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OPTIMAL RESOURCE DEPLETION: FREE END TIME PROBLEMS

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1. Introduction

Many current world problems are concerned with the depletion of natural resources. In the past decade there has been much interest in determining optimal rates of depletion for both renewable (e.g. fish) and non-renewable (e.g. oil and coal) resources. In the case of non-renewable resources, optimal depletion rates depend on

(i) the discount given to future generations;

(ii) the time at which an alternative technology may become available.

Problems involving such factors have been investigated by Anderson [1], Vousden [2], Dasgupta and Heal [3] and Davison [4].

In this article we consider a particular aspect of resource depletion, namely: the problem of finding optimal depletion rates when the final time is not specified, but is determined by fixing the capital required at the time when the resource is exhausted. The model is described in section 2, the equations governing the optimal path are given in section 3, a particular analytical solution is determined in section 4, and numerical solutions are presented in section 5. The results are discussed and conclusions drawn in section 6.

2. Model

We consider an idealised economy in which the production, F, depends on two inputs, capital stock K and resource depletion rate R, and the output is either consumed or used to augment the capital stock. These assumptions are given in mathematical terms by the equation

$$\frac{dK}{dt} = F(K, R) - C, \qquad (1)$$

where C is the consumption rate. The resource depletion is constrained by

$$\frac{dS}{dt} = -R, \qquad (2)$$

where S is the resource stock. Initially capital and resource stocks are specified, say K_0 and S_0. At the final time, T, which is not fixed, we assume that all the resource stock is totally depleted and that the capital stock takes a specified value, say K_T. This level of capital can be interpreted as the capital stock required to maintain a new state for all time $t > T$, when an alternative technology is expected to be readily available. To summarise, the appropriate boundary conditions for the problem are:

$$K(0) = K_0, \quad S(0) = S_0; \tag{3}$$

$$K(T) = K_T, \quad S(T) = 0; \tag{4}$$

$$T \text{ not specified.} \tag{5}$$

We are interested in finding the depletion rate and consumption path which maximises the welfare integral

$$W = \int_0^T e^{-\delta t}\, U(C)dt, \tag{6}$$

where δ is the discount rate and U is the utility function.

3. Equations governing the optimal path

The problem described in section 2 is an optimal control problem (see Intriligator [5]) where K and S are the state variables, R and C the controls. The usual procedure is to form the Hamiltonian

$$H = e^{-\delta t}U(C) + (e^{-\delta t}p_k)(F(K,R) - C) + (e^{-\delta t}p_s)(-R) \tag{7}$$

where p_k and p_s are the shadow prices of capital and stock respectively. The variables p_k and p_s satisfy the equations

$$\frac{d}{dt}(e^{-\delta t}\, p_k) = -\frac{\partial H}{\partial K} = -e^{-\delta t}\, p_k \frac{\partial F}{\partial K} \tag{8}$$

$$\frac{d}{dt}(e^{-\delta t}\, p_s) = 0 \tag{9}$$

and the maximum principle gives

$$\frac{\partial H}{\partial C} = 0, \quad \frac{\partial H}{\partial R} = 0. \tag{10}$$

Using (7) and (10) we find

$$\frac{dU}{dC} = p_k, \tag{11}$$

and

$$p_k \frac{\partial F}{\partial R} = p_s. \tag{12}$$

From (8), (9) and (12), it can be deduced that

$$\frac{d}{dt}\left(\frac{\partial F}{\partial R}\right)/\frac{\partial F}{\partial R} = \frac{\partial F}{\partial K}, \tag{13}$$

and substituting (11) into (9) gives

$$\frac{d}{dt}\left(\frac{dU}{dC}\, e^{-\delta t}\right)/\frac{dU}{dC}\, e^{-\delta t} = -\frac{\partial F}{\partial K}. \tag{14}$$

The optimal paths are thus governed by equations (13) and (14) together with the boundary conditions (3) and (4). The transversality condition

$$H = 0, \quad \text{at} \quad t = T,$$

must also be satisfied since the final time is unspecified.

We now consider solutions for the case of specific functions F and U. We take the familiar forms

$$F = K^{\alpha}R^{1-\alpha}, \qquad (0 < \alpha < 1), \tag{15}$$

$$U = (C - C^*)^{\beta}, \quad (0 < \beta < 1,\ C^* \text{ constant}). \tag{16}$$

Defining a new variable $x = K/R$, (13) reduces to

$$\frac{dx}{dt} = x^{\alpha}, \tag{17}$$

which has solution

$$x = [(1-\alpha)t + A]^{1/(1-\alpha)}, \tag{18}$$

where A is a constant. Equation (14) now gives

$$C - C^* = B\, e^{-\delta t/(1-\beta)}[(1-\alpha)t + A]^{\alpha/(1-\beta)(1-\alpha)}, \tag{19}$$

(B constant).

To find R, K and S, we note from (1) and (15) that

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{K}{R}\right) = \frac{1}{R}\frac{dK}{dt} - \frac{K}{R^2}\frac{dR}{dt} = x^{\alpha} - \frac{C}{R} - \frac{x}{R}\frac{dR}{dt},$$

and then from (17), we obtain

$$\frac{dR}{dt} = -C/x$$

$$= -Be^{-\delta t/(1-\beta)}[(1-\alpha)t + A]^{(\alpha+\beta-1)/(1-\beta)(1-\alpha)}$$

$$- C^*[(1-\alpha)t + A]^{-1/(1-\alpha)}.$$

In the special case, $\alpha + \beta = 1$, we can integrate explicitly. We find that the depletion rate is given by

$$R = D + \frac{B(1-\beta)}{\delta}e^{-\delta t/(1-\beta)} + \frac{C^*}{\alpha}[(1-\alpha)t + A]^{-\alpha/(1-\alpha)}, \qquad (20)$$

(D constant), and then, using (2) and integrating again, we obtain the stock level

$$S = \begin{cases} E - Dt + \frac{B(1-\beta)}{\delta^2}e^{-\delta t/(1-\beta)} - \frac{C^*}{\alpha(1-2\alpha)}[(1-\alpha)t + A]^{(1-2\alpha)/(1-\alpha)} & (\alpha \neq \frac{1}{2}), \\ E - Dt + \frac{B(1-\beta^2)}{\delta^2}e^{-\delta t/(1-\beta)} - 4C^* \log(\frac{1}{2}t + A), & (\alpha = \frac{1}{2}) \end{cases} \qquad (21)$$

(E constant). Finally, the capital K is determined from $K = Rx$, using (18) and (20).

The constants A, B, D, E introduced into the solutions, and the final time T, are determined from

$$K(0) = K_0, \quad S(0) = S_0, \quad K(T) = K_T, \quad S(T) = 0 \quad \text{and} \quad H = 0, \ \text{at} \ t = T. \qquad (22)$$

Numerical techniques may be used to find the constants for which conditions (22) are satisfied. In the next section we examine the behaviour of the solutions in the special case $\alpha = \beta = \frac{1}{2}$ and $\delta = 0$. The results obtained are used in finding the numerical solution of the general problem for $\alpha + \beta \neq 1$. (See section 5).

4. The special case $\alpha = \frac{1}{2},\ \beta = \frac{1}{2}$

In the special case $\alpha = \frac{1}{2} = \beta$ and $\delta = 0$, the equations (18) and (19) become

$$x = (\tfrac{1}{2}t + A)^2, \tag{23}$$

$$C = C^* + B(\tfrac{1}{2}t + A)^2. \tag{24}$$

Since

$$\frac{dR}{dt} = -\frac{C}{x} = -B - C^*(\tfrac{1}{2}t + A)^{-2},$$

we can integrate to obtain

$$R = D - Bt + 2C^*(\tfrac{1}{2}t + A)^{-1}; \tag{25}$$

then further integration yields

$$S = S_0 - Dt + \tfrac{1}{2}Bt^2 - 4C^*\log[(\tfrac{1}{2}t + A)/A]. \tag{26}$$

We can also determine the shadow prices from (11) and (12) to be

$$p_k = \frac{dU}{dC} = \tfrac{1}{2}(C - C^*)^{-\frac{1}{2}} = \tfrac{1}{2}B^{-\frac{1}{2}}(\tfrac{1}{2}t + A)^{-1}, \tag{27}$$

and

$$p_s = p_k\frac{\partial F}{\partial R} = \tfrac{1}{2}B^{-\frac{1}{2}}(\tfrac{1}{2}t + A)^{-1}\cdot\tfrac{1}{2}x^{\frac{1}{2}} = \tfrac{1}{4}B^{-\frac{1}{2}}. \tag{28}$$

To satisfy $H = 0$ at $t = T$, (7) yields, after some algebra, $B = -\frac{1}{2}(D/A)$. Together with the relation $K = Rx$, the remaining boundary conditions give three equations which determine A, D and T:

(i) $K = K_0$ at $t = 0$: $A^2D + 2C^*A = K_0$, (29)

(ii) $K = K_T$ at $t = T$: $\frac{D}{8A}(T + 2A)^3 + C^*(T + 2A) = K_T$, (30)

(iii) $S = 0$ at $t = T$: $-\frac{DT^2}{4A} - DT + S_0 - 4C^*\log\frac{T + 2A}{2A} = 0.$ (31)

Values of A, D and T satisfying these equations may be determined numerically.

We may also use equations (29)-(31) to determine an upper bound for the possible values of K_T. Clearly if K_T were considerably greater than K_0, then it would not be possible to control the economy at all, and so there would be no optimal solution. The solution fails explicitly when the shadow prices become zero, i.e. when $B = 0$, which implies $D = 0$. In this case, (29), (30) and (31) yield

$$2C^*A = K_0,$$

$$C^*(T + 2A) = K_T,$$

$$S_0 - 4C^*\log[(T + 2A)/2A] = 0,$$

with solutions $A = K_0/2C^*$, $T = (K_T - K_0)/C^*$ and $S_0/4C^* = \log(K_T/K_0)$.

Hence, the limiting value of K_T is given by

$$K_T = K_0 e^{S_0/4C^*}, \qquad (32)$$

and the corresponding time to reach this value is

$$T = \frac{K_0}{C^*}(e^{S_0/4C^*} - 1). \qquad (33)$$

Equation (32) shows that the maximum value of K_T is dependent on K_0, S_0 and C^*. The linear dependence on K_0 is expected, but the exponential dependence on S_0/C^* is more interesting. It means that the maximum value of K_T is very sensitive to the values of S_0 and C^*. A small increase in S_0 will provide a relatively large increase in K_T, whereas a small increase in C^* will give a relatively large decrease in K_T. The following table shows the dependence of K_T and the time T on the parameter $\lambda = S_0/4C^*$:

λ	0.5	1.0	1.5	2.0	3.0
K_T/K_0	1.65	2.72	4.48	7.39	20.09
TC^*/K_0	0.65	1.72	3.48	6.39	19.09

5. Numerical results

To solve the general problem of finding the optimal path which maximises (6) subject to the state equations (1) and (2) and boundary conditions (22) we use the "finite element" technique. The application of this technique to optimal control problems in economics with <u>fixed</u> end time is described in [6] and [7]. In the case of <u>free</u> end time, we transform the independent variable onto the fixed interval $[0, 1]$, and introduce T as a further state variable. Taking $\xi = t/T$, and denoting scaled variables by $\tilde{K}$, $\tilde{S}$, $\tilde{R}$ and $\tilde{C}$, the problem becomes

$$\max \int_0^1 e^{-\delta T \xi}(\tilde{C} - C^*)^{\beta} d\xi,$$

subject to

$$d\tilde{K}/d\xi = T(\tilde{K}^{\alpha}\tilde{R}^{1-\alpha} - \tilde{C}),$$

$$d\tilde{S}/d\xi = -T\tilde{R},$$

$$dT/d\xi = 0,$$

and boundary conditions

$$\tilde{K}(0) = K_0, \quad \tilde{S}(0) = S_0, \quad \tilde{K}(1) = K_T, \quad \tilde{S}(1) = 0.$$

This is now a fixed end time problem which is solved as in [6]. We observe that the transversality condition on the new adjoint variable associated with T is equivalent to the transversality condition $H = 0$ at $t = T$ in the free end time problem. Details of the transformation and numerical techniques appear in [7]. The solutions obtained in the special case $\alpha = \beta = \frac{1}{2}$, $\delta = 0$ are used to provide initial approximations for the general case.

The problem has been solved for a wide range of the parameters α, β, δ and C^* of the model and the boundary parameters K_0, S_0 and K_T. For the results illustrated in Figures 1 and 2, we have taken:

$$\beta = \tfrac{1}{2}, \quad C^* = 5, \quad \delta = 0.1, \quad K_0 = 1 \text{ unit}, \quad S_0 = 10 \text{ units}.$$

We have determined the optimal paths for various values of the terminal capital stock, K_T, and the production exponent, α. Figure 1 illustrates the variation in the optimal paths for stock, capital, consumption and depletion rates when $\alpha = 0.65$ and $K_T = 1.00, 0.75$ and 0.50. Figure 2 illustrates the same variables for fixed terminal capital, $K_T = 0.5$, and $\alpha = 0.5, 0.65$ and 0.85.

5. Interpretation of results

(i) Varying terminal capital stock

The most striking result is that varying the terminal capital stock, K_T, although not fundamentally altering the depletion paths, does give a substantial change in the terminal time, T. For example, decreasing the terminal capital from 1 to 0.5 more than doubles the time interval. Decreasing K_T means that more capital is available for transferance to consumption and so more time is allowed for consumption of the available resource. This, of course, means a slower depletion rate, R, for the resource stock, although the rate of change of R appears to have fairly constant level.

The results indicate that a knowledge of the capital level required at the changeover to an alternative technology makes it possible to plan the present consumption and resource depletion rates. The smaller the required level, K_T, the slower the resource depletion rate.

(ii) Varying production exponent

It is important to determine how robust the model is with respect to changes in the basic parameters. The most important parameter is α, the exponent of the capital in the production function. As in (i) the most striking result is the rapid change in the time horizon, T, as α changes. As the value of α becomes smaller more capital is available

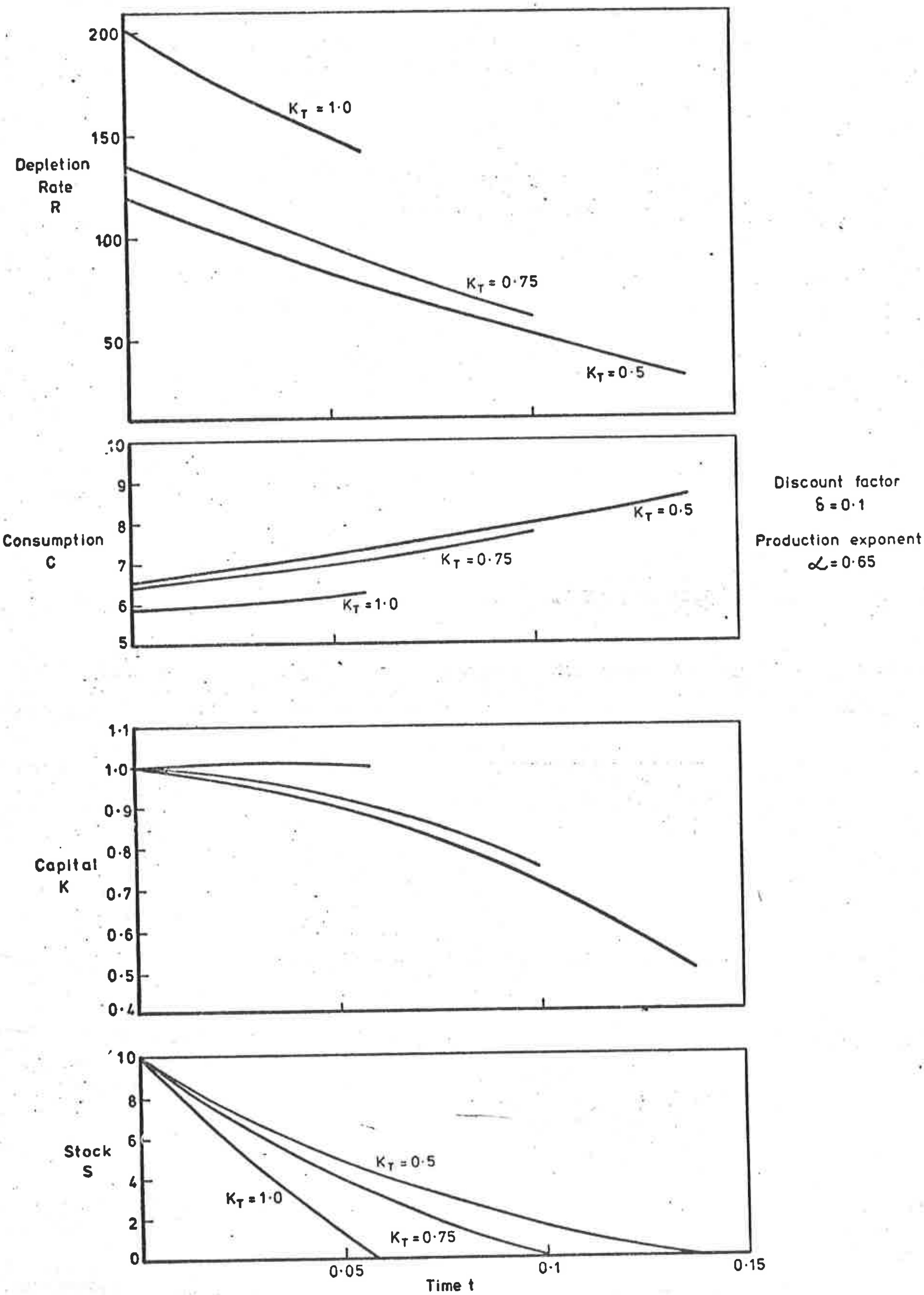
200
150
100
50
Depletion Rate R
K_T = 1·0
K_T = 0·75
K_T = 0·5
10
9
8
7
6
5
Consumption C
K_T = 0·5
K_T = 0·75
K_T = 1·0
Discount factor δ = 0·1
Production exponent α = 0·65
1·1
1·0
0·9
0·8
0·7
0·6
0·5
0·4
Capital K
10
8
6
4
2
0
Stock S
K_T = 0·5
K_T = 1·0
K_T = 0·75
0·05
0·1
0·15
Time t

Fig. 1.

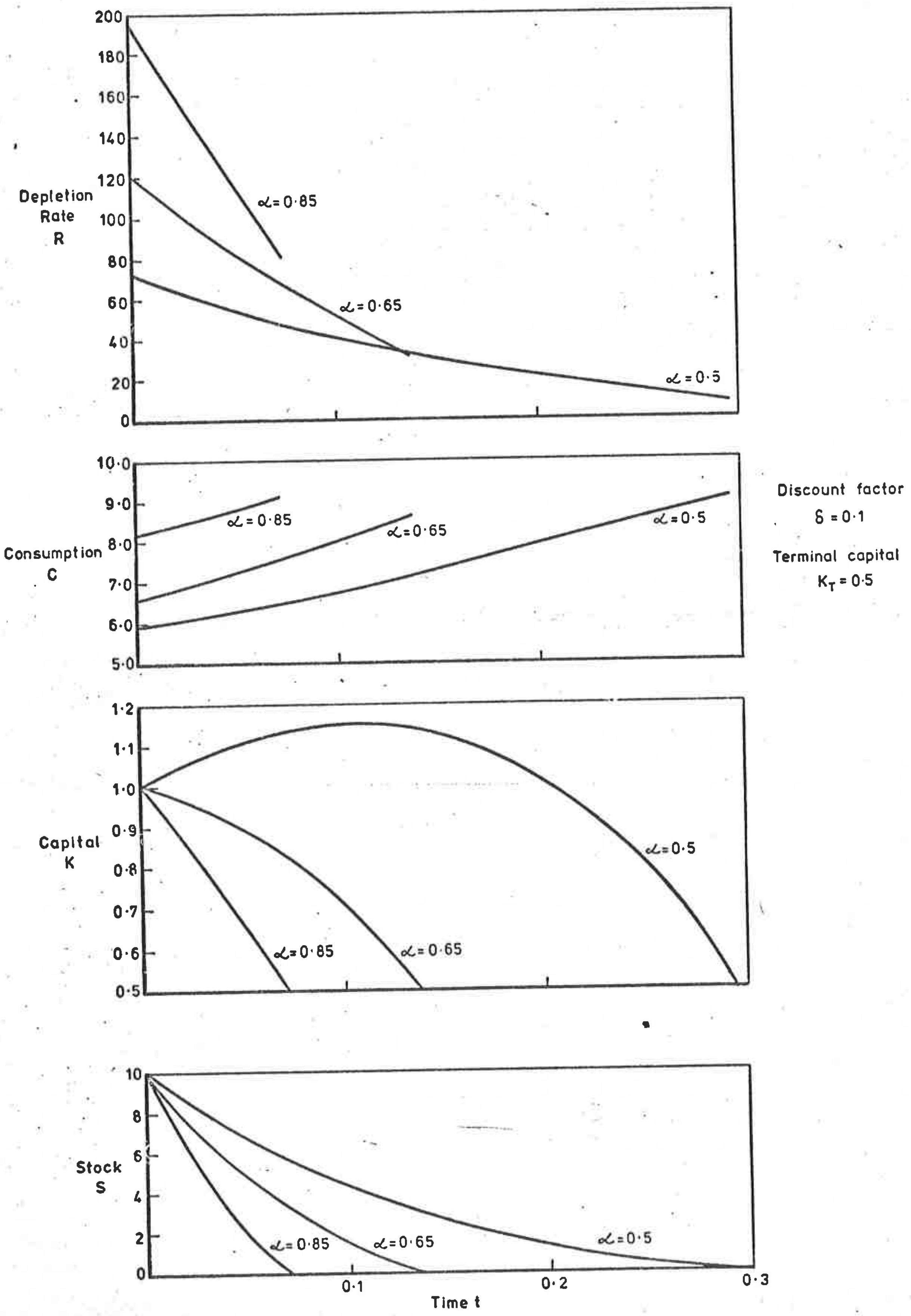
200
180
160
140
120
100
80
60
40
20
0
Depletion
Rate
R
α = 0·85
α = 0·65
α = 0·5
10·0
9·0
8·0
7·0
6·0
5·0
Consumption
C
α = 0·85
α = 0·65
α = 0·5
Discount factor
δ = 0·1
Terminal capital
K_T = 0·5
1·2
1·1
1·0
0·9
0·8
0·7
0·6
0·5
Capital
K
α = 0·5
α = 0·85
α = 0·65
10
8
6
4
2
0
Stock
S
α = 0·85
α = 0·65
α = 0·5
0·1
0·2
0·3
Time t

Fig. 2.

for transference to consumption, and so longer time is allowed for the depletion of the resource. For $\alpha = 0.5$, the capital stock is even able to increase initially (above the final prescribed value $K_T = 0.5$) before it is transferred to consumption. On the other hand, for $\alpha = 0.85$, the capital stock rapidly decreases to $K_T = 0.5$.

(iii) Varying discount factor

Another important parameter of the model is δ, the discount factor. Rather surprisingly, changing δ, where δ is small, causes little variation in the paths. For example, there are virtually no changes from the zero discount level solution when $\delta = 0.1$ and $\delta = 0.2$. One possible explanation for this is that the initial conditions on K and S give rise to relatively small values for the final time, T.

In conclusion, we stress that although changing α and K_T does not alter the form of the optimal solutions, it does cause rapid changes in the time horizon, T. The model is sensitive to change in the capital exponent, α, and terminal capital, K_T, whilst it is insensitive to changes in the discount factor, δ, provided it remains small.

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