \), where \( G_1(x) = \int_0^x u^{-1-1/s} \log(\sum_{x^s} u^{-1/2}) \, du \).
by (2.9). Thus
\[
\frac{\|\varphi_{1,a}\|}{\|a\|} \leq e^\gamma \left( \log_2 T + \log_3 T + \int_0^A \frac{g(u)}{u} \, du - \int_a^1 \frac{1}{u} \, du + \frac{2}{\sqrt{A}} + o(1) \right)
\]
for all \( A > 1 > a > 0 \). We need to minimise the constant term. Since \( g(u) < 1 \), the minimum occurs for \( a \) arbitrarily small. On the other hand \( \int_A^\infty \frac{g(u)}{u} \, du \leq \frac{1}{A} \int_0^\infty g^2(u) \, du = o(1/\sqrt{A}) \), so the constant is minimized for arbitrarily large \( A \); i.e. it is at most \( \int_1^\infty \frac{g(u)}{u} \, du - \int_0^1 \frac{1-g(u)}{u} \, du \). Thus
\[
M_1(T) \leq e^\gamma \left( \log \log T + \log \log T + \kappa + o(1) \right)
\]
where \( \kappa = \sup \{ L(g) : g \in G \} \).

Here \( L(g) = \int_1^\infty \frac{g(u)}{u} \, du - \int_0^1 \frac{1-g(u)}{u} \, du \) and \( G \) is the set of all decreasing \( g : (0, \infty) \to (0, 1) \) for which \( \int_0^\infty \log \frac{1}{1-g(u)} \, du \leq 2 \). As in the proof of Theorem 2.3, let \( h = \log \frac{1}{1-g(u)} \) so that \( g = s \circ h \) where \( s(x) = \sqrt{1-e^{-x}} \). Now we show \( \kappa = 2 \log 2 - 1 \). Trivially, by Cauchy-Schwarz, we have
\[
L(g) \leq \sqrt{\int_1^\infty \frac{1}{u^2} \, du \int_1^\infty g(u)^2 \, du} \leq \sqrt{\int_0^\infty h \leq \sqrt{2}},
\]
so \( \kappa \leq \sqrt{2} \).

Note that the supremum is achieved for \( \int_0^\infty h = 2 \). For if \( \int_0^\infty h < 2 \), then we can always increase \( g \) by a small amount while keeping it less than 1 and decreasing, while \( \int h \) is increased by a prescribed amount – just take \( g_1 = k \circ g \) where \( k : (0, 1) \to (0, 1) \) is increasing and \( k(x) > x \). With \( k(x) - x \) sufficiently small, \( \int h_1 \leq 2 \) while \( L(g_1) > L(g) \).

Further, we may take the supremum over \( g \) for which \( g \) is continuously differentiable and strictly decreasing, since they can approximate functions in \( G \) arbitrarily closely.

Now, for \( L(g) \) to be finite (i.e. \( \sigma = -\infty \)), we need \( \int_0^1 \frac{1-g(u)}{u} \, du \) to converge. For \( x \in (0, 1) \),
\[
\int_x^{\sqrt{x}} \frac{1-g(u)}{u} \, du \geq (1-g(x)) \int_x^{\sqrt{x}} \frac{1}{u} \, du = \frac{1}{2} (1-g(x)) \log \frac{1}{x}.
\]
The LHS tends to 0 as \( x \to 0^+ \), so we must have
\[
(1-g(x)) \log x \to 0 \quad \text{as} \quad x \to 0^+.
\]
In particular, \( g(x) \to 1 \) as \( x \to 0^+ \) (so \( h(x) \to \infty \) as \( x \to 0^+ \)). Also, as in Theorem 2.3, \( x h(x) \to 0 \) as \( x \to \infty \). Now, with \( g = s \circ h \),
\[
\int_1^\infty \frac{g(u)}{u} \, du = [g(u) \log u]_1^\infty - \int_1^\infty s'(h(u))h'(u) \log u \, du = \int_{h(1)} h(u) \log l(y) \, dy,
\]
where \( l = h^{-1} \) is the inverse function of \( h \). Also,
\[
\int_0^1 \frac{1-g(u)}{u} \, du = [(1-g(u)) \log u]_0^1 + \int_0^1 s'(h(u))h'(u) \log u \, du = - \int_{h(1)}^\infty s'(y) \log l(y) \, dy.
\]
Hence \( L(g) = \int_0^\infty s' \log l \) and \( \int_0^\infty l = 2 \).

Now, using Jensen’s inequality \( \int \log f \, d\mu \leq \log(\int f \, d\mu) \) for \( \mu \) a probability measure ([11], p.62), we have
\[
\int_0^\infty s' \log(l/s') \, ds \leq \log \left( \int_0^\infty l/s' \, ds \right) = \log \left( \int_0^\infty l \right) = \log 2. \quad (3.3)
\]
Hence
\[
\int_0^\infty s' \log l \leq \log 2 + \int_0^\infty s' \log s' = \log 2 + \int_0^1 \log \left( \frac{1-u^2}{2u} \right) \, du = 2 \log 2 - 1,
\]
after some calculation.

The proof of the upper bound leads to the optimum choice for \( g \) and the lower bound. We note that we have equality in (3.3) if \( l/s' \) is constant; i.e. \( l(x) = cs'(x) \) for some constant \( c > 0 \) — chosen so that \( \int_0^\infty l = 2 \) (i.e. we take \( c = 2 \)). Thus, actually \( \kappa = 2 \log 2 - 1 \) and the supremum is achieved for the function \( g_0 \), where

\[
g_0(x) = \sqrt{1 - \frac{2}{1 + \sqrt{1 + \left(\frac{x}{2}\right)^2}}}
\]

In fact, we show that we get the required lower bound by just considering \( a \) completely multiplicative. To this end we use Corollary 2.5, and define \( a_p \) by:

\[
a_p = g_0\left(\frac{P}{p}\right),
\]

where \( P = \log T \log \log T \). As such, by the same methods as before, we have \( \|a\| = T^{1+o(1)} \). Let \( a > 0 \) and \( P = \log T \log \log T \). By Corollary 2.5

\[
\frac{||\varphi_1 a||}{\|a\|} \geq \prod_p \frac{1 - \frac{a_p}{p}}{1 - \frac{1}{p}} \prod_{p \leq aP} \frac{1 + \frac{1 - a_p}{p - T}}{1 - \frac{1 - a_p}{p - T}} \prod_{p > aP} \frac{1 - \frac{a_p}{p}}{1 - \frac{a_p}{p}}.
\] (3.4)

Using Merten’s Theorem, the first product on the right is \( e^\gamma (\log aP + o(1)) \), while the second product is greater than

\[
\exp\left\{\sum_{p \leq aP} a_p \right\} \geq 1 - 2 \sum_{p \leq aP} \frac{1 - g_0(p/P)}{p}.
\]

The sum is asymptotic to \( \frac{a}{\log a} \int_0^a \frac{1 - g_0(u)}{u} du < \frac{\varepsilon}{\log P} \), for any given \( \varepsilon > 0 \), for sufficiently small \( a \).

The third product in (3.4) is greater than

\[
\exp\left\{\sum_{p > aP} a_p \right\} = \exp\left\{ \frac{(1 + o(1))}{\log P} \int_a^\infty \frac{g_0(u)}{u} du \right\}
\]

by (2.9). Thus

\[
\frac{||\varphi_1 a||}{\|a\|} \geq e^\gamma \left( \log P + \int_a^\infty \frac{g_0(u)}{u} du + \log a - \varepsilon \right) \geq e^\gamma \left( \log P + L(g_0) - \varepsilon \right)
\]

for \( a \) sufficiently small. As \( L(g_0) = 2 \log 2 - 1 \) and \( \varepsilon \) arbitrary, this gives the required lower bound.

\( \square \)

**Lower bounds for \( \varphi_\alpha \) and some further speculations**

We can study lower bounds of \( \varphi_\alpha \) via the function

\[
m_\alpha(T) = \inf_{a \in M^2} \frac{||\varphi_\alpha a||}{\|a\|}.
\]

Using very similar techniques, one obtains analogous results to Theorem 3.1:

\[
\frac{1}{m_1(T)} = \frac{6e^\gamma}{\pi^2} (\log \log T + \log \log T + 2 \log 2 - 1 + o(1))
\]

and

\[
\log \frac{1}{m_\alpha(T)} \sim \frac{B\left(\frac{1}{2}, 1 - \frac{1}{2\alpha}\right)\alpha (\log T)^{1-\alpha}}{(1-\alpha)2^{\alpha} (\log \log T)^{\alpha}} \quad \text{for} \quad \frac{1}{2} < \alpha < 1.
\]

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We see that $m_\alpha(T)$ corresponds closely to the conjectured minimal order of $|\zeta(\alpha + iT)|$ (see [3] and [9]). We omit the proofs, but just point out that for an upper bound (for $1/m_\alpha(T)$) we use

$$\frac{||a||}{||\varphi_\alpha a||} \leq \prod_{p} \left( 1 + \frac{\gamma_p}{p^\alpha} \right),$$

which can be obtained in much the same way as (2.5). For the lower bound, we choose $a_p$ as $-1$ times the choice in Theorem 3.1 and use Corollary 2.5.

The above formulae suggest that the supremum (respectively infimum) of $||\varphi_\alpha a||/||a||$ with $a \in M^2$ and $||a|| = T$ are close to the supremum (resp. infimum) of $|\zeta_\alpha|$ on $[1, T]$. One could therefore speculate further that there is a close connection between $||\varphi_\alpha a||/||a||$ (for such $a$) and $|\zeta(\alpha + iT)|$, and hence between $Z_\alpha(T)$ and $M_\alpha(T)$. Recent papers by Gonek [4] and Gonek and Keating [5] suggest this may be possible, or at least that $M_\alpha$ is a lower bound for $Z_\alpha$. On the Riemann Hypothesis, it was shown in [4] (Theorem 3.5) that $\zeta(s)$ may be approximated for $\sigma > \frac{1}{2}$ up to height $T$ by the truncated Euler product

$$\prod_{p \leq P} \frac{1}{1 - p^{-s}}$$

for $P \ll T$.

Thus one might expect that, with $a \in M^2_\alpha$ maximizing $||\varphi_\alpha a||/||a||$ subject to $||a|| = T$, and $A(s) = \prod_{p \leq P} \frac{1}{1 - a_p}$ (with $P \ll T$),

$$\int_{-T}^T |\zeta(\alpha - it)|^2 |A(it)|^2 \ dt \sim \int_{-T}^T \prod_{p \leq P} \left( 1 - \frac{1}{p^\alpha} \right) dt = \int_{-T}^T \prod_{p \leq P} |B_p(it)|^2 \ dt$$

where $B_p(s) = \sum_{k \geq 0} b_{p, k} p^{-ks}$. The heuristics of Gonek and Keating now suggests this is asymptotic to

$$2T \prod_{p \leq P} \sum_{k \geq 0} b_{p, k}^2 \sim 2T||\varphi_\alpha a||^2$$

if $P \gg \log T \log \log T$ (for the last step). Thus it would follow that

$$Z_\alpha(T)^2 \geq \frac{\int_{-T}^T |\zeta(\alpha - it)|^2 |A(it)|^2 \ dt}{\int_{-T}^T |A(it)|^2 \ dt} \sim \frac{2T||\varphi_\alpha a||^2}{2T||a||^2} \sim M_\alpha(T)^2$$

and hence $Z_\alpha(T) \geq M_\alpha(T)$.

As mentioned before, this would contradict Lamzouri’s suggestion (that $\log Z_\alpha(T) \sim C(\alpha) (\log T)^{1-\alpha} (\log \log T)^{-\alpha}$) since $C(\alpha) < c(\alpha)$ (notation from Theorem 2.3) for $\alpha$ sufficiently close to $\frac{1}{2}$ at least. It is unclear to the author which possibility is more likely.

References

Appendix
Here we show that if $f \not\in l^2$, we cannot hope to ‘capture’ $\varphi_f$ by considering the mapping on some non-trivial subset of $l^2$.

**Proposition A1**

Suppose $\sum_p |f(p)|^2$ diverges, where $p$ ranges over the primes. Then $\varphi_f a \in l^2$ for $a \in l^2$ if and only if $a = 0$.

**Proof.** Suppose there exists $a \in l^2$ with $a \neq 0$ such that $\varphi_f a \in l^2$. Let $a_m$ be the first non-zero coordinate for $a$. Let $b = (b_n) = \varphi_f a \in l^2$. Consider $b_{pm}$ for $p$ prime such that $p \nmid m$. We have

$$b_{pm} = \sum_{d | pm} f(d) a_{pm/d} = a_m f(p) + k(p),$$

where $k(p) = \sum_{d | m} f(d) a_{pm/d}$. Since

$$\sum_p |k(p)|^2 \leq \sum_p \left( \sum_{d | m} |f(d)|^2 \sum_{d | m} |a_{pm/d}|^2 \right) \leq A \sum_{d | m} \sum_p |a_{pm/d}|^2 < \infty,$$

and $\sum_p |b_{pm}|^2$ converges, we must have

$$|a_m|^2 \sum_p |f(p)|^2 < \infty.$$

This is a contradiction. $\square$