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PROPERTIES OF A GRID MOVEMENT ALGORITHM

by

M J Baines

Numerical Analysis Report 8/95

DEPARTMENT OF MATHEMATICS

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1 Introduction

In this report we discuss the grid movement algorithm

$$x_j^{n+1} = \frac{\sum_{k=1}^{K_j} w_k x_{Gk}^n}{\sum_{k=1}^{K_j} w_k} \tag{1}$$

or, equivalently,

$$x_j^{n+1} - x_j^n = \frac{\sum_{k=1}^{K_j} w_k (x_{Gk}^n - x_j^n)}{\sum_{k=1}^{K_j} w_k}, \tag{2}$$

where x_j^n and x_j^{n+1} are the old and the new space vectors corresponding to the position of node j, the summation k being over the K_j elements surrounding x_j^n, and x_{Gk}^n is the centroid of element k (see Fig. 1). The non-negative weights w_k (with $\sum_{k=1}^{K_j} w_k \neq 0$) are initially assumed to be constant although in a later section we shall allow them to vary. On boundaries it is assumed that x_j is fixed, i.e. that x_j^{n+1} is overwritten by x_j^n. It is evident that x_j^{n+1} lies within the convex hull of the x_{Gk}^n : in 1-D this implies that the ordering of the x_j is preserved but the same is not true of higher dimensions.

Let us write (1) in the matrix form

$$\mathbf{x}_{n+1} = C\mathbf{x}_n + \mathbf{b}. \tag{3}$$

Here $\mathbf{x}$ is a vector of (the space vectors) x_j and C is a matrix which, in 1-D, takes a tridiagonal form with

$$\left(0 \cdots 0 \ \frac{1}{2}\frac{w_{j-\frac{1}{2}}}{\sum w} \ \frac{1}{2} \ \frac{1}{2}\frac{w_{j+\frac{1}{2}}}{\sum w} \ 0 \cdots 0\right) \tag{4}$$

as the j'th row (where $\sum w = w_{j-\frac{1}{2}} + w_{j+\frac{1}{2}} \neq 0$). In 2-D C is not tridiagonal but has the similar property that the diagonal term is $\frac{1}{3}$ and the sum of the off-diagonal terms is $\frac{2}{3}$. In d dimensions the corresponding fractions are $\frac{1}{1+d}$ and $\frac{d}{1+d}$. The vector $\mathbf{b}$ comes from overwriting the boundary points.

2 Convergence Property

We first prove a convergence property of the iteration in the form (3).

Theorem 1: The eigenvalues of C are real and have modulus strictly less than 1.

Proof: Note that $C = W^{-1}S$ where $W^{-1} = diag\{(\sum w)^{-1}\}$ and S is a symmetric matrix (equal to C with the rows multiplied by $\sum w$) . Since

$$|C - \lambda I| = \left|W^{-1}S - \lambda I\right| = \left|W^{-\frac{1}{2}}\right|\left|W^{-\frac{1}{2}}SW^{-\frac{1}{2}} - \lambda I\right|\left|W^{\frac{1}{2}}\right| \tag{5}$$

the matrices C and $W^{-\frac{1}{2}}SW^{-\frac{1}{2}}$ have the same eigenvalues and since the latter is symmetric the eigenvalues of both are real.

Since the w's are non-negative, by the Gerschgorin Theorem the eigenvalues of C lie inside or on a circle, centre $\frac{1}{1+d}$ and radius $\frac{d}{1+d}$: thus they lie inside the unit circle except possibly for an eigenvalue at 1.

Suppose now that $\lambda = 1$ is an eigenvalue of C. Then the matrix $C - I$, with diagonal entries $\frac{-d}{1+d}$, is singular. But this matrix is strongly connected and irreducibly diagonally dominant (by virtue of being strictly diagonally dominant in the rows corresponding to the boundary points). The contradiction proves that $\lambda = 1$ is not an eigenvalue of $C - I$ and that the eigenvalues of C therefore lie within the unit circle. #

Theorem 2: $\mathbf{x}^n \to (I - C)^{-1}\mathbf{b}$ as $n \to \infty$.

Proof: By Theorem 1 $(I - C)$ is non-singular. Define

$$\mathbf{x} = (I - C)^{-1}\mathbf{b} \tag{6}$$

i.e.

$$\mathbf{x} = C\mathbf{x} + \mathbf{b}. \tag{7}$$

Subtracting (7) from (3) gives

$$\mathbf{x}^{n+1} - \mathbf{x} = C(\mathbf{x}^n - \mathbf{x}). \tag{8}$$

Let

$$\mathbf{e}^n = \mathbf{x}^n - \mathbf{x}. \tag{9}$$

Then (8) becomes

$$\mathbf{e}^{n+1} = C\mathbf{e}^n \tag{10}$$

and, since C is a constant matrix,

$$\mathbf{e}^n = C^n \mathbf{e}^0. \tag{11}$$

Hence $\mathbf{e}^n \to 0$ and $\mathbf{x}^n \to \mathbf{x}$ as $n \to \infty$. #

The iteration (1) or (3) is therefore convergent. The rate of convergence may be very slow, however. In the limit $\mathbf{x}$ is given by (6), which can also be written

$$\mathbf{x} = (W - S)W^{-1}\mathbf{b} \quad . \tag{12}$$

The matrix $W - S$ is symmetric and, if required, $\mathbf{x}$ can readily be found by the conjugate gradient method (see also section 6).

3 Equidistribution Property

The limiting $\mathbf{x}$ also has equidistribution properties. Since the components of $\mathbf{x}$ satisfy

$$x_j = \frac{\sum_{k=1}^{K_j} w_k x_{Gk}}{\sum_{k=1}^{K_j} w_k}, \tag{13}$$

i.e. in 1-D

$$(w_{j-\frac{1}{2}} + w_{j+\frac{1}{2}})x_j = \frac{1}{2}w_{j-\frac{1}{2}}(x_{j-1} + x_j) + \frac{1}{2}w_{j+\frac{1}{2}}(x_j + x_{j+1}) \tag{14}$$

or

$$w_{j-\frac{1}{2}}(x_j - x_{j-1}) = w_{j+\frac{1}{2}}(x_{j+1} - x_j) \tag{15}$$

we have

$$w_{j-\frac{1}{2}}\Delta x_{j-\frac{1}{2}} = w_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}. \tag{16}$$

If x is the piecewise linear interpolant of x_j and w is the piecewise constant interpolant of w_k, then

$$\int_{x_{j-1}}^{x_j} w dx = \int_{x_j}^{x_{j+1}} w dx = const., \tag{17}$$

in other words w is an equidistributing function. If w is constant then x is equidistributed. If $w = \frac{dE}{dx}$ (or, in discrete form, $\frac{\Delta E}{\Delta x}$) then E is equidistributed. Since w is assumed to be non-negative E must be chosen to be monotonic.

The iteration in 1-D is then

$$x_j^{n+1} = \frac{\sum_{k=j-\frac{1}{2}}^{j+\frac{1}{2}} \left(\frac{\Delta E_k}{\Delta x_k}\right)^n x_{Gk}^n}{\sum_{k=j-\frac{1}{2}}^{j+\frac{1}{2}} \left(\frac{\Delta E_k}{\Delta x_k}\right)^n} \tag{18}$$

or

$$x_j^{n+1} = x_j^n + \frac{1}{2}\left(\frac{\Delta E_{j+\frac{1}{2}} - \Delta E_{j-\frac{1}{2}}}{\frac{\Delta E_{j+\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}} + \frac{\Delta E_{j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}}}\right)^n . \tag{19}$$

Care is needed when computing the update if ΔE is very small.

In particular, E may be taken to be the arclength associated with a given differentiable function $f(x)$, namely

$$E = s = \int^x \sqrt{1 + f'^2 dx} \tag{20}$$

or, discretely, as

$$\Delta E = \Delta s = \Delta x \sqrt{1 + f'^2}. \tag{21}$$

In higher dimensions there is no corresponding equidistribution principle, but since x_j then satisfies

$$\left(\sum_{k=1}^{K_j} w_k\right) x_j = \sum_{k=1}^{K_j} w_k x_{Gk}, \tag{22}$$

we have

$$\sum_{k=1}^{K_j} w_k (x_{Gk} - x_j) = 0. \tag{23}$$

4 Variable Weights

Now suppose that the w_k in (1) depend on n (whilst remaining non-negative with $\sum_{k=1}^{K_j} w_k \neq 0$) and write (3) as

$$\mathbf{x}^{n+1} = C_n \mathbf{x}^n + \mathbf{b}_n \tag{24}$$

where C_n and $\mathbf{b}_n$ are defined as the C and $\mathbf{b}$ in section 1 with w_k replaced by w_k^n. We extend the convergence proof of section 2 to this case.

Theorem 3: If $w_k^n \to w_k\ \forall k$ and $C_n \to C$ as $n \to \infty$ then $I - C$ is non-singular.

Proof: Since the w_k^n are non-negative, then so are the w_k and, by Theorem 1, it follows that $I - C$ is non-singular. #

Theorem 4: Under the conditions of Theorem 3, if $\mathbf{b}_n \to \mathbf{b}$ as $n \to \infty$, then $\mathbf{x}_n \to (I - C)^{-1}\mathbf{b}$ as $n \to \infty$.

Proof: Define

$$\mathbf{x} = (I - C)^{-1}\mathbf{b}, \tag{25}$$

i.e.

$$\mathbf{x} = C\mathbf{x} + \mathbf{b} \tag{26}$$

as in Theorem 2. Subtracting (26) from (24) gives

$$\begin{aligned} \mathbf{x}^{n+1} - \mathbf{x} &= C_n\mathbf{x}^n - C\mathbf{x} + \mathbf{b}_n - \mathbf{b} \\ &= C(\mathbf{x}^n - \mathbf{x}) + (C_n - C)\mathbf{x}^n + \mathbf{b}_n - \mathbf{b} \end{aligned} \tag{27}$$

or

$$\mathbf{e}^{n+1} = C\mathbf{e}^n + \mathbf{z}_n \tag{28}$$

where

$$\mathbf{z}_n = (C_n - C)\mathbf{x}^n + \mathbf{b}_n - \mathbf{b}. \tag{29}$$

Observe that, since $\mathbf{x}_n$ is bounded,

$$\mathbf{z}_n \to 0 \text{ as } n \to \infty.$$

Now, from (28),

$$\mathbf{e}^n = C^n\mathbf{e}^0 + \sum_{m=0}^{n-1} C^\mu \mathbf{z}_m, \tag{30}$$

where

$$\mu = n - m - 1. \tag{31}$$

Since $C = W^{-1}S$,

$$\begin{aligned} C^p &= (W^{-1}S)^p \\ &= W^{-\frac{1}{2}}(W^{-\frac{1}{2}}SW^{-\frac{1}{2}})^p W^{\frac{1}{2}}, \end{aligned} \tag{32}$$

so that

$$\|C^p\| \le \left\|W^{-\frac{1}{2}}\right\| \left\|W^{-\frac{1}{2}}SW^{-\frac{1}{2}}\right\|^p \left\|W^{\frac{1}{2}}\right\|$$

$$\leq \left\|W^{-\frac{1}{2}}\right\| \left\|W^{\frac{1}{2}}\right\| \rho, \tag{33}$$

where $\rho < 1$ is the spectral radius of the symmetric matrix $W^{-\frac{1}{2}}SW^{-\frac{1}{2}}$.

Now write (30) as

$$\mathbf{e}^n = C^n\mathbf{e}^0 + \sum_{m=0}^{m_1} C^\mu \mathbf{z}_m + \sum_{m=m_1+1}^{n-1} C^\mu \mathbf{z}_m, \tag{34}$$

where m_1 is an integer to be chosen.

Since $\mathbf{z}_n \to 0$ as $n \to \infty$, then given ϵ we can find an integer such that $\|\mathbf{z}_n\| \leq \epsilon$ if n is larger than this integer, which we choose as m_1. Then, considering the norms of each term of (34) in turn: first

$$\left\|C^n\mathbf{e}^0\right\| \leq \left\|W^{-\frac{1}{2}}\right\| \left\|W^{\frac{1}{2}}\right\| \left\|\mathbf{e}^0\right\| \rho^n < \epsilon \tag{35}$$

if $n > m_0$ say, since $\rho < 1$; next

$$\left\|\sum_{m=0}^{m_1} C^\mu \mathbf{z}_m\right\| \leq \left\|W^{-\frac{1}{2}}\right\| \left\|W^{\frac{1}{2}}\right\| \max_{m \leq m_1} \|\mathbf{z}_m\| \sum_{m=0}^{m_1} \rho^\mu$$

$$= \left\|W^{-\frac{1}{2}}\right\| \left\|W^{\frac{1}{2}}\right\| \max_{m \leq m_1} \|\mathbf{z}_m\| \rho^{n-m_1-1} \frac{(1-\rho^{m_1+2})}{1-\rho} < \epsilon \tag{36}$$

if $n > m_2$ say, since $\mu = n - m - 1$ and $\rho < 1$; finally

$$\left\|\sum_{m=m_1+1}^{n-1} C^\mu \mathbf{z}_m\right\| \leq \sum_{m=m_1+1}^{n-1} \left\|W^{-\frac{1}{2}}\right\| \left\|W^{\frac{1}{2}}\right\| \|\mathbf{z}_m\| \rho^\mu$$

$$\leq \left\|W^{-\frac{1}{2}}\right\| \left\|W^{\frac{1}{2}}\right\| \epsilon \sum_{m=m_1+1}^{n-1} \rho^\mu$$

$$= \left\|W^{-\frac{1}{2}}\right\| \left\|W^{\frac{1}{2}}\right\| \epsilon \left(\frac{1-\rho^{n-m_1}}{1-\rho}\right)$$

$$\leq \left\|W^{-\frac{1}{2}}\right\| \left\|W^{\frac{1}{2}}\right\| \epsilon(1-\rho)^{-1} \tag{37}$$

provided that $n > m_1$ and $\rho < 1$. Hence, if $n > \max(m_0, m_1, m_2)$,

$$\|\mathbf{e}^n\| \leq \left(2 + \left\|W^{-\frac{1}{2}}\right\| \left\|W^{\frac{1}{2}}\right\| (1-\rho)^{-1}\right) \epsilon \tag{38}$$

and it follows that $\mathbf{e}^n \to 0$ and that $\mathbf{x}^n \to \mathbf{x}$ as $n \to \infty$.#

In the limit the same equidistribution property (17) as in section 3 holds, as well as (23).

Using this result the algorithm still converges when combined with a sequence of variable non-negative weights w_k^n which tends to a limit as $n \to \infty$. In particular, the algorithm may be combined with a sequence of $w's$ coming from, for example, either the evaluation of a given function on the n'th grid or from an entirely separate (convergent) iteration for w^n. Some illustrations are given in Figs. 3-5, taken from references [1] and [2]. In Figs. 3-4 the underlying functions are $tanh\{20\{x+y-1)\}$ and $tanh\{20(x^2+y^2-(0.5)^2)\}$, respectively, on the unit square. Fig. 5 shows the result of interleaving the iteration with a multidimensional upwinding scheme for the circular advection of a square wave profile with velocity $(y,-x)$ on the domain $[-1,1] \times [0,1]$.

5 Properties of the Limit in Higher Dimensions

We have seen that the iteration

$$x_j^{n+1} - x_j^n = \frac{\sum_{k=1}^{K_j} w_k^n (x_{Gk}^n - x_j^n)}{\sum_{k=1}^{K_j} w_k^n} \tag{39}$$

converges to a limit x_j satisfying

$$\sum_{k=1}^{K_j} w_k (x_{Gk} - x_j) = 0. \tag{40}$$

Taking the inner product with a unit vector $\hat{\mathbf{r}}$ in an arbitrary direction gives the displacement

$$\delta r_j^{n+1} = \frac{\sum_{k=1}^{K_j} w_k^n \delta r_{Gk}^n}{\sum_{k=1}^{K_j} w_k^n} \tag{41}$$

in that direction (see Fig. 2), a weighted sum of the projections δr_{Gk}^n of $x_{Gk} - x_j$ onto the line through node j parallel to $\hat{r}$.

Now consider the weight w_k^n in relation to a monitor function $E(x,y)$, say. Suppose first that $\hat{\mathbf{r}}$ is in the direction of ∇E. Then, with $w = |\nabla E|$, the displacement δr_j^{n+1} gives the same form as in (15) with the sum extended over all surrounding elements. In the limit

$$\sum |\nabla E| \, \delta r_{Gk} = 0 \tag{42}$$

which represents a generalised discrete form of the 1-D property

$$\left|\frac{\Delta E_{j-\frac{1}{2}}}{\Delta x_{j-\frac{1}{2}}}\right| (x_j - x_{j-1}) + \left|\frac{\Delta E_{j+\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}}\right| (x_j - x_{j+1}) = 0 \tag{43}$$

(c.f. (15) and (19) in the direction of $\hat{\mathbf{r}}$. Now suppose that $\hat{\mathbf{r}}$ is perpendicular to ∇E. Then, with $w = |\nabla E|$ again, the displacement is once again (41) but this time the weights are evenly spread either side of the line through j in the direction of $\hat{r}$, more closely resembling the $w = const.$ situation, which gives equal spacing in 1-D.

Thus the choice $w = |\nabla E|$ tends to give equidistribution of E in the direction of ∇E and equal spacing perpendicular to ∇E.

If we take the component of (40) in the direction of ∇E, we have

$$\sum_{k=1}^{K_j} |\nabla E| \, (x_{Gk} - x_j) \cdot \frac{\nabla E}{|\nabla E|} = 0, \tag{44}$$

i.e.

$$\sum_{k=1}^{K_j} \nabla E \cdot (x_{Gk} - x_j) = 0 \tag{45}$$

so that, if the function E is assumed to be piecewise linear,

$$\sum_{k=1}^{K_j} (E_{Gk} - E_j) = 0 \tag{46}$$

or

$$\sum_{l=1}^{L_j} (E_l - E_j) = 0 \tag{47}$$

where $l = 1, 2, ..., L_j$ runs over the corners of the elements surrounding the node j. This is precisely the condition (see [1]) that the piecewise constant function

$$E = \frac{1}{(1+d)} \sum_{l=1}^{L_j} E_l \tag{48}$$

is the best fit with variable nodes to the piecewise linear function E in the discrete norm

$$||g|| = \sum_{k=1}^{K} \sum_{c=1}^{N_c} g^2, \tag{49}$$

where $c = 1, 2, ..., N_c$ runs over the corners of the k'th element (out of K).

Similar arguments apply in higher dimensions.

Given a function $f(x, y)$ we may take $|\nabla E|$ to be $|\nabla f|$ or $\sqrt{1 + |\nabla f|^2}$, corresponding to arclength in the direction of ∇f), amongst others. In 1-D a monitor which takes both first and second derivatives into account is

$$\sqrt{1 + \alpha f'^2 + \beta f''^2} \tag{50}$$

where α and β are parameters to be chosen (see [2]), and this generalises to

$$\sqrt{1 + \alpha |\nabla f| + \beta |\nabla^2 f|} \tag{51}$$

in the case of higher dimensions.

6 Continuous Analogues

Observe that (23) is a discretisation of

$$\nabla.(w\nabla X) = 0 \tag{52}$$

where X is any one of the components of the space vector x and that (2) is a relaxed Jacobi iteration for its solution. The latter can also be regarded as a discretisation of the PDE

$$\frac{1}{\tau} x_\tau = \nabla.(w\nabla X) \tag{53}$$

where $\tau = (\Sigma w)^{-1}$ (cf. [3]).

The PDEs need boundary conditions, of course, which, for the component $X(\xi, \eta)$ in 2-D may be taken (on a computational grid ξ, η on the unit square) to be $X = 0, X = 1$ on $\xi = 0, \xi = 1$, respectively, with corresponding conditions for the other component. Rather than choosing the finite difference approximation (23) and the iteration (2), the PDE may be solved by any convenient approximate method, for example by finite elements in which the functional

$$\int w(\nabla X)^2 d\Omega \tag{54}$$

is minimised.

Likewise, the iteration (2) is a discretisation of

$$X_\tau = \nabla.(w\nabla X) \tag{55}$$

with the same boundary conditions and a suitable initial condition (e.g. a uniform grid).

The quasi-equidistribution property of the last section can also be written (in 2-D) as a discretisation of

$$\frac{\partial}{\partial \nu}\left(|\nabla E|\frac{\partial}{\partial \nu}N\right) + \frac{\partial^2 N}{\partial \theta^2} = 0 \tag{56}$$

where N is a coordinate measured in the direction of ∇E and ν, θ are cartesian reference coordinates measured along and perpendicular to the direction of ∇E, respectively.

From this point of view the algorithm (2) or (23) can be replaced by any discretisation of the nonlinear elliptic or parabolic equations (52) or (54) and once again combined if desired with any w coming from say the evaluation of a function or another different iteration. Illustrations are given in Figs. 6-8.

In figures 6-7 the former procedure is used with functions

(a) $tanh\{200(x + y - 1)\} - tanh\{200(x - y + 1)\}$ and

(b) $tanh(200(x^2 + y^2 - (0.5)^2)\}$, respectively.

In Fig. 8 a Poisson problem with suitable load and boundary conditions for a solution $u = tanh\{200((x - 0.5)^2 + (y - 0.5)^2 - (0.25)^2)\}$ was solved on each grid before passing on to the next grid iteration. Convergence of the double iteration in this case required the use of under-relaxation.

There is another interpretation of the grid adapter in the case of the monitor $\sqrt{1 + |\nabla f|^2}$. Suppose that the aim is to have a uniform grid on a monitor surface. Consider

$$\nabla^2 \chi = 0 \tag{57}$$

where χ is any coordinate in the monitor surface. A discretisation of (57) should lead to a uniform grid. Taking Σ (in 2-D) to be the vector of orthogonal coordinates (σ, τ) on the monitor surface (corresponding to coordinates (ν, θ) on the xy plane), we have

$$\nabla \Sigma = J \nabla \Upsilon \tag{58}$$

where

$$J = \frac{\partial(\sigma, \tau)}{\partial(\nu, \theta)} \tag{59}$$

and Υ is the vector (ν, θ). With σ in the direction of ∇f we have (for a piecewise linear f)

$$J = \begin{pmatrix} \frac{\Delta\sigma}{\Delta x} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{1 + |\nabla f|^2} & 0 \\ 0 & 1 \end{pmatrix} \tag{60}$$

Now, bearing in mind the argument in section 5, discretise

$$\nabla . (J \nabla \Upsilon) = 0 \tag{61}$$

in the manner of (23), giving

$$\sum \left(\sqrt{1 + |\nabla f|^2} \right)_k (x_{Gk} - x_j) = 0 \tag{62}$$

for the vector x at each point j. An iteration for the solution of (62) is

$$x_j^{n+1} - x_j^n = \left(\frac{\Sigma \left(\sqrt{1 + |\nabla f|^2} \right)_k (x_{Gk} - x_j)}{\Sigma \left(\sqrt{1 + |\nabla f|^2} \right)_k} \right)^n, \tag{63}$$

as in the algorithm considered in this report. At convergence we achieve a uniform grid on the monitor surface.

Given this interpretation it is not so surprising that the grids shown in figs. 6-7 have such strong convex properties; they are the projection onto the xy plane of a regular grid drawn on the surface manifold of f. As long as f is single-valued the projected grid cannot be tangled.

The monitor function does not have to be f but could be a combination of f and $|\nabla f|$, for example (see (51).

7 Conclusions

We have seen that the iteration (1) and its continuous analogue is a powerful tool for generating equidistributed and quasi-equidistributed grids. Its convergence and other properties allow it to be used as a useful grid generator or grid adapter for steady problems. Further work is under way to apply the algorithm to time-dependent problems.

8 Acknowledgements

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9 References

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[3] **Huang, W., Ren, Y.** and **Russell, R.D.**, Moving Mesh Partial Differential Equations (MMPDEs) based on the Equidistribution Principle. SIAM J. Num. An. (to appear) (1995).

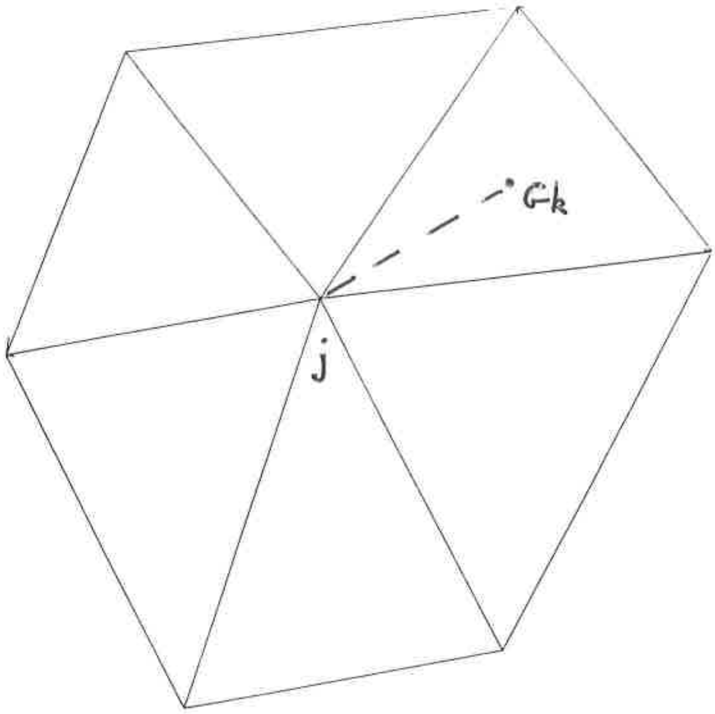

Fig. 1

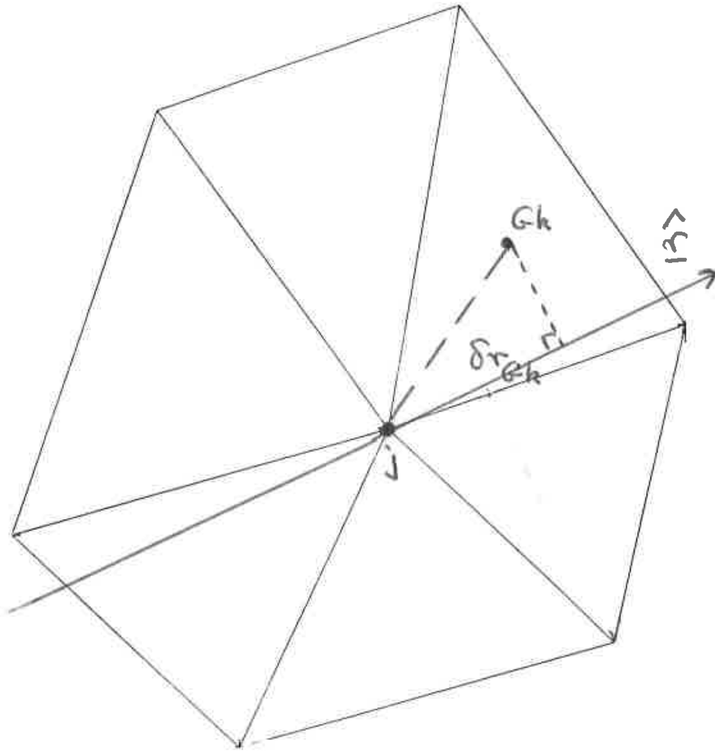

Fig. 2

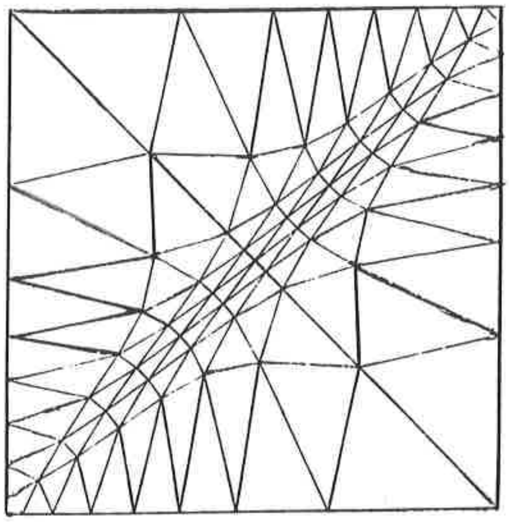

Fig. 3

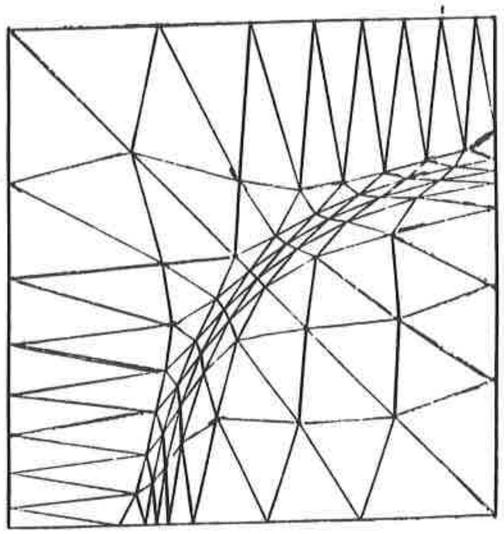

Fig. 4

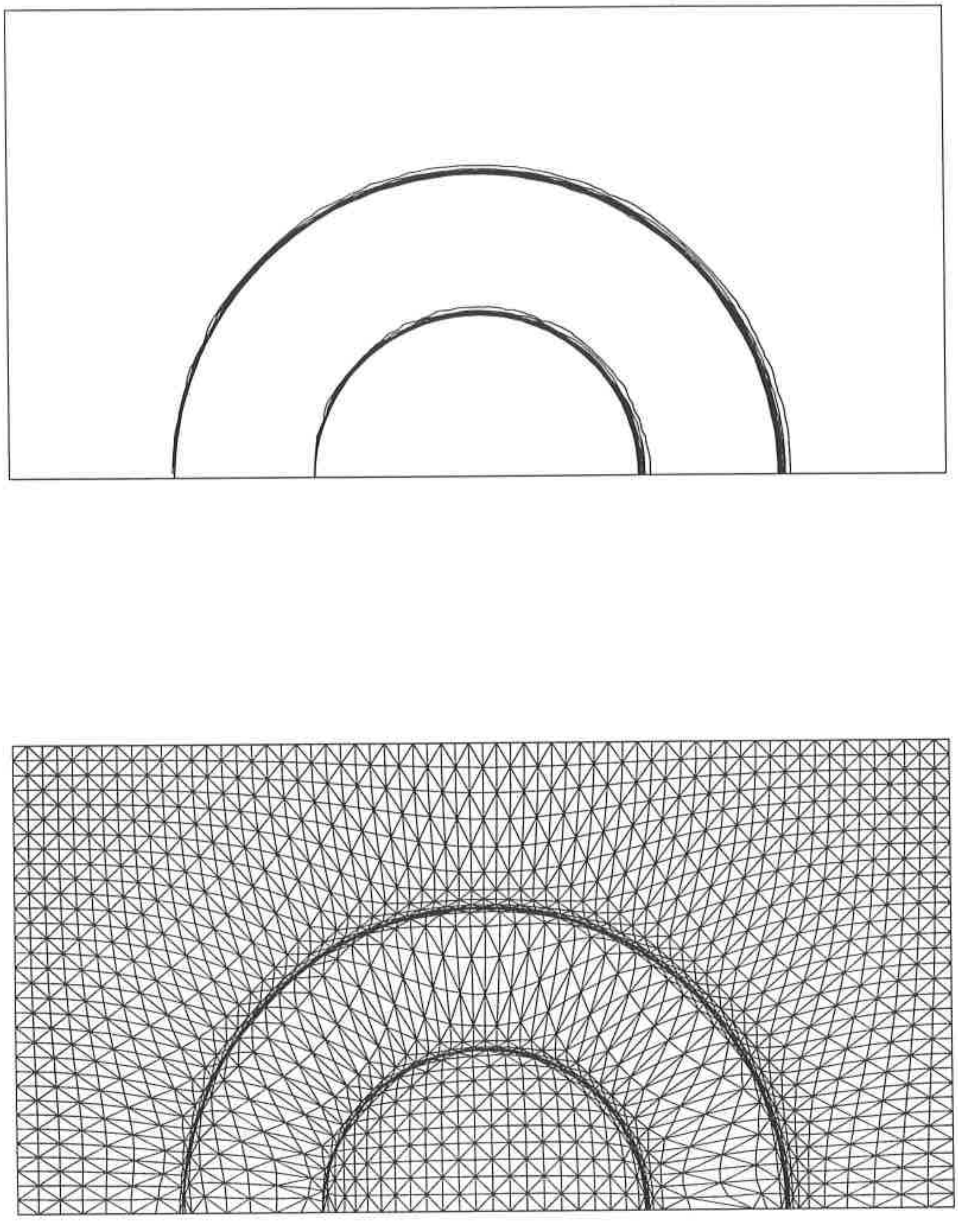

Fig. 5

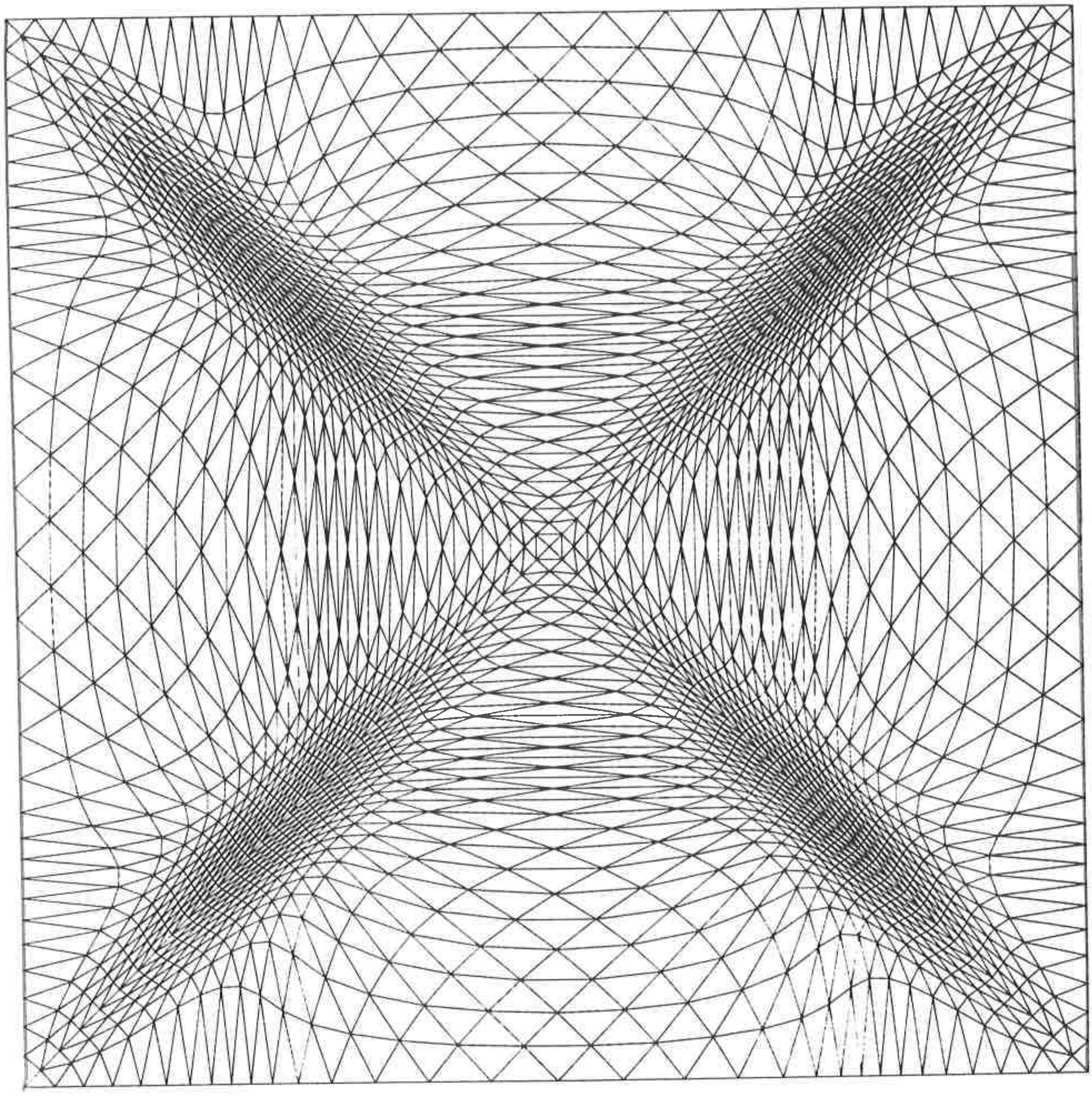

Fig. 6

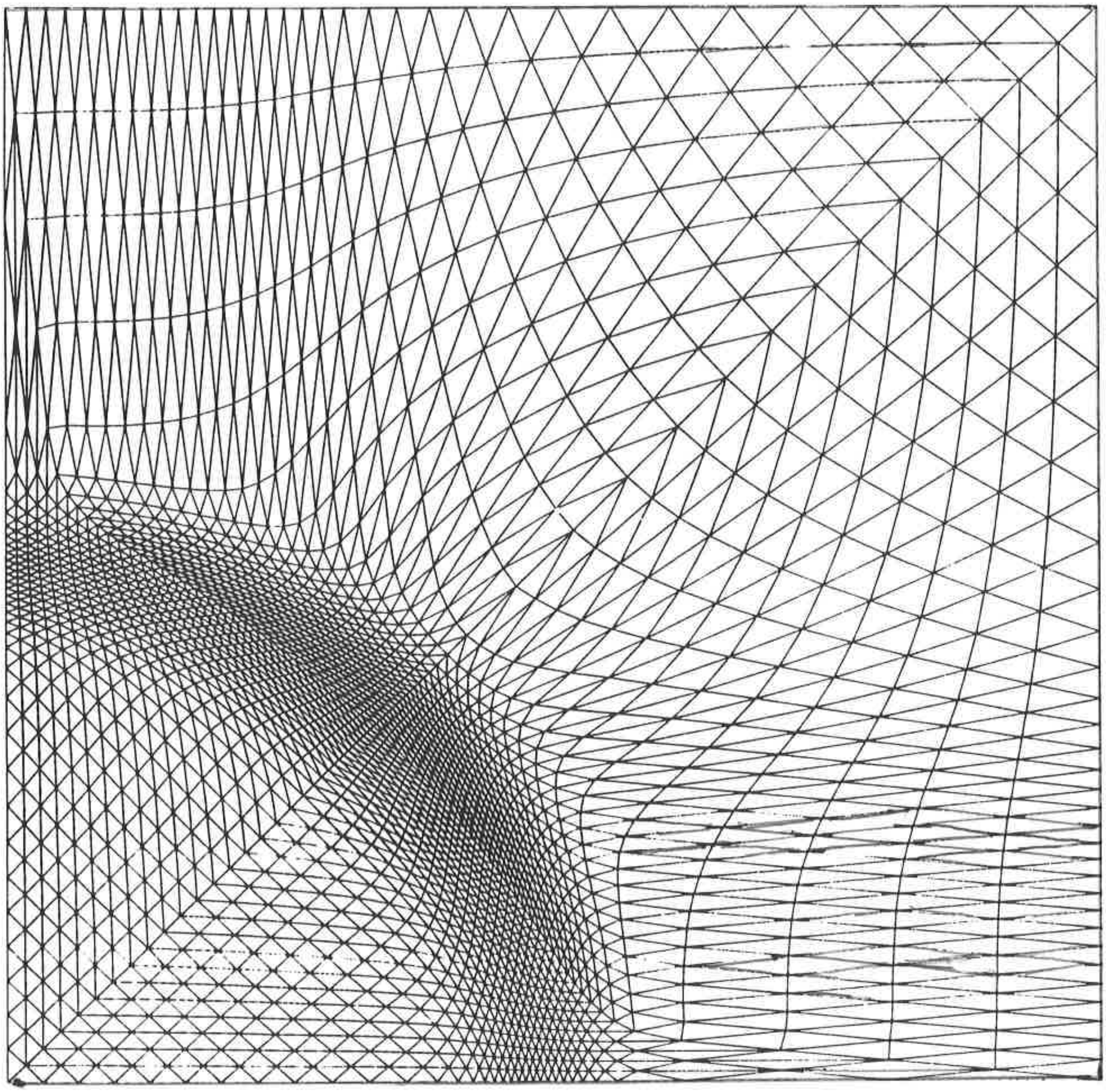

Fig. 7

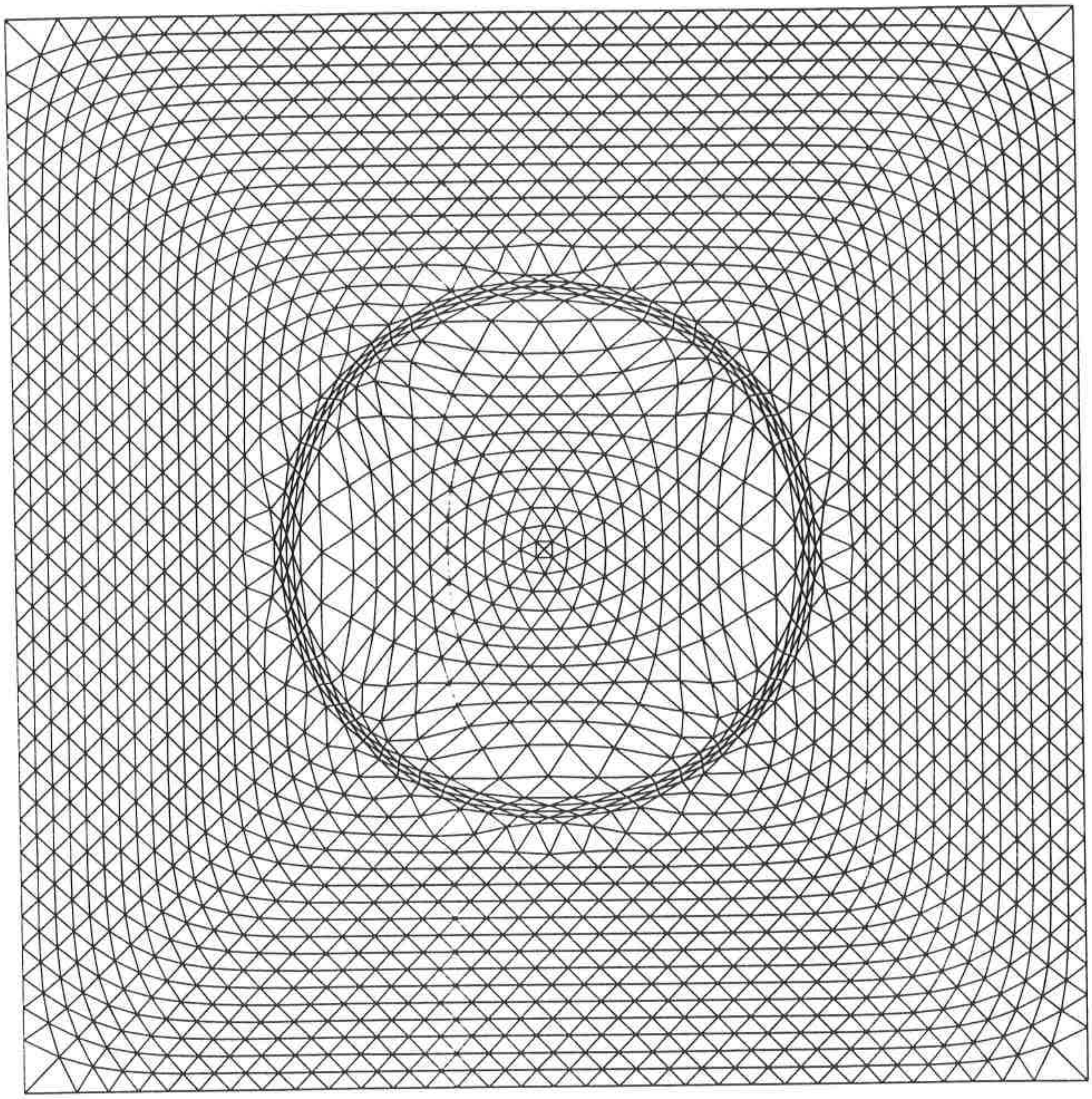

Fig. 8