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An Analysis of Properties of the Shock Tip in Two-Dimensional Steady Compressible Flow

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Numerical Analysis Report 7/88

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Contents

Preface

After many hours hard toil (especially by Mrs. Brigitte Calderon who kindly typed out my appalling notes) I here present my second numerical analysis report on the analytic behaviour of the weak end of shock waves.

I would like to express thanks to Dr. Mike Baines for supervising me and also to Dr. Peter Sweby, Prof. John Hunt and Prof. Bill Morton for the useful contribution that each of them provided.

Abstract

In my first report (see [1]), I analysed the qualitative structure of weak shock waves in one dimension. This second report is an attempt to tackle the problem of the "weak end" from several other angles.

The first two chapters of this report give the groundwork for the proof of the generalised shock slope conditions (similar to the Rankine Hugoniot Jump Conditions but in space not in space and time), the proof of which will appear in a later report. The fluid motion equations are derived in divergence form in the two-dimensional steady case. A sketch of the proof and its constructions is then given.

Chapter three shows how this result is consistent with the stationary Rankine-Hugoniot Jump Conditions, but has a better theoretical basis and expresses the information in a way more appropriate to this context. It also deals with the relationship between the shock and the sonic line.

On a different track, in the fourth chapter the shock tip curvature is categorised and modelled using a simple asymptotic result concerning the shock slope derivable from the Steady Jump Conditions.

Finally, in chapter five, three example models for a single conservation law with different geometry and analytical data type are analysed, looking at the asymptotic shock tip curvature and flux jump strength. It is hoped that these models will prove to be a suitable foundation for analytical models of greater complexity which have a more meaningful physical interpretation (e.g. multiple conservations laws, more realistic jacobian matrices and a domain with an inner boundary).

1. Derivation of the Fluid Motion Equations in Differential Divergence Form.

1.0 Background

No originality is claimed for this chapter, but no one reference found gives the equations in the same form. For a similar account, see [2].

The assumption is made that a suitably small volume of fluid $\delta V(t)$ (with boundary $\delta S(t)$ moving can be regarded as passing through successive states of thermodynamic equilibrium (see fig. 1). Therefore we can use thermodynamic variables without restriction provided suitably small fluid volumes are considered. The standard three conservation laws are now derived.

1.1 Conservation of Mass

The mass of δV is given by

$$\delta m = \int_{\delta V(t)} \rho dV \qquad \text{where } \rho \text{ is the fluid density.}$$

Conservation of mass $\Rightarrow \dfrac{d}{dt}\delta m = 0$

$$\Rightarrow \frac{d}{dt}\int_{\delta V(t)} \rho dV = 0$$

$$\Rightarrow \int_{\delta V(t)} \frac{\partial \rho}{\partial t} dV + \int_{\delta S(t)} \rho \underline{q}.d\underline{S} = 0$$

where $\underline{q}$ is the fluid velocity.

$$\Rightarrow \int_{\delta V(t)} \left[\frac{\partial \rho}{\partial t} + \nabla.(\rho \underline{q})\right] dV = 0 \qquad \text{by the divergence theorem.}$$

Hence by taking $\delta V(t) \to 0$:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \underline{q}) = 0 \qquad \text{at all points in the flow, at all times,}$$

or $\rho,_t + (\rho q_i),_i = 0$ in compact notation. (1.1)

1.2 Conservation of Momentum

For the sake of brevity, the explicit time dependence of δV and δS is dropped henceforth.

The conservation law here is:

force on δV = rate of change of momentum of δV,

where, force on $\delta V = \int_{\delta S} \underline{\tau}\, dS$, where $\underline{\tau}$ is the traction.

Cauchy's relation gives $\underline{\tau} = \underline{\underline{\sigma}}\,\underline{n}$ in the absence of a couple, where $\underline{\underline{\sigma}}$ is the stress tensor and $\underline{n}$ is the surface normal.

Therefore

$$\text{force on } \delta V = \int_{\delta S} \underline{\underline{\sigma}}.\underline{n}\, dS = \underline{F} \quad \text{say,}$$

so

$$F_i = \int_{\delta S} \sigma_{ij}\, n_j\, dS$$

$$= \int_{\delta V} \sigma_{ij},_j\, dV.$$

$$\text{Momentum of } \delta V = \int_{\delta V} \rho \underline{q}\, dV = \underline{I} \quad \text{say.}$$

$$\text{Rate of change of momentum} = \frac{d}{dt}\int_{\delta V} \rho \underline{q}\, dV$$

$$= \int_{\delta V} (\rho \underline{q}),_t dV + \int_{\delta V} (\rho \underline{q})\, \underline{q}.\underline{n}\, dS$$

$$= \dot{\underline{I}}$$

therefore

$$\dot{I}_i = \int_{\delta V} (\rho q_i),_t dV + \int_{\delta S} \rho q_i q_j \, n_j \, dS$$

$$= \int_{\delta V} (\rho q_i),_t dV + \int_{\delta V} (\rho q_i q_j),_j \, dV$$

But, by the conservation law, $\dot{I}_i = F_i$.

Hence $$\int_{\delta V} \left[(\rho q_i),_t + (\rho q_i q_j),_j\right] dV = \int_{\delta v} \sigma_{ij},_j \, dV$$

So, again in the limit $\delta V \rightarrow 0$ we get

$$(\rho q_i),_t + (\rho q_i q_j - \sigma_{ij}),_j = 0 \tag{1.2}$$

Now, by elasticity theory,

$$\sigma_{ij} = - P\delta_{ij} + 2\mu(e_{ij} - \frac{1}{3}q_{k,k} \, \delta_{ij}), \tag{1.3}$$

where

$$e_{ij} = \frac{1}{2}(q_{i,j} + q_{j,i}) \tag{1.4}$$

is the rate of strain tensor, and P is the pressure.

1.3 Conservation of Energy

The conservation law for energy is:

rate of change of total energy = energy gained by surface tractions

+ energy gained by heat conduction.

Total energy $= \int_{\delta V} \rho(U + \frac{1}{2}q^2) \, dV$ where U is the internal energy/unit mass.

Let $E = U + \frac{1}{2}q^2$ = total energy/unit mass.

Then total energy $= \int_{\delta V} \rho E dV$.

But, energy gained by surface tractions
$$= \int_{\delta S} \underline{T}.\underline{q}\ dS$$
$$= \int_{\delta S} \sigma_{ij} n_j q_i dS$$
$$= \int_{\delta V} (\sigma_{ij} q_i),_j\ dV,$$

and energy gained by heat conduction
$$= \int_{\delta S} \kappa \nabla T.\underline{n}\ \underline{dS}$$
$$= \int_{\delta V} \nabla.(\kappa \nabla T)\ dV$$
$$= \int_{\delta V} (\kappa T,_i),_i\ dV\ ,$$

therefore

$$\frac{d}{dt}\int_{\delta V} \rho E\ dV = \int_{\delta V} \left[(\sigma_{ij} q_i),_j + (\kappa T,_i),_i\right] dV$$

$$\frac{d}{dt}\int_{\delta V} \rho E\ dV = \int_{\delta V} (\rho E),_t\ dV + \int_{\delta S} \rho E\ q_i n_i dS$$

$$= \int_{\delta V} \left[(\rho E),_t + (\rho E q_i),_i\right] dV.$$

Hence

$$\int_{\delta V} \left[(\rho E),_t + (\rho E q_j - \sigma_{ij} q_i - kT,_j),_j\right] dV = 0.$$

So, again in the limit $\delta V \to 0$ we obtain:

$$(\rho E),_t + (\rho E q_j - \sigma_{ij} q_i - kT,_j),_j = 0 \tag{1.5}$$

For an ideal gas, $U = \dfrac{P}{(\gamma-1)\rho}$ and $T = \dfrac{P}{\rho R}$

Therefore

$$\rho E = \frac{P}{(\gamma-1)} + \frac{1}{2}\rho q^2 . \tag{1.6}$$

2. Shock Slope Conditions for the Viscous, Conductive Limit of the Fluid Equations for the Two-Dimensional Steady System

2.0 Overview

Initially, the fluid equations from §1 need to be reduced to those for the two-dimensional steady system. It is then shown that these equations can be written in a simple divergence form with the fluxes each consisting of a viscous, inviscid and thermal component.

The shock slope theorem is then assumed and its relevant applications to the full two-dimensional steady system are derived.

2.1 Derivation of the Two-Dimensional Steady Conservation Laws

For 2D flow, let $\underline{q} = (u,v)$ $\underline{x} = (x,y)$. As the flow is steady, we also have $\partial/\partial t \equiv 0$. So, (1.1) $\Rightarrow$

$$(\rho u)_{,x} + (\rho v)_{,y} = 0. \tag{2.1}$$

$$(1.2) \Rightarrow \begin{cases} (\rho u^2 - \sigma_{xx})_{,x} + (\rho uv - \sigma_{xy})_{,y} = 0 \\ \\ (\rho uv - \sigma_{yx})_{,x} + (\rho v^2 - \sigma_{yy})_{,y} = 0 \end{cases}$$

where $\sigma_{xx} = -p + 2\mu\left[u_{,x} - \frac{1}{3}(u_{,x} + v_{,y})\right]$,

$\sigma_{xy} = \sigma_{yx} = \mu(u_{,y} + v_{,x})$,

$\sigma_{yy} = -p + 2\mu\left[v_{,y} - \frac{1}{3}(u_{,x} + v_{,y})\right]$ from (1.3) and (1.4).

Hence we obtain:

$$\left.\begin{aligned} \left[p + \rho u^2 - \tfrac{2}{3}\mu\left(2u,_x - v,_y\right)\right],_x + \left[\rho uv - \mu\left(u,_y + v,_x\right)\right],_y &= 0 \\ \left[\rho uv - \mu\left(u,_y + v,_x\right)\right],_x + \left[p + \rho v^2 - \tfrac{2}{3}\mu\left(2v,_y - u,_x\right)\right],_y &= 0 \end{aligned}\right\} \quad (2.2)$$

$$(1.5) \Rightarrow \left(\rho Eu - \sigma_{xx}u - \sigma_{yx}v - \kappa T,_x\right),_x + \left(\rho Ev - \sigma_{xy}u - \sigma_{yy}v - \kappa T,_y\right),_y = 0.$$

$$\Rightarrow \left[\rho u(p + \rho E) - \tfrac{2}{3}\mu v\left(2u,_x - v,_y\right) - \mu v\left(u,_y + v,_x\right) - \kappa T,_x\right],_x$$

$$+ \left[\rho v(p + \rho E) - \mu u\left(u,_y + v,_x\right) - \tfrac{2}{3}\mu v\left(2v,_y - u,_x\right) - \kappa T,_y\right],_y = 0 \quad (2.3)$$

Dropping the comma notation for derivatives, (2.1), (2.2) and (2.3) can now be combined in the form:

$$\underline{F}_x + \underline{G}_y = \underline{0} \quad (2.4)$$

where

$$\underline{F} = \begin{bmatrix} \rho u \\ P+\rho u^2 \\ \rho uv \\ u(p+\rho E) \end{bmatrix} - \mu \begin{bmatrix} 0 \\ \tfrac{2}{3}\left(2u_x - v_y\right) \\ u_y + v_x \\ \tfrac{2}{3}u\left(2u_x - v_y\right) + v\left(u_y + v_x\right) \end{bmatrix} - \kappa \begin{bmatrix} 0 \\ 0 \\ 0 \\ T_x \end{bmatrix} \quad (2.5)$$

and

$$\underline{G} = \begin{bmatrix} \rho v \\ \rho uv \\ P+\rho v^2 \\ v(p+\rho E) \end{bmatrix} - \mu \begin{bmatrix} 0 \\ u_y + v_x \\ \tfrac{2}{3}\left(2v_y - u_x\right) \\ u\left(u_y + v_x\right) + \tfrac{2}{3}v\left(2v_y - u_x\right) \end{bmatrix} - \kappa \begin{bmatrix} 0 \\ 0 \\ 0 \\ T_y \end{bmatrix} \quad (2.6)$$

2.2. Statement of the Shock Slope Theorem

Define the n^{th} system, Σ_n, by:

a) A fixed finite domain $\Delta \subset \mathbb{R}^2$

b) A vector $\underline{\omega}_n$ obeying

$$\nabla . \underline{\omega}_n = 0 \qquad \forall \; \underline{x} \in \Delta,$$

and having the same boundary conditions on $\partial\Delta$ for all n.

c) A general domain $D_n \subseteq \Delta$

d) A general viscous shock centre portion Γ_n, in D_n, see fig. 2.

e) The set of functions $C_0^1(D_n)$, which are zero on ∂D_n and C^1 in D_n.

f) The jump function $[\cdot]_n$.

Suppose, as $n \to \infty$, Σ_n tends uniformly to the viscous system Σ_∞, in some sense. So, for consistency, assume Γ_n becomes a portion of inviscid shock and $[\cdot]_n$ tends uniformly to the jump function $[\cdot]$.

Suppose also that general curves Γ_n and functions ϕ_n in $C_0^1(D_n)$ can be identified as n changes.

The shock slope theorem then states:

$\exists([\cdot]_n)$ such that $\forall \; \Gamma_n$ under certain constrictions, $\forall \; \phi_n \in C_0^1(D_n)$ identified in a certain way (giving uniform continuity to the inviscid system),

$$\lim_{n \to \infty} \int_{\Gamma_n} [\underline{\omega}_n \phi_n]_n . \underline{dr} = 0. \tag{2.7}$$

2.3. Application of the Shock Slope Theorem to the 2D Steady System

Putting $n = \infty$ in (2.7) we obtain

$$\int_{\Gamma_\infty} [\underline{\omega}_\infty \phi_\infty] . \underline{dr} = 0.$$

But ϕ_∞ is C^1 on Γ_∞. Thus

$$\int_{\Gamma_\infty} \phi_\infty [\underline{\omega}_\infty] . \underline{dr} = 0.$$

Then by letting ϕ_∞ be localised to an arbitrary point on Γ_∞, we obtain

$$\forall\ \xi \in \cup\, \Gamma_\infty\ , \quad [\underline{\omega}_\infty] . \underline{\nu} = 0\ , \tag{2.8}$$

where ν is the normal at ξ and $\cup\, \Gamma_\infty$ is the union of shock portions satisfying the constraint of the theorem.

Now, if we let $\underline{\omega}_n = (F_i^{(n)}, G_i^{(n)})$ for some i, where $F_i^{(n)}, G_i^{(n)}$ are the same fluxes as before, except that μ, κ, have been replaced by μ_n, κ_n, then, if the sequences of μ_n and κ_n are chosen suitably so that they obey the uniformity conditions of the theorem, (2.8) gives

$$[(F_i^{(\infty)}, G_i^{(\infty)})] . \underline{\nu} = 0$$

where $\underline{F}^{(\infty)}, \underline{G}^{(\infty)}$ are the inviscid conductive components of $\underline{F}$ and $\underline{G}$. If we then let the shock at ξ have slope θ,

$$\underline{\nu} = \pm(\sin\theta, -\cos\theta).$$

Hence
$$\sin\theta[F_i^{(\infty)}] = \cos\theta[G_i^{(\infty)}]$$

i.e.
$$\tan\theta = \frac{[G_i^{(\infty)}]}{[F_i^{(\infty)}]}\ , \tag{2.9}$$

from whence the name "shock slope theorem" is taken. It will hopefully turn out that $\cup\Gamma_{\infty}$ is all points on shock-waves apart from shock interactions, kinks and tips.

3. Further Results from the Shock Slope Conditions.

3.1 The Two-Dimensional Steady Rankine-Hugoniot Jump Conditions.

Consider any point on a shock wave where $\theta = 0$; the shock slope theorem then yields

$$[G_i^{(\infty)}] = 0 \qquad i = 1, \ldots, 4 .$$

Hence by considering a frame rotation to align the x-axis with the shock tangent, we will always have

$$[G_i^{\prime(\infty)}] = 0 \qquad i = 1, \ldots, 4 ,$$

where G_i' is the flux in the new co-ordinate system which only differs from $\underset{\sim}{G}$ by replacing (u,v) by (q_n, q_t), where n stands for normal and t stands for tangential (see fig. 3). Thus we have

$$\begin{aligned} [\rho q_n] &= 0 \\ [\rho q_n q_t] &= 0 \\ [p+\rho q_n^2] &= 0 \\ [q_n(p+\rho E)] &= 0 . \end{aligned}$$

Denoting the upstream side of the shock by the suffix 0, and the downstream side by the suffix 1,

$$\rho_1 q_{n1} - \rho_0 q_{n0} = 0 \tag{3.1}$$

$$\rho_1 q_{t1} q_{n1} - \rho_0 q_{t0} q_{n0} = 0$$

$$p_1 + \rho_1 q_{n1}^2 - (p_0 + \rho_0 q_{n0}^2) = 0 \tag{3.2}$$

$$q_{n1}(p_1 + \rho_1 E_1) - q_{n0}(p_0 + \rho_0 E_0) = 0 .$$

The first two equations give

$$q_{t0} = q_{t1} . \tag{3.3}$$

The last equation is (for an ideal gas)

$$q_{n1}p_1 + q_{n1}\left[\frac{p_1}{(\gamma-1)} + \frac{1}{2}\rho_1 q_1^2\right] = q_{n0}p_0 + q_{n0}\left[\frac{p_0}{\gamma-1} + \frac{1}{2}\rho_0 q_0^2\right]$$

using (1.6)

$$\Rightarrow \quad \frac{\gamma q_{n1}p_1}{(\gamma-1)} + \frac{1}{2}\rho_1 q_{n1}^3 = \frac{\gamma q_{n0}p_0}{(\gamma-1)} + \frac{1}{2}\rho_0 q_{n0}^3 \qquad (3.4)$$

as $\rho_1 q_{n1} q_{t1}^2 = \rho_0 q_{n0} q_{t0}^3$ from (3.1) and (3.3).

Equations (3.1),...,(3.4) we know are the one-dimensional Rankine-Hugoniot jump conditions with velocity replaced by normal velocity along with the condition of conservation of tangential velocity.

By suitable choice of the initial frame of reference, these conditions could have been obtained for every point on the shock.

3.2 The Flow Speed Near the Shock Tip.

The sound speed for an ideal gas is

$$a = \left[\frac{\gamma p}{\rho}\right]^{\frac{1}{2}} . \qquad (3.5)$$

Let $\underline{M} = \underline{q}/a$ - the mach velocity . (3.6)

At a shock $\mathscr{S}$, let $\underline{M} = (M_n, M_t)$, see fig. 3.

For one dimensional flow it is possible to show that

$$\left.\begin{aligned} M_0^2 &= \frac{\gamma+1}{2\gamma}\frac{p_1}{p_0} + \frac{\gamma-1}{2\gamma} \\ M_1^2 &= \frac{\gamma+1}{2\gamma}\frac{p_0}{p_1} + \frac{\gamma-1}{2\gamma} \end{aligned}\right\} , \qquad (3.7)$$

where $M = q_n/a$.

So for a compressive (usual) shock, $(p_1/p_0 \geq 1)$,

$$M_0^2 \geq 1$$

$$M_1^2 \leq 1 .$$

Now, for a two-dimensional flow, by the argument in Section 3.1, we have

$$\left.\begin{aligned} M_{n0}^2 &= \frac{\gamma+1}{2\gamma}\frac{p_1}{p_0} + \frac{\gamma-1}{2\gamma} \\ M_{n1}^2 &= \frac{\gamma+1}{2\gamma}\frac{p_0}{p_1} + \frac{\gamma-1}{2\gamma} \\ q_{t0} &= q_{t1} \end{aligned}\right\} . \qquad (3.8)$$

Clearly, $M_0^2 \geq M_{n0}^2 \geq 1$ as $(p_1 \geq p_0)$ as before.

But $M_1^2 = M_{n1}^2 + M_{t1}^2$

$$= \left(\frac{\partial + 1}{2\gamma}\right)\frac{p_0}{p_1} + \frac{\gamma - 1}{2\gamma} + M_{t1}^2$$

Thus $M_1^2 \leq 1$ if and only if

$$M_{t1}^2 \leq 1 - \frac{\gamma - 1}{2\gamma} - \left(\frac{\gamma + 1}{2\gamma}\right)\frac{p_0}{p_1}$$

i.e. $$M_{t1}^2 \leq \left(\frac{\gamma + 1}{2\gamma}\right)\left(1 - \frac{p_0}{p_1}\right) \qquad (3.9)$$

If this condition is violated at a point before the shock tip, the shock will definitely deviate from the sonic line. There is no reason within this theory why it should not re-join the sonic line later on. A necessary condition for the shock to remain on the sonic line is that $M_{t1} = 0$ at the shock tip. This seems to be a rather contrived and unlikely circumstance. So the most likely behaviour is for the shock to leave the sonic line and not re-join it, as in figure 4.

Alternatively, for a shock to remain on the sonic line, the streamline at the tip must be normal to the sonic line there; and at successive positions near the tip, they may only deviate from this orthogonal direction by an amount corresponding to (3.9) (see [3]).

4. Results Concerning Shock Curvature

4.0 Overview

Let ξ be the arc length measured from the tip of the shock. Let $\rho(\xi)$ be the radius of curvature of the shock corresponding to ξ (see fig 5). At this point, the jump quantities take on certain values ($[u(\xi)]$, etc.). The object of this chapter is to calculate the asymptotic behaviour of $\rho(\xi)$ in the form:

$$\rho(\xi) \sim f([\underline{q}(\xi)]\ ,\ [\nabla\underline{q}(\xi)]\ ,\ [\nabla\nabla\underline{q}(\xi)]\ ,\ \ldots) \qquad (4.1)$$

where f is some function to be determined and $\sim$ has the meaning:

$$\rho(\xi) \sim f(\xi) \Longleftrightarrow \lim_{\xi\to 0} \frac{\rho(\xi)}{f(\xi)} = 1 \qquad (4.2)$$

This has proved to be a somewhat recursive problem because the asymptotic behaviour of $\rho(\xi)$ will dictate the model required to model it. It has been discovered that the three generic cases:

i) $\rho(0) = 0$

ii) $0 < |\rho(0)| < \infty$

iii) $|\rho(0)| = \infty$,

need to be treated independently and a different model needs to be constructed for each.

The major objective is to come up with asymptotic equations of the form (4.1) which by virtue of their behaviour will enable one to distinguish which of the three generic cases matches a given shock.

Of the three cases, the most difficult to model and to distinguish is the first case, $\rho(0) = 0$.

4.1 Preliminary Results

All the models in this chapter make use of the result (see §3.1)

$$[\underline{q}.\underline{e}_\xi] = 0 , \qquad (4.3)$$

where ξ is the shock length parameter. If the shock is parametrised by

$$y = Y(x) ,$$

then it can be shown that

$$\underline{e}_\xi = (1 + Y'^2)^{-\frac{1}{2}}(1, Y') . \qquad (4.4)$$

So, in particular,

$$\frac{\partial}{\partial\xi} \equiv \underline{e}_\xi.\nabla \equiv (1 + Y'^2)^{-\frac{1}{2}}\left[\frac{\partial}{\partial x} + Y' \frac{\partial}{\partial y} \right] \qquad (4.5)$$

(4.3) gives immediately,

$$[u + Y'v] = 0$$

$$\Rightarrow \qquad Y' = - \frac{[u]}{[v]}. \qquad (4.6)$$

Also, since (4.3) is true for all ξ, it may be differentiated with respect to ξ (corresponding to along the shock) to give

$$[u_x + Y'(u_y + v_x) + Y''v + Y'^2v_y] = 0$$

$$\Rightarrow \qquad Y'' = - \frac{[u_x + Y'(u_y + v_x) + Y'^2v_y]}{[v]} . \qquad (4.7)$$

Now, of course, §2.2 shows us that

$$Y' = \frac{[G_i]}{[F_i]} \quad \forall\xi\forall i ,$$

but this result was considered too complicated to use here; it may of course be used and differentiated as above to give Y''.

Differential geometry gives the well-known result:

$$\rho = -\frac{[1 + Y'^2]^{3/2}}{Y''} . \tag{4.8}$$

Also, if the shock is parametrised by $r = R(\theta)$, then

$$\rho = \frac{[R^2 + R'^2]^{3/2}}{R^2 + 2R'^2 - RR^2} . \tag{4.9}$$

Taking the shock tip to be at the pole, then

$$R(0) = 0 . \tag{4.10}$$

The most general analysable assumption found so far is to take

$$\left.\begin{array}{l} R(\theta) = k\theta^{\alpha} + O(\theta^{\beta}) \quad , \\ \text{where } \alpha > 0 , \quad \beta > \alpha, \quad k \neq 0, \quad \alpha,\beta \in \mathbb{R} \end{array}\right\} \tag{4.11}$$

Then,

$$R'(\theta) = \alpha k\theta^{\alpha-1} + O(\theta^{\beta-1})$$

$$R''(\theta) = \alpha(\alpha-1)R\theta^{\alpha-2} + O(\theta^{\beta-2})$$

So $$R'(\theta)^2 = \alpha^2k^2\theta^{2(\alpha-1)} + O(\theta^{\alpha+\beta-2})$$

and $$R(\theta)R''(\theta) = \alpha(\alpha-1)k^2\theta^{2(\alpha-1)} + O(\theta^{\alpha+\beta-2}) .$$

It is necessary now to distinguish the two cases $\alpha = 1$ and $\alpha \neq 1$.

Let $\gamma = \min\{2\alpha, \alpha + \beta - 2\}$. Since $R(\theta)^2 = O(\theta^{2\alpha})$,

$$\rho = \frac{\alpha^3k^3\theta^{3(\alpha-1)}\left[1 + O(\theta^{\alpha-2(\alpha-1)})\right]}{\left[2\alpha^2k^2 - \alpha(\alpha-1)k^2\right]\theta^{2(\alpha-1)} + O(\theta^{\gamma})} \qquad \alpha \neq 1$$

$$= \frac{k\alpha^2}{\alpha+1}\,\theta^{(\alpha-1)}\left[1 + O(\theta^{\alpha-2(\alpha-1)})\right] \tag{4.12}$$

If $\alpha = 1$, $R''(\theta) = 0$, and

$$\rho = \frac{[k^2 + O(\theta^\gamma)]^{3/2}}{2k^2 + O(\theta^\gamma)}$$

$$= \frac{k}{2} + O(\theta^\gamma) \quad , \quad \gamma = \min\{2, \beta-1\} \tag{4.13}$$

So, (4.12) $\Rightarrow \lim_{\theta \to 0} \rho = \infty$ for $\alpha < 1$

and $\lim_{\theta \to 0} \rho = 0$ for $\alpha > 1$ (4.14)

And (4.13) $\Rightarrow \lim_{\theta \to 0} \rho = \frac{k}{2}$ for $\alpha = 1$

Now, let $\lim_{\theta \to 0} \rho = \rho_0 \left[= \lim_{\xi \to 0} \rho \right]$.

The polar-cartesian conversion equations for the shock are:

$$\left.\begin{aligned} x &= R(\theta)\cos\theta \\ Y(x) &= -R(\theta)\sin\theta \end{aligned}\right\} \tag{4.15}$$

(see fig. 5).

Hence, $$Y' = \frac{dY}{dx} = \frac{dY}{d\theta}\frac{d\theta}{dx}$$

$$= \frac{dY}{d\theta} \Big/ \frac{dx}{d\theta}$$

$$\frac{dY}{d\theta} = -(R'\sin\theta + R\cos\theta)$$

$$\frac{dx}{d\theta} = R'\cos\theta - R\sin\theta \; .$$

Thus, $$Y' = -\frac{R'\sin\theta + R\cos\theta}{R'\cos\theta - R\sin\theta} \; . \tag{4.16}$$

Also, $$Y'' = \frac{dx'}{d\theta} \Big/ \frac{dx}{d\theta}$$

$$= \frac{dY'/d\theta}{R'\cos\theta - R\sin\theta} \tag{4.17}$$

4.2 First Case: Zero Shock Tip Curvature - $\rho_0 = 0$

Consistent with §4.1, assume

$$R(\theta) = k\theta^{\alpha} + 0(\theta^{\beta}) \ , \quad \alpha > 1 \ , \ \beta > \alpha \ , \quad \alpha,\beta \in \mathbb{R} \ , \quad k \neq 0 \ .$$

Then, as before, from (4.12),

$$\rho(\theta) = \frac{k\alpha^2}{\alpha H}\theta^{(\alpha-1)}\left[1 + 0(\theta^{\delta})\right] .$$

Where $\delta = \min\{2, \beta - \alpha\} \implies \delta > 0$.

So (4.16) $\implies$

$$Y' = -\frac{k\alpha\theta^{\alpha} + k\theta^{\alpha} + 0(\theta^{\delta+\alpha})}{k\alpha\theta^{\alpha-1} + 0\left[\theta^{\delta+(\alpha-1)}\right]}$$

$$= -\frac{(\alpha+1)}{\alpha}\theta + 0\left[\theta^{\delta+1}\right] \qquad (4.18)$$

And (4.17) $\implies$

$$Y'' = \left\{-\frac{(\alpha+1)}{\alpha} + 0(\theta^{\delta})\right\}\left\{k\alpha\theta^{\alpha-1} + 0\left[\theta^{\delta+\alpha-1}\right]\right\}^{-1}$$

$$= -\frac{(\alpha+1)}{\alpha^2}\theta^{-(\alpha-1)}(1 + 0(\theta^{\delta})) \qquad (4.19)$$

Differential geometry also gives

$$\xi = \int_0^{\theta} [R(\phi)^2 + R'(\phi)^2]^{1/2}\, d\phi$$

$$= \int_0^{\theta} \{k\alpha\phi^{\alpha-1} + 0(\phi^{\alpha-1+\delta})\}\, d\phi$$

$$= k\theta^{\alpha} + 0(\theta^{\alpha+\delta}) \ .$$

This is correct as $\alpha-1 > 0$ and $\delta > 0$ from above. So $\alpha-1+\delta > 0$.

$$\implies \theta^{\alpha} = \frac{\xi}{k} + 0(\theta^{\alpha+\delta})$$

$$\implies \theta = \left[\frac{\xi}{k}\right]^{1/\alpha} + 0(\theta^{\alpha+\delta})$$

$$= \left[\frac{\xi}{k}\right]^{1/\alpha} + 0\left[\xi^{\frac{\alpha+\delta}{\alpha}}\right] \qquad (4.20)$$

Hence,
$$\rho(\xi) \sim \frac{k\alpha^2}{\alpha+1}\left[\frac{\xi}{k}\right]^{\frac{\alpha-1}{\alpha}} = \frac{\alpha^2}{\alpha+1}\, k^{1/\alpha}\, \xi^{\frac{\alpha-1}{\alpha}} \qquad (4.21)$$

Now
$$\frac{d}{d\theta} \equiv \frac{dx}{d\theta}\frac{\partial}{\partial x} + \frac{dY}{d\theta}\frac{\partial}{\partial y} \quad \text{along the shock}$$

$$\equiv \left\{k\alpha\theta^{\alpha-1} + O(\theta^{\delta+\alpha-1})\right\}\frac{\partial}{\partial x} - \left\{k(\alpha+1)\theta^{\alpha} + O(\theta^{\delta+\alpha})\right\}\frac{\partial}{\partial y} \qquad (4.22)$$

$$[f(\theta)] = \left[f(0) + \theta\frac{d}{d\theta}f\Big|_{\theta=0} + \frac{\theta^2}{2!}\frac{d^2}{d\theta^2}f\Big|_{\theta=0} + O(\theta^3)\right]$$

But (4.22) $\Rightarrow$ $\dfrac{d^n}{d\theta^n}f\Big|_{\theta=0} = 0$ as $\alpha > 1$ (so all the θ powers are zero at $\theta = 0$).

Hence it is not possible to calculate Taylor expansions for $Y'(\theta)$ or $Y''(\theta)$ about $\theta = 0$ from equations (4.6) and (4.7). Thus $\rho(\xi)$ cannot be derived asymptotically in the form (4.1).

4.3 Second Case: Finite Shock Tip Curvature – $0 < |\rho_0| < \infty$

This case is modelled in two ways; firstly with intrinsic shock parametrisation (ξ), and secondly with polar shock parametrisation (R,θ). In the first model, it was found convenient to use both a single cartesian reference frame orientated with the shock tip and a sequence of cartesian frames orientated at continually decreasing angles to the fixed frame of reference. Two Taylor expansions were then instigated; one to move from away from the shock tip to the tip, and another from the rotated frame to the fixed frame.

The second model is altogether simpler and the model assumptions are equivalent.

4.3.1 The First Model - A sequence of Cartesian Frames

Let (x,y) be the fixed frame and let $(x^{(n)},y^{(n)})$ be the sequence of rotated frames, inclined at angles θ_n to (x,y), (see Fig. 6).

Let $\xi = \xi_n$ when the shock has curved through an angle θ_n. Let $(u^{(n)},v^{(n)})$ be the fluid velocity in the frame $(x^{(n)},y^{(n)})$ and let Y'_n be the corresponding shock slope. Define the sequence of frames so that ξ_n is monotonic in n and ξ_n tends to the shock tip as $n \to \infty$.

Under the assumption that $\rho(\xi_n)$ varies linearly and $\rho_o \neq 0$, we obtain

$$\theta_n = \frac{\xi_n}{\rho_o} + O(\xi^2{}_n). \qquad (4.23)$$

It is also easy to show that

$$\left.\begin{aligned}
&\begin{bmatrix} u^{(n)} \\ v^{(n)} \end{bmatrix} = R_n \begin{bmatrix} u \\ v \end{bmatrix} \\
&\begin{bmatrix} \partial/\partial x^{(n)} \\ \partial/\partial y^{(n)} \end{bmatrix} \equiv R_n \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix}, \\
&\text{where} \\
&R_n = \begin{bmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{bmatrix}
\end{aligned}\right\} \qquad (4.24)$$

As $\rho(\xi)$ is frame invariant, (4.8) also holds in frame $(x^{(n)},y^{(n)})$ $\forall n$, so

$$\rho(\xi) = -\frac{[1+Y'^2_n]^{3/2}}{Y''_n} \qquad (4.25)$$

and by the construction of $(x^{(n)},y^{(n)})$ and ξ_n,

$$Y'_n(\xi_n) = 0. \qquad (4.26)$$

Hence $$\rho(\xi_n) = -\frac{1}{Y''(\xi_n)}. \tag{4.27}$$

(4.7), (4.26) $\Rightarrow$ $$Y''_n(\xi_n) = -\frac{\left[\frac{\partial u^{(n)}}{\partial x^{(n)}}(\xi_n)\right]}{\left[v^{(n)}(\xi_n)\right]}$$

So, from (4.27),

$$\rho(\xi_n) = \frac{[v^{(n)}(\xi_n)]}{\left[\frac{\partial u^{(n)}}{\partial x^{(n)}}(\xi_n)\right]} \tag{4.28}$$

The objective now is to find an asymptotic expansion for $\rho(\xi_n)$ where the terms written down may be assumed $O(1)$. This is done by initially taking the two Taylor expansions mentioned at the beginning of §4.3.

The initial result:

$$[\underline{q}(0)] = \underline{0} \tag{4.29}$$

must hold by the definition of the shock tip.

Performing a Taylor expansion of $v^{(n)}(\xi_n)$ about $\xi = 0$:

$$v^{(n)}(\xi_n) = v^{(n)}(0) + \xi_n \frac{\partial v^{(n)}}{\partial \xi}\bigg|_{\xi=0} + O(\xi^2{}_n).$$

From (4.5),

$$\frac{\partial v^{(n)}}{\partial \xi} = (1 + Y'^2(\xi))^{-\frac{1}{2}} \left[\frac{\partial v^{(n)}}{\partial x}(\xi) + Y'(\xi)\frac{\partial v^{(n)}}{\partial y}(\xi)\right]$$

Therefore,

$$\frac{\partial v^{(n)}}{\partial \xi}\bigg|_{\xi=0} = \frac{\partial v^{(n)}}{\partial x}(0),$$

and so, $$v^{(n)}(\xi_n) = v^{(n)}(0) + \xi_n \frac{\partial v^{(n)}}{\partial x}(0) + O(\xi^2{}_n). \tag{4.30}$$

Also, (4.24) $\Rightarrow$ $$\frac{\partial u^{(n)}}{\partial x^{(n)}} = (\cos\theta_n \frac{\partial}{\partial x} - \sin\theta_n \frac{\partial}{\partial y})(u\cos\theta_n - v\sin\theta_n)$$

θ_n is a constant, so

$$\frac{\partial u^{(u)}}{\partial x^{(u)}} = \cos^2\theta_n U_x - \sin\theta_n \cos\theta_n(u_y + v_x) + \sin^2\theta_n v_y.$$

Now, from (4.23),

$$\cos\theta_n = 1 + O(\xi_n^2)$$

$$\sin\theta_n = \frac{\xi_n}{\rho_o} + O(\xi_n^2)$$

$$= \frac{\xi_n}{\rho(\xi_n)} + O(\xi_n^2) \qquad \text{since } \rho(\xi_n) \text{ varies linearly.}$$

Therefore,

$$\frac{\partial u^{(n)}}{\partial x^{(n)}} = u_x - \frac{\xi_n}{\rho(\xi_n)}(u_y + v_x) + O(\xi_n^2) \qquad (4.31)$$

So, $U_x(\xi_n)$ is now needed. By a similar argument to that obtaining (4.30) it is found that

$$u_x(\xi_n) = u_x(0) + \xi_n u_{xx}(0) + O(\xi_n^2) \qquad (4.32)$$

Also $u_{xx}(\xi_n) = u_{xx}(0) + O(\xi_n)$. Hence (4.32) implies

$$u_x(\xi_n) = u_x(0) + \xi_n u_{xx}(\xi_n) + O(\xi_n^2) \qquad (4.33)$$

By the same argument, (4.30) implies

$$v^{(n)}(\xi_n) = v^{(n)}(0) + \xi_n \frac{\partial v^{(n)}}{\partial x}(\xi_n) + O(\xi_n^2).$$

Using (4.24) to change frames,

$$v^{(n)}(\xi_n) = v^{(n)}(0) + \xi_n \frac{\partial v}{\partial x}(\xi_n) + O(\xi_n^2).$$

Taking the jump, using (4.29),

$$[v^{(n)}(\xi_n)] = \xi_n\left[\frac{\partial v}{\partial x}(\xi_n)\right] + O(\xi_n^2) \qquad (4.34)$$

Substituting (4.33) in (4.31) at $\xi = \xi_n$:

$$\frac{\partial u^{(n)}}{\partial x^{(n)}}(\xi_n) = u_x(0) + \xi_n u_{xx}(\xi_n) - \frac{\xi_n}{\rho(\xi_n)}\left[u_y(\xi_n) + v_x(\xi_n)\right] + O(\xi_n^2) \tag{4.35}$$

Next, it shall be proved that $[u_x(o)] = 0$

$$(4.3) \Rightarrow [u^{(n)}(\xi_n)] = 0$$

and $$u^{(n)}(\xi_n) = u^{(n)}(0) + \xi_n u_x(0) + O(\xi_n^2),$$

but $$u^{(n)}(0) = u(0) - \frac{\xi_n}{\rho_o} v(0) + O(\xi_n^2),$$

therefore,

$$u^{(n)}(\xi_n) = u(0) + \xi_n \left[u_x(0) - \frac{v(0)}{\rho_o}\right] + O(\xi_n^2),$$

so, $$\left[u(0) + \xi_n \left[u_x(0) - \frac{v(0)}{\rho_o}\right] + O(\xi_n^2)\right] = 0.$$

Dividing by ξ_n as $[u(0)] = 0$,

$$\left[u_x(0) - \frac{v(0)}{\rho_o} + O(\xi_n)\right] = 0$$

Letting $n \to \infty$, as $[v(0)] = 0$,

$$[u_x(0)] = 0 \quad \text{as required.}$$

So, taking the jump of (4.35):

$$\left[\frac{\partial u^{(n)}}{\partial x^{(n)}} (\xi_n)\right] = \xi_n\left[u_{xx}(\xi_n) - \frac{1}{\rho(\xi_n)} \left[u_y(\xi_n) + v_x(\xi_n)\right]\right] + O(\xi_n^2) \tag{4.36}$$

Combining (4.28), (4.34) and (4.36):

$$\rho(\xi_n) = \frac{[v_x(\xi_n)]}{\left[u_{xx}(\xi_n) - \frac{1}{\rho(\xi_n)}\left[u_y(\xi_n) + v_x(\xi_n)\right]\right]} + O(\xi_n)$$

Now $\rho(\xi_n) = O(1)$ by the original assumption, so it is necessary only to assume that $[v_x(\xi_n)] = O(1)$. This is equivalent to

$$[v^{(n)}(\xi_n)] = O(\xi_n) \tag{4.37}$$

So, assuming (4.37)

$$\rho(\xi_n)\,[u_{xx}(\xi_n)] - [u_y(\xi_n) + v_x(\xi_n)] = [v_x(\xi_n)] + O(\xi_n)$$

$$\Rightarrow \quad \rho(\xi_n) \sim \frac{[2v_x(\xi_n) + u_y(\xi_n)]}{[u_{xx}(\xi_n)]} .$$

As the choice of the sequence of frames $(x^{(n)},y^{(n)})$ is not restricted, ξ_n may be replaced by ξ, giving

$$\rho(\xi) \sim \frac{[2v_x(\xi) + u_y(\xi)]}{[u_{xx}(\xi)]} \tag{4.38}$$

- which is in the form of (4.1).

4.3.2 The Second Model - A Polar Frame

Putting $\alpha = 1$ in (4.20) yields

$$\theta = O(\xi). \tag{4.39}$$

It has already been shown that α must equal 1 in (4.11) in order that $\rho(\theta) = O(1)$ (or $0 < |\rho_o| < \infty$). Therefore, (4.11) here is

$$R = k\theta + O(\theta^{\beta}) \quad \text{where} \quad \beta > 1 \quad , \quad \beta \in \mathbb{R}.$$

(4.13) is $\rho = \frac{k}{2} + O(\theta^{\gamma}) \quad , \quad \gamma = \min\{2, \beta-1\}$

$$(4.16) \Rightarrow \quad Y' = -\frac{2k\theta + 0(\theta^{\gamma+1})}{k + 0(\theta^{\gamma})}$$

$$= -2\theta + 0(\theta^{\gamma+1}) \qquad (4.40)$$

Hence $(4.17) \Rightarrow Y'' = \dfrac{-2 + 0(\theta^{\gamma})}{k + 0(\theta^{\gamma})}$

$$= -\frac{2}{k} + 0(\theta^{\gamma}) \qquad (4.41)$$

As in (4.22) with $\alpha = 1$,

$$\frac{d}{d\theta} \equiv \left\{ k + 0(\theta^{\gamma}) \right\} \frac{\partial}{\partial x} + 0(\theta)\frac{\partial}{\partial y} \qquad (4.42)$$

$Y''(\theta)$ is required. From (4.7),

$$Y''(\theta) = -\frac{\left[u_x(\theta) + Y'(\theta)\left[u_y(\theta) + v_x(\theta)\right] + Y'(\theta)^2 v_y(\theta)\right]}{[v(\theta)]}$$

$$v_x(\theta) = 0(v_x(\xi)) = 0(1)$$

$$Y'(\theta) = 0(\theta)$$

Using (4.42),

$$u_x(\theta) = u_x(0) + \theta k u_{xx}(0) + 0(\theta^{\lambda+1})$$
$$= u_x(0) + \theta k u_{xx}(0) + 0(\theta^{\lambda+1})$$

By (4.38),

$$[u_{xx}(\theta)] = 0([u_{xx}(\xi)]) = 0(1).$$

Let $\lambda = \min\{2, \gamma\}$.

Hence, using (4.42)

$$v(\theta) = v(0) + k\theta v_x(0) + 0(\theta^{\lambda+1})$$
$$= v(0) + k\theta v_x(0) + 0(\theta^{\lambda+1}) \; .$$

$$\text{Hence}\quad Y''(\theta) = -\frac{\left[\theta k u_{xx}(\theta) + Y'(\theta)\left[u_y(\theta) + v_x(\theta)\right] + O(\theta^{\lambda+1})\right]}{[k\theta v_x(\theta) + O(\theta^{\lambda+1})]}$$

$$= -\frac{\left[k u_{xx}(\theta) - \left[u_y(\theta) + v_x(\theta)\right] + O(\theta^{\lambda})\right]}{[k v_x(\theta) + O(\theta^{\lambda})]} \qquad \text{Using (4.40)}$$

Hence from (4.41) and (4.13)

$$Y^2(\theta) = -\frac{2}{k} + O(\theta^{\gamma}) = -\frac{1}{\rho} + O(\theta^{\gamma})$$

$$-\frac{\left[\frac{k}{2} u_{xx}(\theta) - \left[u_y(\theta) + v_x(\theta)\right]\right]}{\left[\frac{k}{2} v_x(\theta)\right]} + O(\theta^{\gamma})$$

$$\Rightarrow \quad \frac{1}{\rho} + O(\theta^{\gamma}) = \frac{\left[\rho u_{xx}(\theta) - \left[u_y(\theta) + v_x(\theta)\right]\right]}{\rho[v_x(\theta)]}$$

$$\Rightarrow \quad [v_x(\theta)] \sim \rho[u_{xx}(\theta)] - [u_y(\theta) + v_x(\theta)]$$

$$\Rightarrow \quad \rho(\theta) \sim \frac{[2v_x(\theta) + u_y(\theta)]}{[u_{xx}(\theta)]}$$

So, with (4.39),

$$\rho(\xi) \sim \frac{[2v_x(\xi) + u_y(\xi)]}{[u_{xx}(\xi)]},$$

which is (4.38).

4.4 Third Case: Infinite Shock Curvature – $|\rho_o| = \infty$

Let $Y(0) = Y_o'$, $Y''(0) = Y_o''$.

Clearly, from (4.8),

$$|\rho_o| = \infty \quad \Rightarrow \quad Y_o'' = 0.$$

The cartesian frame (x,y) has been chosen such that $Y'_o = 0$. Now, performing a Taylor expansion on (4.6),

$$Y'(\xi) = \frac{\left[u(0) + \xi \frac{\partial}{\partial \xi} u|_{\xi=0} + 0(\xi^2)\right]}{\left[v(0) + \xi \frac{\partial}{\partial \xi} v|_{\xi=0} + 0(\xi^2)\right]}$$

again treating u and v as functions of ξ.

By the definition of the shock tip,

$$[u(0)] = 0 \quad , \quad [v(0)] = 0 \, .$$

Therefore,

$$Y'(\xi) = \frac{\left[\frac{\partial}{\partial \xi} u|_{\xi=0} + \frac{\xi}{2} \frac{\partial^2}{\partial \xi^2} u|_{\xi=0} + 0(\xi^2)\right]}{\left[\frac{\partial}{\partial \xi} v|_{\xi=0} + 0(\xi)\right]}$$

$$\frac{\partial}{\partial \xi} u = (1+Y'^2)^{-\frac{1}{2}}(u_x + Y'u_y)$$

Therefore $\frac{\partial}{\partial \xi} u|_{\xi=0} = u_x(0)$

Therefore $Y'(\xi) = - \frac{[u_x(0) + 0(\xi)]}{[v_x(0) + 0(\xi)]}$

But $Y'_o = 0$

Hence,

$$[u_x(0)] = 0 \qquad (4.43)$$

We thus need $\frac{\partial^2}{\partial \xi^2} u|_{\xi=0}$

$$\frac{\partial^2}{\partial \xi^2} u = (1+Y'^2)^{-\frac{1}{2}} \left(\frac{\partial}{\partial x} + Y'\frac{\partial}{\partial y}\right) (1 + Y'^2)^{-\frac{1}{2}}\left(\frac{\partial}{\partial x} + Y'\frac{\partial}{\partial y}\right) u$$

$$= (1+Y'^2)^{-\frac{1}{2}} \left\{ \left(\frac{\partial}{\partial x} + Y'\frac{\partial}{\partial y}\right) (1 + Y'^2)^{-\frac{1}{2}}(u_x + Y'u_y)\right\}$$

$$= (1+Y'^2)^{-\frac{1}{2}} \left\{ - Y'Y''(1 + Y'^2)^{-\frac{3}{2}}(u_x + Y'u_y) + (1 + Y'^2)^{-\frac{1}{2}} \right.$$

$$\left. (u_{xx} + Y''u_y + Y'u_{xy}) + (1 + Y'^2)^{-\frac{1}{2}} Y'(u_{xy} + Y'u_{yy}) \right\}$$

Therefore

$$\frac{\partial^2}{\partial\xi^2} u\Big|_{\xi=0} = u_{xx}(0) \quad \text{as} \quad Y'(0) = Y''(0) = 0.$$

So
$$Y'(\xi) = -\frac{\xi}{2}\frac{[u_{xx}(0) + 0(\xi)]}{[v_x(0)]} \tag{4.44}$$

But
$$Y'(\xi) = Y'_o + \xi\frac{\partial}{\partial\xi} Y'\Big|_{\xi=0} + 0(\xi^2)$$

$$\frac{\partial Y'}{\partial\xi}\Big|_{\xi=0} = Y''_o = 0$$

Hence
$$Y'(\xi) = 0(\xi^2)$$

So (4.44) $\Rightarrow$ $0(\xi^2) = -\dfrac{\xi}{2}\dfrac{[u_{xx}(0)]}{[v_x(0)]} + 0(\xi^2)$

$$\Rightarrow \frac{[u_{xx}(0)]}{[v_x(0)]} = 0$$

$$\Rightarrow [u_{xx}(0)] = 0 \tag{4.45}$$

Equation (4.45) is the required condition.

It is also possible to derive a similar result to (4.21) using $\alpha < 1$ in (4.11), but this is less useful.

4.5 Summary and Conclusion

First case, $\rho_o = 0$

$R(\theta) = k\theta^\alpha + 0(\theta^\beta)$, $\alpha > 1$, $\beta > \alpha$ is consistent and yields

$$\rho(\xi) \sim \frac{\alpha^2}{\alpha+1} k^{1/\alpha} \xi^{\frac{\alpha-1}{\alpha}}$$

No equation of the form (4.1) can be produced because all Taylor expansions fail near the shock tip.

Second Case. $0 < |\rho_o| < \infty$. Under the assumption (4.37),

$$\left[v^{(n)}(\xi_n)\right] = O(\xi_n) ,$$

both models give

$$\rho(\xi) \sim \frac{[2v_x(\xi) + u_y(\xi)]}{[u_{xx}(\xi)]} ,$$

which is of the required form.

Other assumptions (such as $\left[\frac{\partial v^{(n)}}{\partial x^{(n)}}(\xi_n) = O(\xi_n)\right]$) will probably yield similar results in higher derivatives of u and v.

Third Case. $|\rho_o| = \infty$. It is shown that

$$\lim_{\xi \to 0} [u_{xx}(\xi)] = 0.$$

The above assumption (4.37) is critical. If it holds, the cases ii) and iii) are distinguishable as if $\rho(\xi) = O(1)$ and $[v_x(\xi)] = O(1)$ then it seems reasonable to assume $[2v_x(\xi) + u_y(\xi)] = O(1)$ and therefore from (4.38),

$[u_{xx}(\xi)] = O(1)$, not giving $[u_{xx}(0)] = 0$ in general.

Also, as Taylor expansions fail near the shock tip in case i), it seems reasonable to propose that

$$\lim_{\xi \to 0} |[u_{xx}(\xi)]| = \infty \quad \text{for this case.}$$

It is hoped that this critical condition (4.37) will be ratified in the future, rather than calculating a weaker generic condition.

5. Theory and Examples of Flux Jump Order and Tip Curvature using Characteristics

5.0 Overview

In this section three analytical models are presented using continuous data for a single conservation law. The over-riding heuristic is simplicity, so only straight line characteristic base curves and simple flux functions are considered. It is considered necessary that the flux functions supplied as data belong to the same functional family (e.g. linear, logarithmic etc.) without degeneracy.

As we are concerned about producing a shock, we must have meeting characteristics. This adds further constraints to the allowable models. Also the simple physical condition of boundedness is considered and the geometrical qualitative structure is described. Furthermore, the asymptotic flux strength and shock tip curvature are calculated, tying in with the last chapter.

5.1 General Theory

As in §2.1, consider a single conservation law of the form

$$F_x + G_y = 0 \qquad (5.1)$$

where F and G are both functions of x and y, and have discontinuities across the shock $\mathcal{S}$ parametrised by

$$\mathcal{S}: \quad y = Y(x)$$

It has been shown in §2.3 that

$$(2.15): \quad Y' = \frac{[G]}{[F]} .$$

Now let G be a function of F, and let A be its derivative with respect to F:

$$G = G(F)$$
$$\frac{dG}{dF} = A(F)$$

Then the full functional-dependent form of (2.15) is

$$Y'(x) = \frac{[G(F(x,Y(x)))]}{[F(x,Y(x))]} \qquad (5.2)$$

Let the two sides of the shock be denoted by suffixes $_0$ and $_1$ as in §3.1. Consider two lines:

$$\Gamma_0 : \frac{dy}{dx} = A_0$$
$$\Gamma_1 : \frac{dy}{dx} = A_1 ,$$

both constrained to pass through a fixed point on the shock $(x, Y(x))$, and with A_0, A_1 to be determined, (see fig. 7).

Now $$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{dy}{dx}\frac{\partial F}{\partial y} ,$$

and $$\frac{\partial G}{\partial y} = \frac{dG}{dF}\frac{\partial F}{\partial y}$$
$$= A(F)\frac{\partial F}{\partial y} .$$

Hence, $$\frac{dy}{dx} = A(F) \Rightarrow \frac{dF}{dx} = 0 \quad \text{using (5.1)}$$
$$\Rightarrow F = \text{constant}$$
$$\Rightarrow G = \text{constant}, \quad A = \text{constant}.$$

So F,G and A are all constant on Γ_0 and on Γ_1 (not necessarily having the same values on both). Γ_0 and Γ_1 are called "characteristics".

Data is supplied along characteristic base curves which may be curves or lines, single or multiple (see fig. 8).

It is required to find [F] and [G] along $\mathcal{S}$ as a function of ξ, the arc length. A quantity needs to be constructed which is orientation-invariant and is a measure of shock strength.

For a fixed point on the shock let $\tilde{x}$ be parallel to the tangent and $\tilde{y}$ perpendicular to $\tilde{x}$. Let the rotation from (x,y) to $(\tilde{x},\tilde{y})$ be θ (see fig. 9). Then,

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{\partial}{\partial x} \equiv \frac{\partial \tilde{x}}{\partial x}\frac{\partial}{\partial \tilde{x}} + \frac{\partial \tilde{y}}{\partial x}\frac{\partial}{\partial \tilde{y}}$$

$$\equiv \cos\theta\frac{\partial}{\partial \tilde{x}} - \sin\theta\frac{\partial}{\partial \tilde{y}}$$

Hence, (5.1) $\Rightarrow$

$$\left[\cos\theta\frac{\partial}{\partial \tilde{x}} - \sin\theta\frac{\partial}{\partial \tilde{y}}\right] F + \left[\sin\theta\frac{\partial}{\partial \tilde{x}} - \cos\theta\frac{\partial}{\partial \tilde{y}}\right] G = 0$$

$$\Rightarrow \tilde{F}_{\tilde{x}} + \tilde{G}_{\tilde{y}} = 0 ,$$

where

$$\tilde{F} = F\cos\theta + G\sin\theta$$

and $\tilde{G} = G\cos\theta - F\sin\theta$.

Let $\tilde{y} = \tilde{Y}(\tilde{x})$ also parametrise the shock. By the construction, for this fixed point,

$$\tilde{Y}' = 0 .$$

Hence (5.2) $\Rightarrow$ $[\tilde{G}] = 0$.

Therefore, a good invariant measure of shock strength is

$$[\tilde{F}] = [F\cos\theta + G\sin\theta].$$

$$\text{But} \quad Y' = \tan\theta = \frac{[G]}{[F]}, \quad \text{so}$$

$$\left.\begin{aligned} [\tilde{F}] &= ([F]^2 + [G]^2)^{1/2} \\ &= J(\xi), \text{ say.} \end{aligned}\right\} \qquad (5.3)$$

At the shock tip, $\xi = \xi^*$ say and $J(\xi^*) = 0$. It is required to find

$$J(\xi^* \pm \delta\xi),$$

as an expansion in $\delta\xi$, where the $\pm$ sign is chosen so that J is real and non-zero.

5.2 <u>First Model:</u> <u>Single Straight Characteristic Base Curve with Quadratic Data.</u>

Without loss of generality, the characteristic base curve may be taken as the y-axis.

$$\text{Let} \quad \begin{aligned} F(0,y) &= \hat{F}(y) \\ G(0,y) &= \hat{G}(y) \end{aligned}$$

Let Γ_0, Γ_1 start from $x = 0$, $y = y_0$, y_1, respectively. Hence

$$\left.\begin{aligned} \Gamma_0: \quad y &= y_0 + A(\hat{F}(y_0))x \\ \Gamma_1: \quad y &= y_1 + A(\hat{F}(y_1))x \end{aligned}\right\} \qquad (5.4)$$

$$\text{Also, (5.2)} \Rightarrow \quad Y'(x) = \frac{\hat{G}(y_1(x)) - \hat{G}(y_0(x))}{\hat{F}(y_1(x)) - \hat{F}(y_0(x))} \qquad (5.5)$$

For this model, let

$$\left.\begin{aligned} \hat{F} &= ay^2 + by + c \\ \hat{G} &= dy^2 + ey + f \end{aligned}\right\} \qquad (5.6)$$

Now $dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy$

$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$

Therefore, $$\left.\frac{dG}{dF}\right|_{x=0} = \left.\frac{dG}{dF}\right|_{dx=0} = \left.\frac{G_y}{F_y}\right|_{x=0}$$

$$= \frac{d\hat{G}/dy}{d\hat{F}/dy}$$

$$= \frac{2dy + e}{2ay + b} \quad \text{from (5.6)}$$

$$= \hat{A}(y) .$$

So (5.4) gives y_0, y_1 as the roots of

$$Y = y + \left[\frac{2dy + e}{2ay + b}\right]x , \tag{5.7}$$

since Γ_0, Γ_1 coincide at (x,Y).

Also, (5.5) $\Rightarrow$ $$Y' = \frac{d(y_0 + y_1) - e}{a(y_0 + y_1) - b} \tag{5.8}$$

Hence, it is not necessary to solve (5.7) here, only to express it in the normal form for a quadratic equation:

$$(5.7) \Rightarrow (y - Y)(2ay + b) + (2dy + e)x = 0$$

$$\Rightarrow 2ay^2 + \{b - 2aY + 2dx\}y + ex - bY = 0 \tag{5.9}$$

Therefore, $$y_0 + y_1 = \frac{2aY - b - 2dx}{2a} = Y - \frac{b + 2dx}{2a} \tag{5.10}$$

So (5.8) $\Rightarrow$ $$Y' = \frac{d(2aY - b - 2dx) - 2ae}{a(2aY - b - 2dx) - 2ab}$$

$$= \frac{2adY - 2d^2x - bd - 2ae}{2a^2Y - 2adx - 3ab} \tag{5.11}$$

It is possible to integrate (5.11) once by a suitable change of variable, but this is not necessary for calculating $J(\xi_0 \pm \delta\xi)$.

$$J(\xi) = ([F]^2 + [G]^2)^{1/2}$$
$$= |y_1 - y_0| \{(d(y_0 + y_1) - e)^2 + (a(y_0 + y_1) - b)^2\}^{1/2}$$

Hence, $J(\xi) = 0 \iff \xi = \xi^* \iff y_0 = y_1$, or

$$\left[d(y_0 + y_1) - e = 0 \quad \text{and} \quad a(y_0 + y_1) - b = 0 \right]$$

But this latter condition generates the following condition between the coefficients:

$$\frac{e}{d} = \frac{b}{a},$$

which in turn implies that $\hat{A}$ is constant, which implies (5.4) only has one root everywhere. So this case is disallowed. The other case, $y_0 = y_1$ implies (5.9) is a perfect square. If x^* and Y^* are the values of x and Y when $J(\xi) = 0$, we may complete the square in (5.9) and obtain a relationship between them:

$$(5.9) \Rightarrow y^2 + \frac{1}{2a}\left\{b-2aY+2bc\right\} y + \frac{1}{2a}\left\{ex-bY\right\} = 0$$

$$\Rightarrow \left\{y - \frac{1}{4a}(2aY-b-2dx)\right\}^2 + \frac{1}{2a}(ex-bY) - \frac{1}{16a^2}(2aY-b-2dx)^2 = 0 .$$

So, from the above argument,

$$ex^* - bY^* - \frac{1}{8a}(2aY^*-b-2dx^*)^2 = 0. \tag{5.12}$$

Note, the condition $y_0 = y_1$ also implies that the shock tip must coincide with this point, i.e.

$$\left.\begin{array}{lll} x^* & = & 0 \\ Y^* & = & y_0 = y_1 \end{array}\right\} \qquad (5.13)$$

$$(5.12),\ (5.13) \quad \Rightarrow \quad -bY^* - \frac{a}{2}Y^{*2} = 0$$

$$\Rightarrow \quad Y^* = 0 \quad \text{or} \quad -\frac{b}{2a}$$

$$\text{Also } (5.10) \quad \Rightarrow \quad 2Y^* = Y^* - \frac{b}{2a}$$

$$\Rightarrow \quad Y^* = -\frac{b}{2a}, \quad \text{so this root is consistent.} \qquad (5.14)$$

Substituting in (5.11):

$$Y^{*\prime} = Y'(x^*) = \frac{-2bd - 2ae}{-ab - 3ab}$$

$$= \frac{ae + bd}{2ab} \qquad (5.15)$$

Now, consider a linear approximation to the shock near its tip (x^*, Y^*):

$$x = x^* + \delta dx$$

$$Y = Y^* + Y^{*\prime}\delta x + 0(\delta x^2)$$

Now, (5.9) $\Rightarrow$

$$y = -\frac{1}{4a}\left\{b - 2aY + 2dx\right\} \pm \frac{1}{4a}\sqrt{\left[\left\{b - 2aY + 2dx\right\}^2 - 8a(ex - bY)\right]}$$

So $$|y_1 - y_0| = \frac{1}{2a}\left[\left\{2aY - b - 2dx\right\}^2 - 8a(ex - bY)\right]^{1/2}.$$

Therefore, near the shock tip,

$$|y_1 - y_0| = \frac{1}{2a}\left[\left\{-b + \frac{(ae+bd)}{b}\delta x - b - 2d\delta x\right\}^2 - 8a(e\delta x + \frac{b^2}{2a} - \frac{(ae+bd)}{2a}\delta x) + 0(\delta x^2)\right]^{1/2}$$

$$= \frac{1}{2a}\left[4b^2 + 4b(d - \frac{ae}{b})\delta x - 4b^2 - 8ae\delta x + 4(ae+bd)\delta x + 0(\delta x^2)\right]^{1/2}$$

$$= \frac{1}{2a}\left[(4bd - 4ae - 8ae + 4ae + 4bd)\delta x + 0(\delta x^2)\right]^{1/2}$$

$$= \frac{(\delta x)^{1/2}}{a}[2(bd - ae)]^{1/2} + 0(\delta x^{3/2})$$

$$(5.10) \Rightarrow \quad y_0 + y_1 = -\frac{b}{2a} + \frac{(ae + bd)}{2ab}\delta x - \frac{b}{2a} - \frac{d}{a}\delta x + 0(\delta x^2)$$

$$= -\frac{b}{a} + \frac{(ae - bd)}{2ab}\delta x + 0(\delta x^2)$$

Thus, $$J(\xi^* + \delta\xi) = \frac{1}{a}[2(bd - ae)\ \delta x]^{1/2}\left\{\left[-\frac{db}{a} + \frac{d}{2ab}(ae - bd)\delta x - e\right]^2 + \left[-b + \frac{1}{2b}(ae - bd)\delta x - b\right]^2 + 0(\delta x^2)\right\} + 0(\delta x^{3/2})$$

$$= \frac{1}{a}[2(bd - ae)\delta x]^{1/2}\left\{\left[\frac{ae + bd}{a}\right]^2 + 4b^2\right\} + 0(\delta x^{3/2}).$$

Also, $$\delta\xi = (1 + Y'^2)^{1/2}\delta x + 0(\delta x^2)$$

$$= \left[1 + Y^{*'2}\right]^{1/2}\delta x + 0(\delta x^2)$$

$$= \frac{[(ae + bd)^2 + 4a^2b^2]^{1/2}}{2ab}\delta x + 0(\delta x^2).$$

So, $$\delta x = \frac{2ab\ \delta\xi}{[(ae+bd)^2 + 4a^2b^2]^{1/2}} + O(\delta\xi^2)$$

Hence, $$J(\xi^* + \delta\xi) = \frac{1}{a}\left\{\frac{4ab(bd-ae)\delta\xi}{[(ae+bd)^2 + 4a^2b^2]^{1/2}}\right\}^{1/2}\left\{\frac{(ae+bd)^2 + 4b^2a^2}{a^2}\right\} + O(|\delta\xi|^{3/2})$$

$$= \frac{1}{a^3}[(ae+bd)^2 + 4a^2b^2]^{1/4}[4ab(bd-ae)\delta\xi]^{1/2} + O(|\partial\xi|^{3/2})$$

....(5.16)

So, for this model, the shock only exists on one side of the characteristic base curve (as $[4ab(bd - ae)\delta\xi]^{1/2}$ is only real for one of $(\delta\xi < 0), (\delta\xi > 0)$) and

$$J(\xi^* + \delta\xi) = O(|\delta\xi|^{1/2}). \qquad (5.16)$$

It is also possible to calculate the tip curvature. Let

$$Y^{*''} = Y''(x^*) .$$

Provided $Y^{*''} \neq 0$, the tip curvature is

$$\rho(\xi^*) = -\frac{\left[1 + Y^{*'2}\right]^{3/2}}{Y^{*''}} \qquad \text{(see (4.8))}.$$

Now $$Y' = Y^{*'} + \delta x\ Y^{*''} + O(\delta x^2)$$

$$= \frac{(ae+bd)}{2ab} + Y^{*''}\delta x + O(\delta x^2)$$

Also $$Y = Y^* + \delta x\ Y^{*'} + O(\delta x^2)$$

$$= -\frac{b}{2a} + \frac{(ae+bd)}{2ab}\delta x + O(\delta x^2)$$

So, from (5.11):

$$\frac{(ae+bd)}{2ab} + Y^{*''}\delta x + O(\delta x^2) = \left[- bd + \frac{d}{b}(ae+bd)\delta x - 2d^2\delta x - bd - 2ae + O(\delta x^2)\right].$$

$$\left[- ab + \frac{a}{b}(ae+bd)\delta x - 2ad\ \delta x - 3ab + O(\delta x^2)\right]^{-1}$$

$$= \left[bd + ae - \frac{d}{2b}(ae-bd)\delta x + O(\delta x^2) \right].\left[2ab - \frac{a}{2b}(ae-bd)\delta x + O(\delta x^2) \right]^{-1}$$

$$= \frac{(bd+ae)}{2ab}\left[1 - \frac{d(ae-bd)}{2b(ae+bd)}\ \delta x + O(\delta x^2) \right]\left[1 + \frac{(ae-bd)}{4b^2}\ \delta x + O(\delta x^2) \right]$$

$$= \frac{(ae+bd)}{2ab}\left[1 + \frac{a^2e^2 - b^2d^2 - 2bd(ae-bd)}{4b^2(ae+bd)}\ \delta x + O(\delta x^2) \right]$$

$$= \frac{ae + bd}{2ab} + \frac{(ae-bd)^2}{8ab^3}\ \delta x + O(\delta x^2)$$

$$\Rightarrow \quad Y^{*''} = \frac{(ae-bd)^2}{8ab^3}$$

Hence, $$\rho(\xi^*) = \frac{-8ab^3}{(ae-bd)^2}\ \frac{[(ae+bd)^2 + 4a^2b^2]^{3/2}}{8a^3b^3}$$

$$= -\frac{[(ae+bd)^2 + 4a^2b^2]^{3/2}}{a^2(ae-bd)^2} \qquad (5.17)$$

5.3 Second Model: Single Straight Characteristic Base Curve with Logarithmic Data.

As in §5.2, the first model, let the characteristic base curve be the y-axis and let $\hat{F}$ and $\hat{G}$ be the intitial data for F and G.

$$\text{Let} \quad \left.\begin{aligned} \hat{F} &= a\ \ln(y+b) + c \\ \hat{G} &= d\ \ln(y+e) + f \end{aligned}\right\} \qquad (5.18)$$

Then, $\left.\frac{dG}{dF}\right|_{x=0} = \left.\frac{d\hat{G}/dy}{d\hat{F}/dy}\right|_{x=0}$

$$= \frac{d(y+b)}{a(y+e)} = \hat{A}(y) \quad \text{as before}$$

So y_0, y_1 are the roots of

$$Y = y + \frac{d(y+b)}{a(y+e)}\,x \tag{5.19}$$

Define the shock tip by (x^*, Y^*) as before, and let $\xi = \xi^*$ there. Now,

$$[F] = \{a \ln(y_1+b) + c\} - \{ a \ln(y_0+b) + c \}$$

$$= a \ln\left[\frac{y_1 + b}{y_0 + b}\right], \quad \text{and}$$

$$[G] = d \ln\left[\frac{y_1 + e}{y_0 + e}\right] \quad \text{similarly}$$

Thus, $J(\xi) = \left\{[F]'' + [G]^2\right\}^{1/2}$

$$= \left\{a^2\left[\ln\left[\frac{y_1+ b}{y_0+ b}\right]\right]^2 + d^2\left[\ln\left[\frac{y_1+ e}{y_0+ e}\right]\right]^2\right\}^{1/2}$$

$$\xi = \xi^* \iff J(\xi) = 0 \iff a \ln\left[\frac{y_1+ b}{y_0+ b}\right] = 0 \text{ and } d \ln\left[\frac{y_1+ e}{y_0+ e}\right] = 0$$

Assuming $a, d \neq 0$, $\xi = \xi^* \iff$

$$y_1 + b = y_0 + b \quad \text{and} \quad y_1 + e = y_0 + e$$

i.e., $y_0 = y_1$.

So, again we have

$$\left.\begin{aligned} x^* &= 0 \\ Y^* &= y_0 = y_1 \end{aligned}\right\} \tag{5.20}$$

Y^* is found by completing the square in (5.19):

$$a(y-Y)(y+e) + d(y+b)x = 0$$

$$\Rightarrow \quad ay^2 + \{ae + dx - aY\}\, y + bdx - aey = 0$$

$$\Rightarrow \quad y^2 - \{Y - e - \frac{d}{a}x\}\, y + \frac{d}{a}bx - eY = 0$$

$$\Rightarrow \quad (y - \frac{1}{2}\{Y - e - \frac{e}{a}x\})^2 - \frac{1}{4}\{Y - e - \frac{d}{a}x\}^2 + \frac{d}{a}bx - eY = 0.$$

So $y_0 = y_1$ when

$$\{Y - e - \frac{d}{a}x\}^2 - 4(\frac{d}{a}bx - eY) = 0.$$

It is also known that $x = 0$ here; hence

$$(Y - e)^2 + 4eY = 0$$

$$\Rightarrow \quad (Y + e)^2 = 0$$

$$\Rightarrow \quad Y^* = -e \qquad (5.21)$$

This model is therefore "unphysical" as G is not bounded at the shock tip (even though $[G] \to 0$).

It can, however, been shown that

$$J(\xi^* + \delta\xi) = 2(1 + \frac{d^2}{a^2})^{1/4}\left[\frac{d\delta\xi}{a(b-e)}\right]^{1/2} + O(\delta\xi), \qquad (5.22)$$

the argument is rather lengthy and tedious, so is not produced here,

$$\left.\begin{array}{ll} \text{and} \quad \rho(\xi^*) = 0 \quad , \quad \text{as} \\ Y''(\xi^* + \delta\xi) = O(\delta\xi^{-2}) \end{array}\right\} . \qquad (5.23)$$

5.4 Third Model Two Straight Characteristic Base Curves with Linear Data

Without loss of generality, define the intersection point of the characteristic base curves to be the origin, let one of the curves be the y-axis and the other to be inclined at an angle θ to the y-axis. Let σ be the length parameter on the latter curve (see fig. 10).

Then, on the latter curve,

$$y = x \cot\theta, \qquad \text{and}$$

$$\sigma = y \cos\theta + x \sin\theta$$

On the first curve, let $F = \hat{F}$, $G = \hat{G}$

On the second curve, let $F = \tilde{F}$, $G = \tilde{G}$.

$$\text{Let} \quad \hat{F}(y) = ay + b$$

$$\hat{G}(y) = cy + d$$

$$\tilde{F}(\sigma) = e\sigma + f$$

$$\tilde{G}(\sigma) = g\sigma + h$$

$$\text{Then} \quad \tilde{A} = \left.\frac{dG}{dF}\right|_{x=0} = \frac{d\hat{G}/dy}{d\hat{F}/dt} = \frac{c}{a} = \lambda, \text{ say}$$

$$\text{and, similarly,} \quad \tilde{A} = \left.\frac{dG}{dF}\right|_{y=x\cot\theta} = \frac{d\tilde{G}/d\sigma}{d\tilde{F}/d\sigma} = \frac{g}{e} = \mu, \quad \text{say.}$$

Let the y-axis characteristic base curve give the 0 suffix information to the shock. Then

$$[G] = \tilde{G}_1 - \hat{G}_0$$

$$= g\sigma_1 + h - (cy_0 + d) ,$$

and

$$[F] = \tilde{F}_1 - \hat{F}_0$$

$$= e\sigma_1 + f - (ay_0 + b).$$

Now, σ can be found as a function of y and θ:

$$\sigma = y\cos\theta + x\sin\theta$$

$$x = y\tan\theta$$

$$\Rightarrow \quad \sigma = y(\cos\theta + \sin\theta\tan\theta)$$

$$= y\sec\theta$$

Hence,

$$\left.\begin{aligned} [F] &= gy_1\sec\theta + h - (cy_0 + d) \\ [G] &= ey_1\sec\theta + f - (ay_0 + b) \end{aligned}\right\} \qquad (5.24)$$

And the equations for $\hat{A}$, $\tilde{A}$ give

$$\left.\begin{aligned} Y &= y_0 + \lambda x \\ Y &= y_1 + \mu x \end{aligned}\right\} \qquad (5.25)$$

$$(5.24) \quad \Rightarrow \quad Y' = \frac{[G]}{[F]} = \frac{ey_1\sec\theta + f - ay_0 - b}{gy_1\sec\theta + h - cy_0 - d} . \qquad (5.26)$$

At the shock tip, [F] = 0 and [G] = 0. Let $y_0 = y_0^*$, $y_1 = y_1^*$ at the tip. Then, (5.24) $\Rightarrow$

$$\left.\begin{aligned} gy_1^*\sec\theta + h &= cy_0^* + d \\ ey_1^*\sec\theta + f &= ay_0^* + b \end{aligned}\right\} \qquad (5.27)$$

So, $\frac{1}{c}\left\{gy_1^*\sec\theta + h - d\right\} = \frac{1}{a}\left\{ey_1^*\sec\theta + f - b\right\}$,

(assuming non-degeneracy throughout)

giving, $\left[\frac{e}{a} - \frac{g}{c}\right]y_1^*\sec\theta = \frac{h-d}{c} - \frac{f-b}{a}$

$$\Rightarrow \quad y_1^* = \frac{\cos\theta}{ce-ag}\left\{a(h-d) - c(f-b)\right\} \qquad (5.28)$$

Also, (5.27) $\Rightarrow$

$$\frac{1}{g}(cy_0^* + d - h) = \frac{1}{e}(ay_0^* + b - f)$$

$$\Rightarrow \quad \left[\frac{a}{e} - \frac{c}{g}\right]y_0^* = \frac{d-h}{g} - \frac{b-f}{e}$$

$$\Rightarrow \quad y_0^* = \frac{-\,e(d-h) + g(b-f)}{ce - ag} \qquad (5.29)$$

Now, (5.25) $\Rightarrow$

$$y_0 + \lambda x = y_1 + \mu x$$

$$\Rightarrow \quad x = \frac{y_1 - y_0}{\lambda - \mu} = \frac{ae(y_1 - y_0)}{ce - ag}$$

$$\Rightarrow \quad x^* = \frac{ae(y_1^* - y_0^*)}{ce - ag}$$

So, (5.28), (5.29) $\Rightarrow$

$$x^* = \frac{ae\{g-c\cos\theta)(f-b) - (e-a\cos\theta)(h-d)\}}{(ce-ay)^2} \tag{5.30}$$

Also, (5.25) $\Rightarrow$ $\dfrac{Y - y_0}{\lambda} = \dfrac{Y - y_1}{\mu}$

$$\Rightarrow \quad \mu(Y-y_0) = \lambda(Y-y_1)$$

$$\Rightarrow \quad Y(\lambda-\mu) = \lambda y_1 - \mu y_0$$

$$\Rightarrow \quad Y = \frac{\lambda y_1 - \mu y_0}{\lambda - \mu}$$

$$\Rightarrow \quad Y = \frac{cey_1 - ag\, y_0}{ce - ag}, \quad \text{giving}$$

$$Y^* = \frac{ce\, y_1^* - ag\, y_0^*}{ce - ag}$$

$$= \frac{(f-b)(gce-c^2e\cos\theta) - (h-d)(age-a^2g\cos\theta)}{(ce - ag)^2} \tag{5.31}$$

$Y^{*'}$ is also required before $J(\xi^* + \delta\xi)$ may be evalued. Clearly, (5.26) will not yield $Y^{*'}$ from a straight substitution at (x^*, Y^*), a $[G] = [F] = 0$. A Taylor expansion is therefore necessary:

$$Y' = Y^{*'} + 0(\delta x) \qquad \text{when} \quad x = x^* + \delta x$$

$$y_0 = y_0^* + Y_0^{*'}\delta x + 0(\delta x^2)$$

$$Y_1 = y_1^* + Y_1^{*'}\delta x + 0(\delta x^2)$$

Clearly, the $0(1)$ terms in the right hand side of (5.26) must cancel, so

$$Y^{*'} + 0(\delta x) \quad = \quad \frac{ey_1^{*'} - ay_0^{*'} + 0(\delta x)}{gy_1^{*'} - cy_0^{*'} + 0(\delta x)}$$

$$y_0' \quad = \quad Y' - \lambda \quad \Rightarrow \quad y_0^{*'} \quad = \quad Y^{*'} - \lambda$$

Similarly, $\qquad y_1^{*'} \quad = \quad Y^{*'} - \mu$

Hence,

$$Y^{*'} + 0(\delta x) \quad = \quad \frac{eY^{*'} - g - aY^{*'} + c + 0(\delta x)}{gY^{*'} - g^2/e - cY^{*'} + c^2/a + 0(\delta x)}$$

So, again assuming, non-degeneracy, the $0(\delta x)$ terms may be removed:

$$Y^{*'} \left\{(g-c)Y^{*'} + \frac{c^2}{a} - \frac{g^2}{e}\right\} \quad = \quad (e-a)Y^{*'} + c - g$$

Let $z = Y^{*'}$, then

$$(g-c)z^2 - \left\{\frac{g^2}{e} - \frac{c^2}{a} + e - a\right\} z + g - c \quad = \quad 0$$

$$\frac{g^2}{e} - \frac{c^2}{a} + e - a \quad = \quad \frac{ag^2 - ec^2 + ae^2 - ea^2}{ae}$$

$$= \quad \frac{a(g^2-e^2) - e(c^2-a^2)}{ae}$$

So, $z = \dfrac{a(g^2-e^2) - e(c^2-a^2)}{2(g-c)ae} \pm \dfrac{1}{2(g-c)} \sqrt{\left[-4(g-c)^2 + \dfrac{a^2(g^2-e^2)^2 - 2ae(g^2-e^2)(c^2-a^2) + e^2(c^2-a^2)^2}{a^2e^2}\right]}$ (5.32)

This gives $Y_+^{*'}$ and $Y_-^{*'}$ - the two roots, generally not equal.

Now, $[F] = [F^*] + [F^{*'}]\delta x + O(\delta x^2)$

$[G] = [G^*] + [G^{*'}]\delta x + O(\delta x^2)$,

where $*$ denotes evaluation at $\xi = \xi^*$

$$[F^{*'}] = gy_1^{*'}\sec\theta - cy_0^{*'} = Y^{*'}(g\sec\theta - c) - (\lambda g\sec\theta - \mu c)$$

$$[G^{*'}] = ey_1^{*'}\sec\theta - ay_0^{*'} = Y^{*'}(e\sec\theta - a) - (\lambda e\sec\theta - \mu a)$$

Hence,

$J(\xi^* + \delta\xi)$ is not $O(\delta x)$ when

$$\left\{z(g\sec\theta-c)-(\lambda g\sec\theta-\mu c)\right\}^2 + \left\{z(e\sec\theta-a)-(\lambda e\sec\theta-\mu a)\right\}^2 = 0$$

- which is another degenerate case.

Otherwise,

$$J(\xi^* + \delta\xi) = \sqrt{\left[\left\{z(g\sec\theta - c) + \frac{gc}{e} - \frac{ga}{e}\sec\theta\right\}^2 + \left\{z(e\sec\theta - a) + \frac{ga}{c} - \frac{ea}{c}\sec\theta\right\}^2\right]}\,\delta x + O(\delta x^2), \quad (5.33)$$

where z is one of the roots of (5.32).

It has not been attempted to find $\rho(\xi^*)$ as, presumably, $Y^{*''}$ is even more complicated than $Y^{*'}$.

5.5 Summmary

The single base curve models both had

$$J(\xi^* + \delta\xi) = O(|\delta\xi|^{1/2}),$$

with the shock tip coincident with the base curve and a "single" shock tip being present (see fig. 11).

The double base curve model had

$$J(\xi^* + \delta\xi) = O(\delta\xi),$$

with the shock tip not coincident with the base curve and a "double" shock tip being present (see fig. 12), containing a kink due to the two roots of (5.32).

It was not possible to find any other consistent models for the single base curve case with the data for F and G of the same analytic (continuous) function from. Formally, this can be expressed by considering

$H \in C[\mathbb{R},\mathbb{R}]$ – the continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Let $\mathcal{F}(H) = \{\phi(\underline{\alpha},\cdot) \in C[\mathbb{R},\mathbb{R}],\ \underline{\alpha} \in \mathbb{R}^n$ with $\exists\ \underline{a} \in \mathbb{R}^n$ such that $H(\cdot) \equiv \phi(\underline{a}, \cdot)\}$

then $\hat{F}$ and $\hat{G}$ are said to have the same functional form if

$\mathcal{F}(\hat{F}) = \mathcal{F}(\hat{G})$ with no degeneracy.

Note that the zero radius of curvature at the shock tip in the second model occurred with infinite fluxes.

6. Conclusion

This report as a whole yields significant advances in a number of different approaches to the problem of the theoretical analysis of the shock tip. The first two chapters provide the background and groundwork to the analysis of the jump conditions in the presence of limiting viscousity. Chapter three summarises the pertinent inviscid implications from these conditions. Chapter four gives an account of quantity (as opposed to system) perturbations relevant to the shock tip. The final chapter provides the basis for a theory of shock-waves modelled by a single divergence equation and having analytic data.

It is hoped that these approaches will be synthesised and extended in further work with a view to giving a complete theoretical picture of this problem.

7. References

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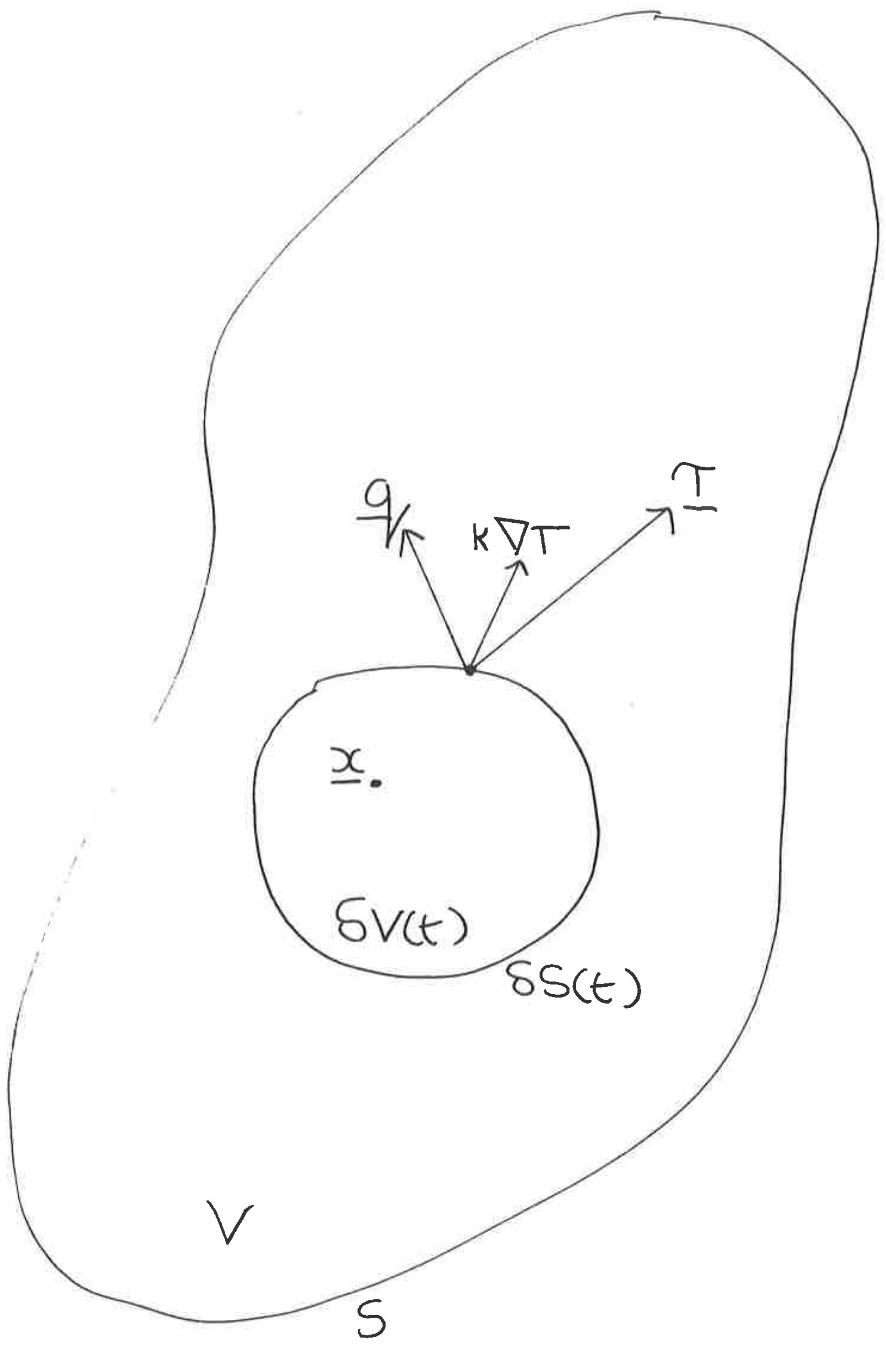


Fig. 1.

Σ_n:

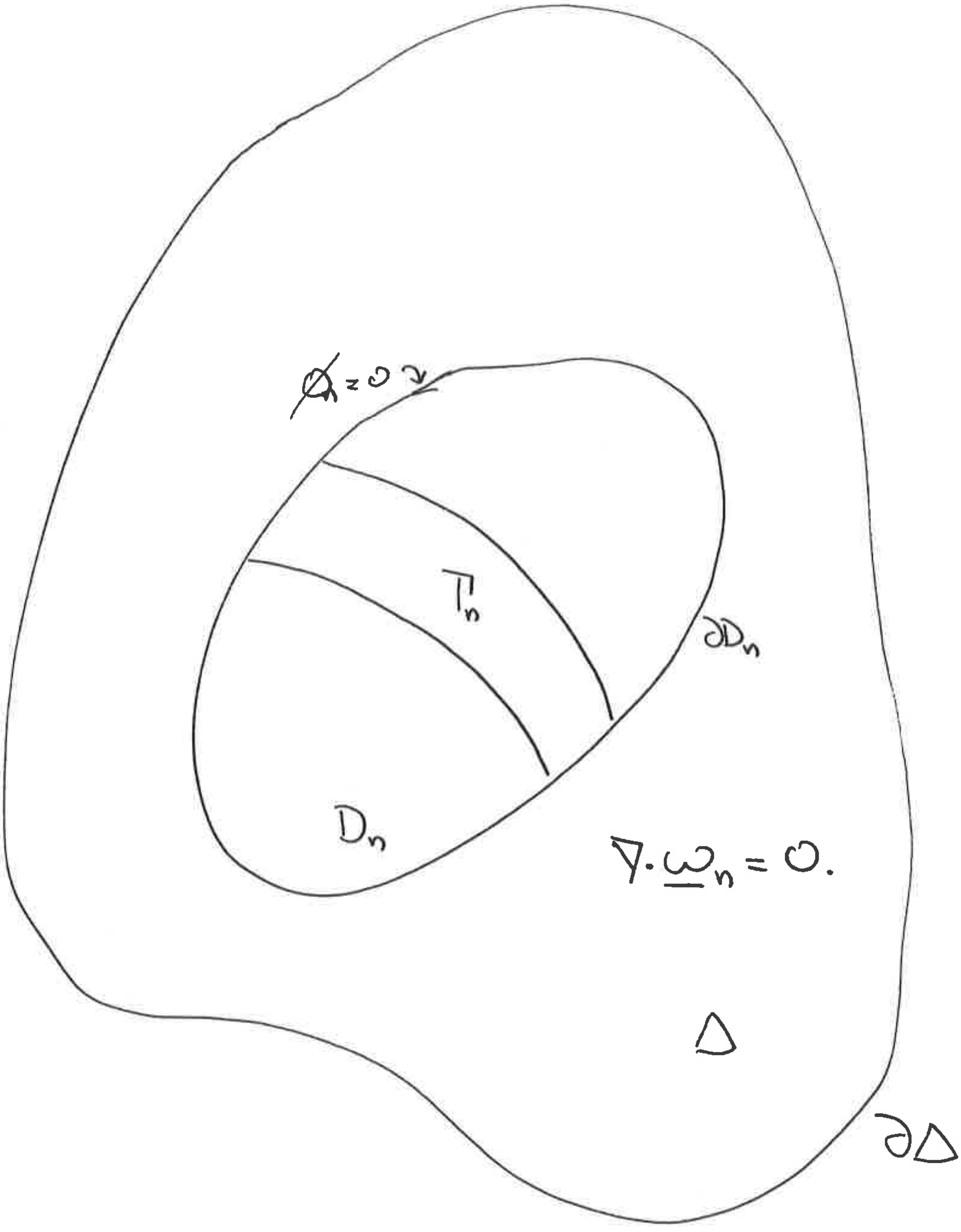

$\phi_n = 0$
Γ_n
∂D_n
D_n
$\nabla \cdot \underline{\omega}_n = 0.$
Δ
$\partial\Delta$

Fig. 2.

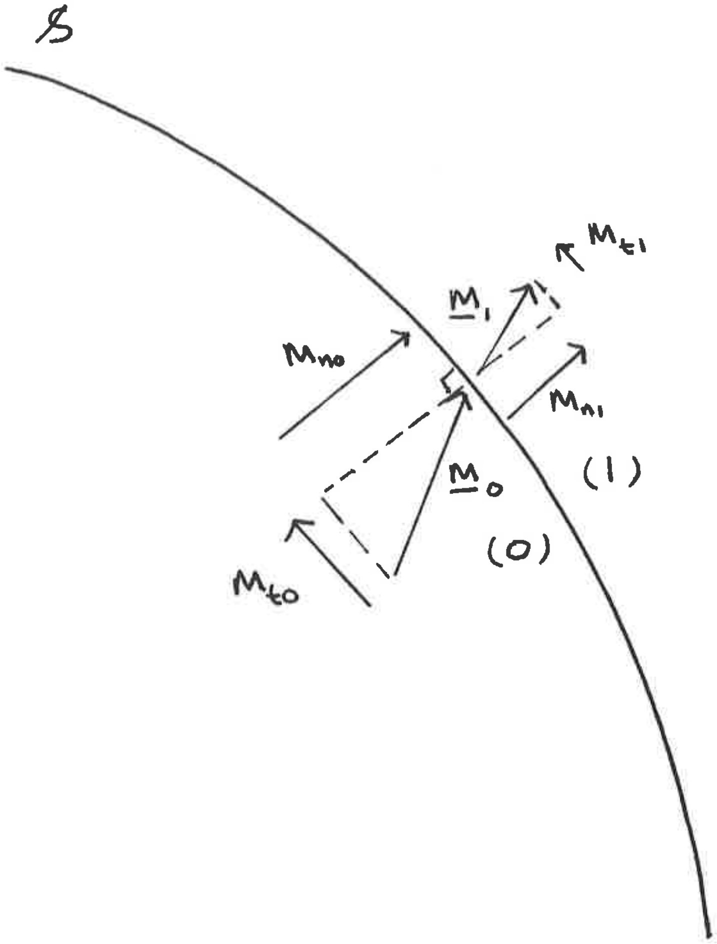


Fig 3.

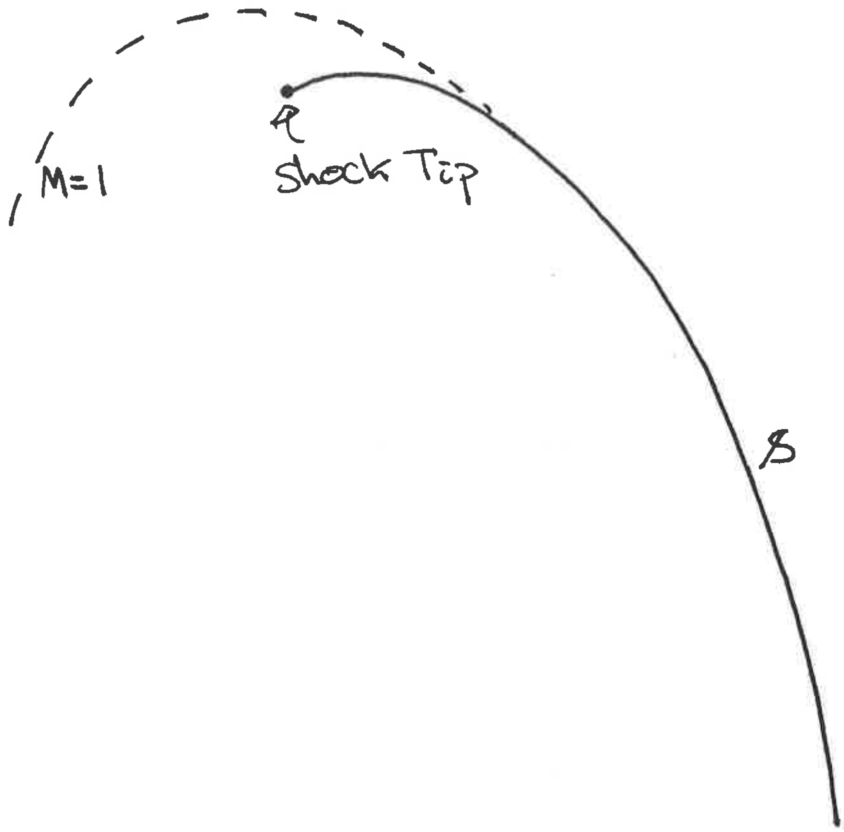
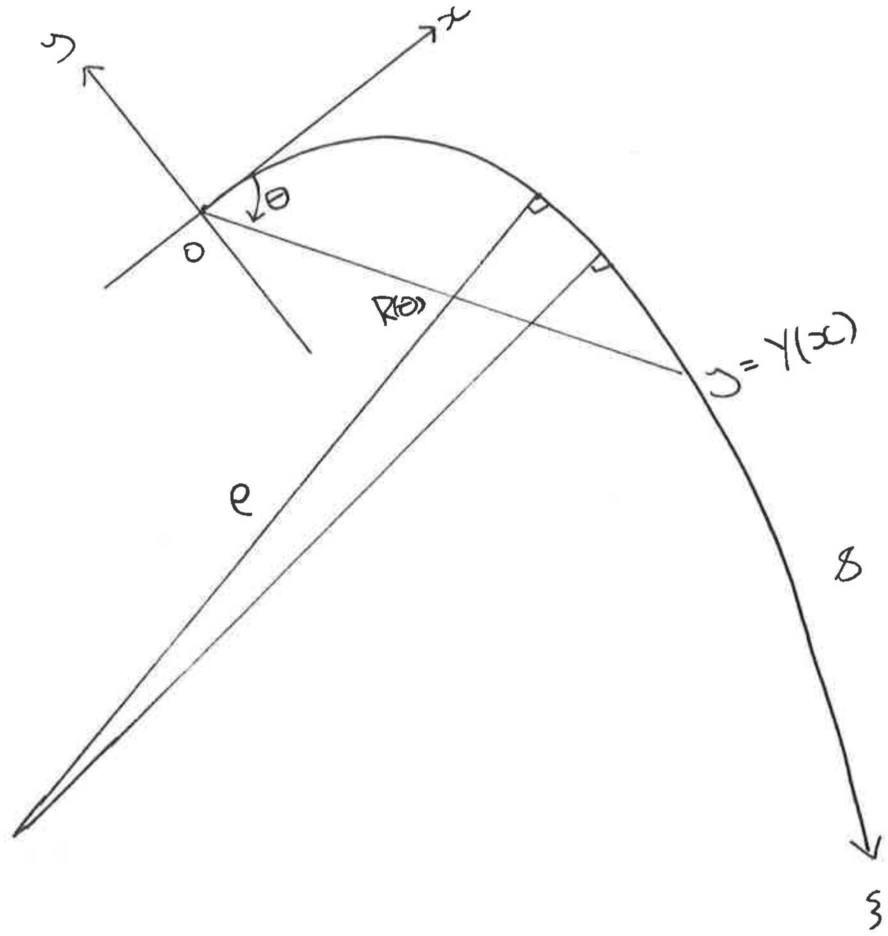


Fig 4.



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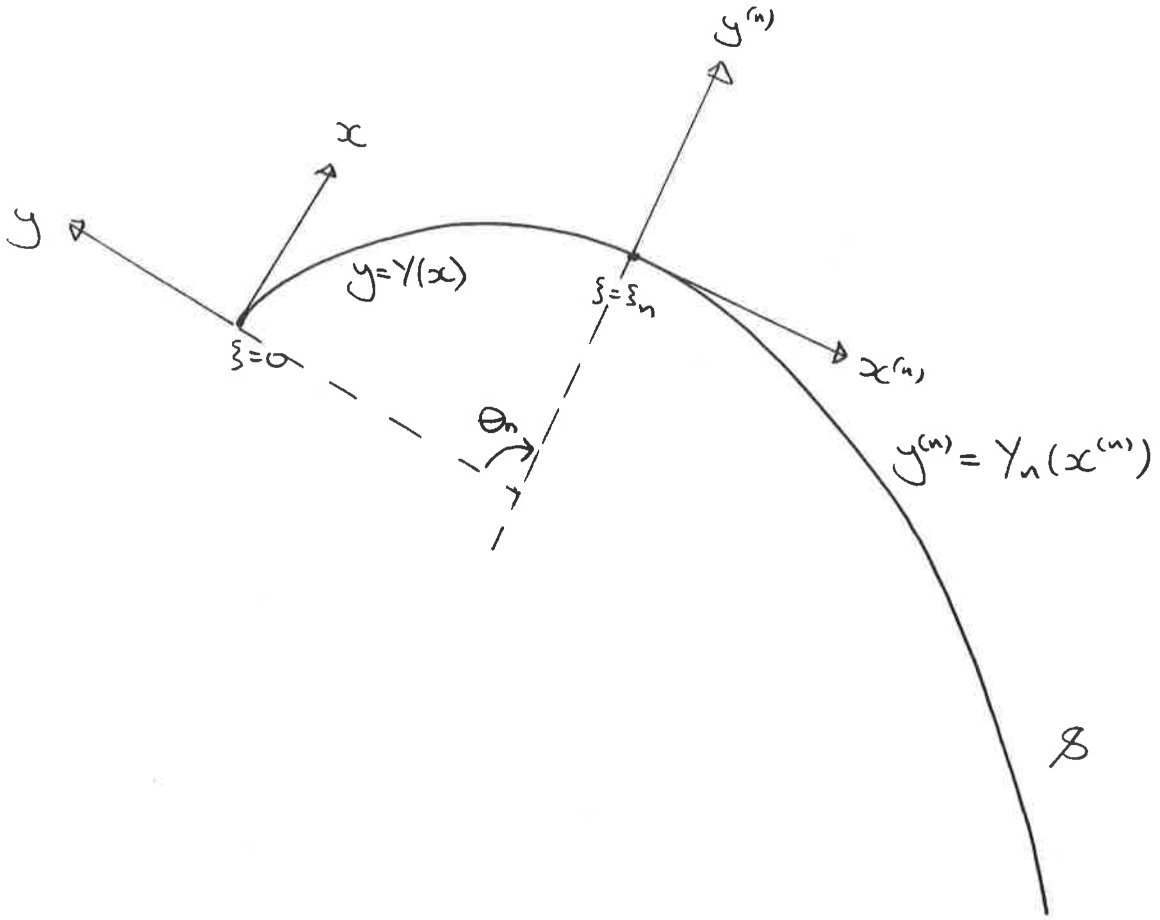

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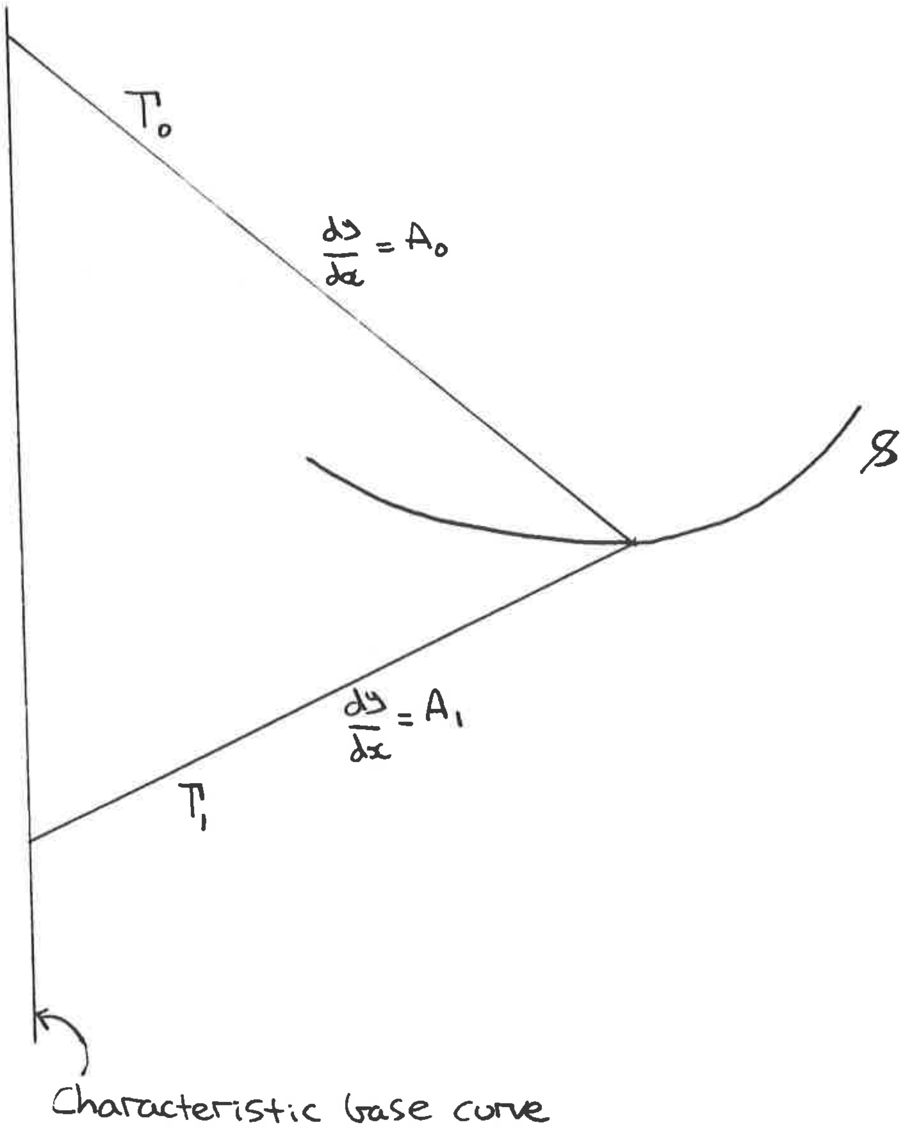

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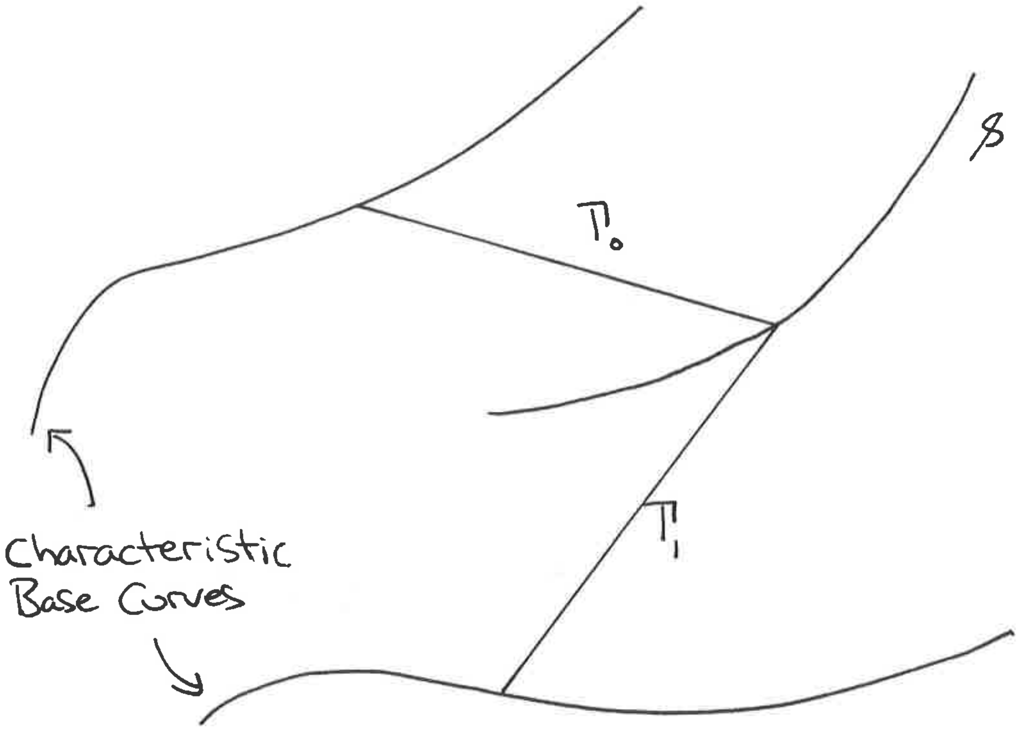
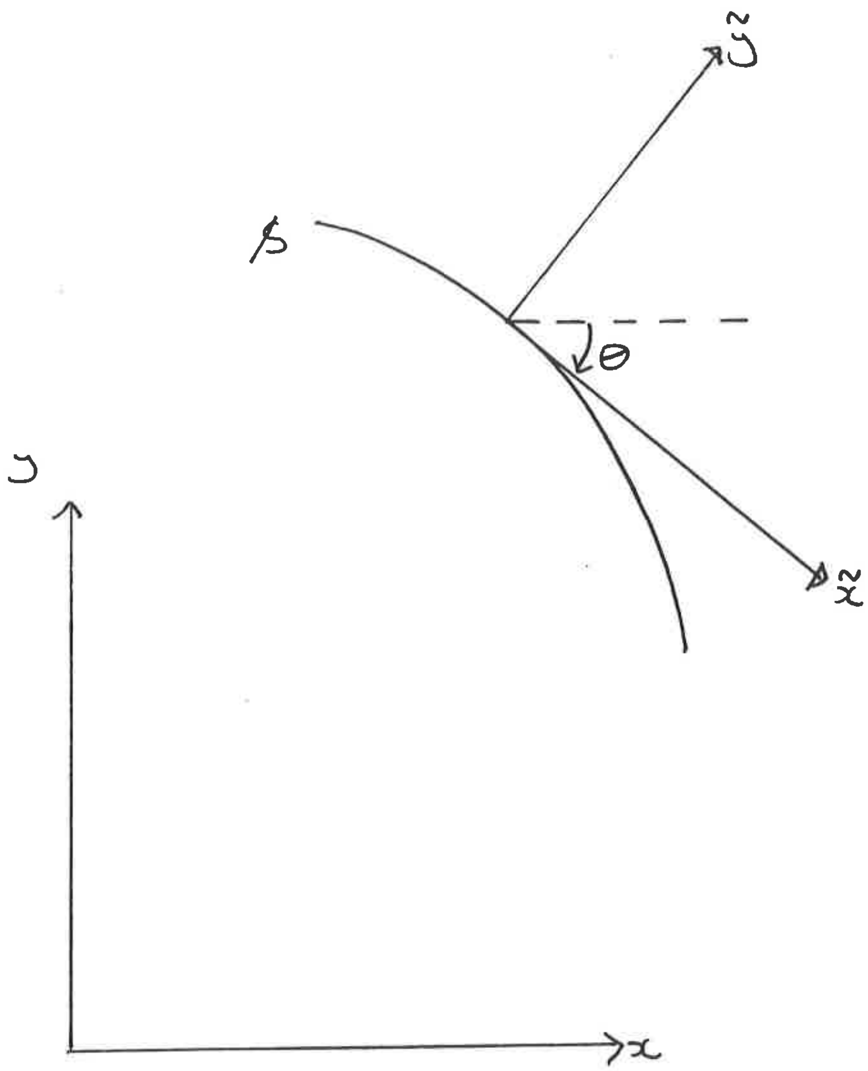


Fig. 8.



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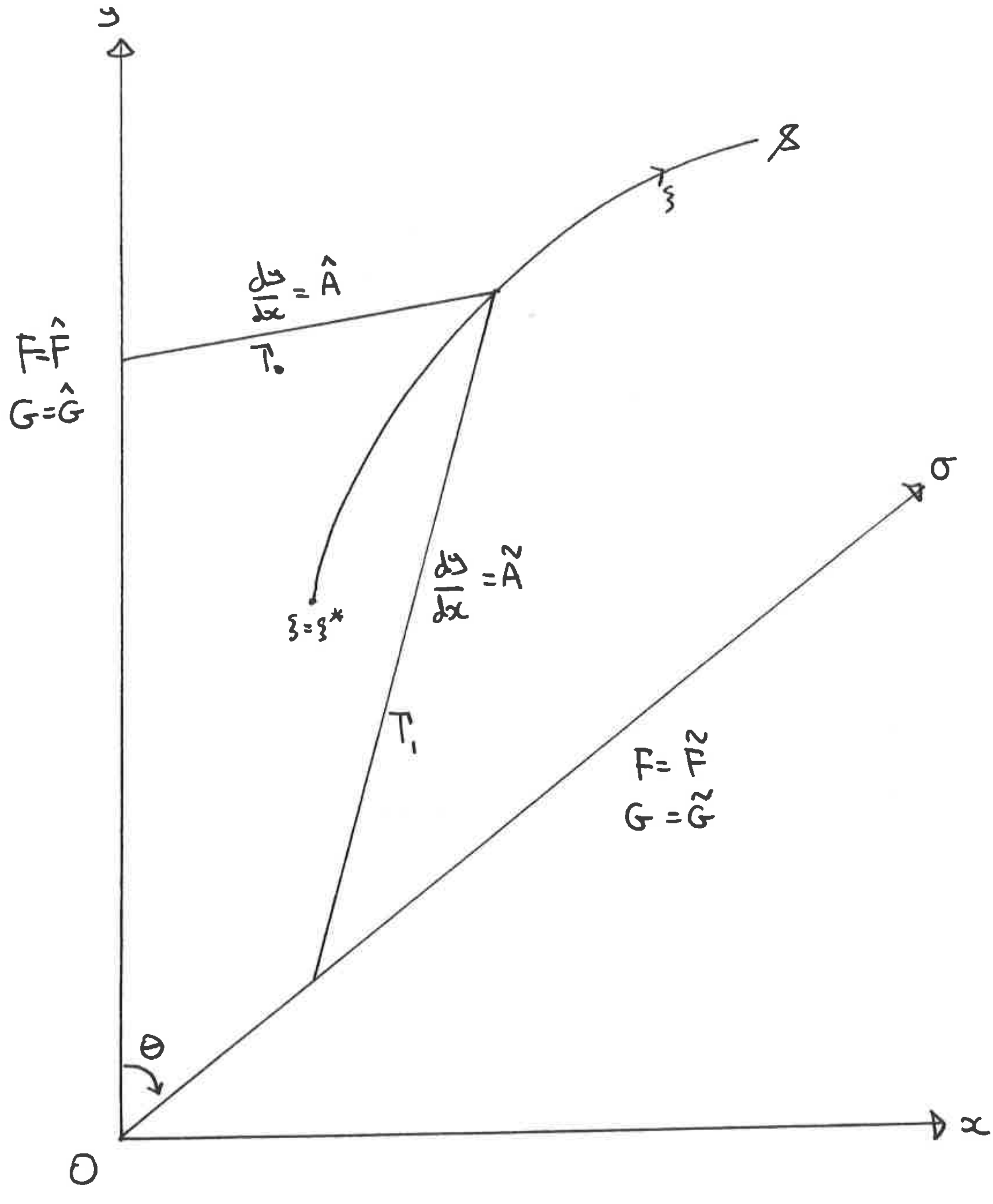
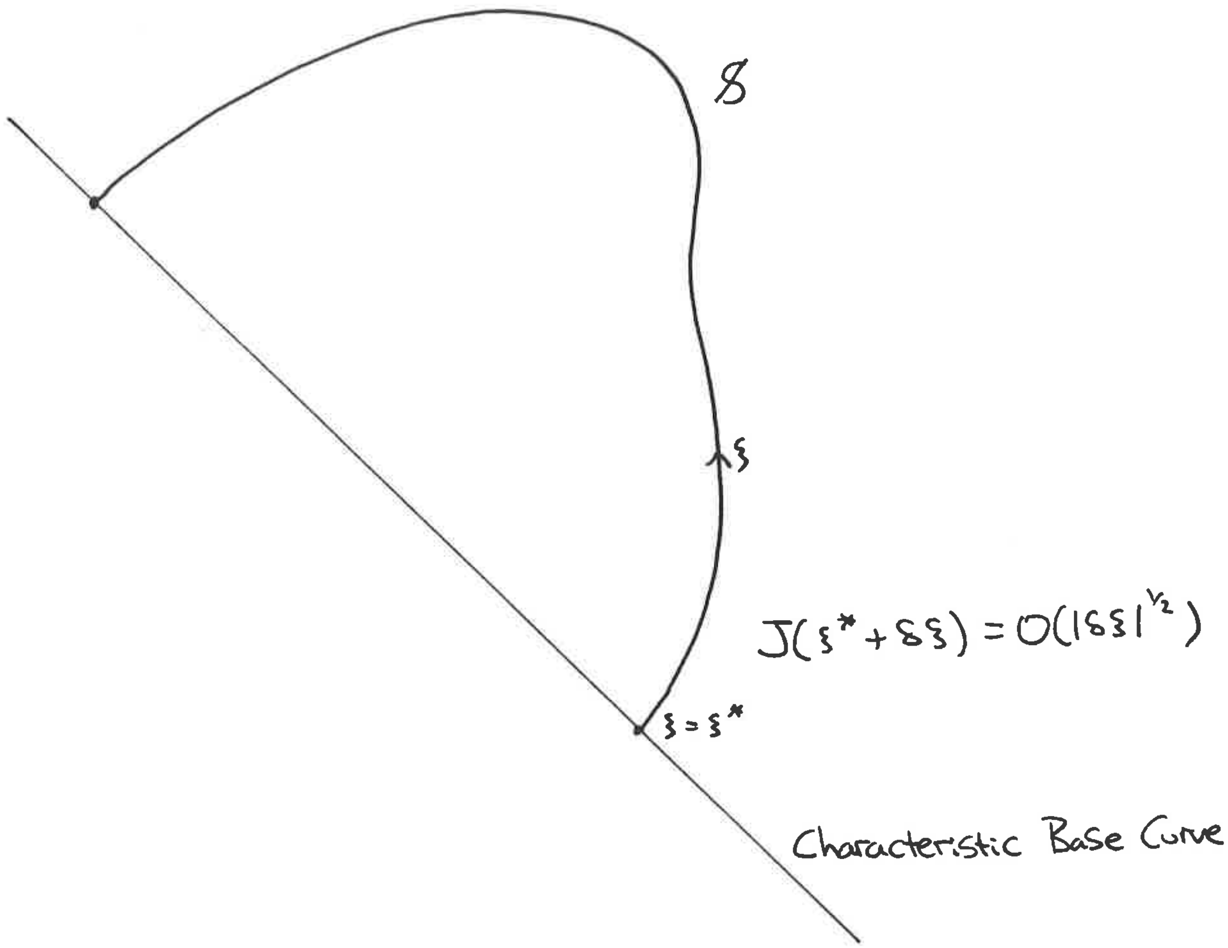


Fig. 10.



$\mathcal{S}$
ξ
$J(\xi^* + \delta\xi) = O(|\delta\xi|^{1/2})$
$\xi = \xi^*$
Characteristic Base Curve

Fig. 11.

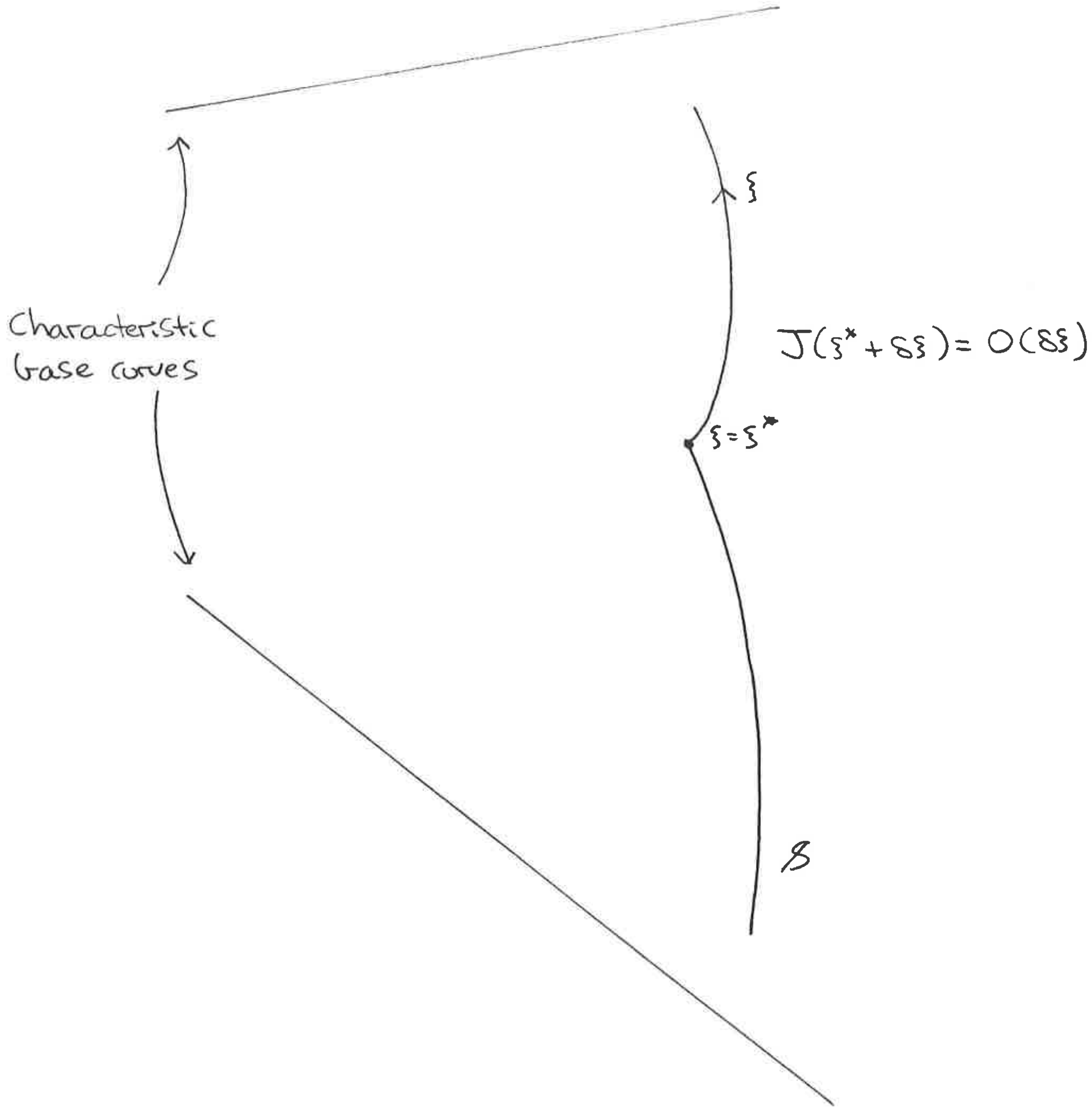


Fig. 12.