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Abstract

Feedback design for a second order control system leads to an eigenstructure assignment problem for a quadratic matrix polynomial. It is desirable that the feedback controller not only assigns specified eigenvalues to the second order closed loop system, but also that the system is *robust*, or insensitive to perturbations. We show that robustness of the quadratic inverse eigenvalue problem can be achieved by solving a generalized linear eigenvalue assignment problem subject to *structured* perturbations. Numerically reliable methods for solving the structured generalized linear problem are derived that take advantage of the special properties of the system in order to minimize the computational work required. In this part of the work we treat the case where the leading coefficient matrix in the quadratic polynomial is nonsingular, which ensures that the polynomial is regular. In a second part we will examine the case where the open loop matrix polynomial is not necessarily regular.

Keywords Second order control systems, quadratic inverse eigenvalue problem, feedback design, robust eigenstructure assignment, structured perturbations.

1   Introduction

The time-invariant second-order control system

$$J\ddot{\mathbf{z}} - D\dot{\mathbf{z}} - C\mathbf{z} = B\mathbf{u}, \quad \mathbf{z}(0), \dot{\mathbf{z}}(0) \text{ given,} \tag{1}$$

where $\mathbf{z}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, $J, D, C \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times m}$, arises naturally in a wide variety of applications, including, for example, the control of large flexible space structures, earthquake engineering, the control of mechanical multi-body systems, stabilization of damped gyroscopic systems, robotics, and vibration control in structural dynamics [1],[2],[12],[13],[19],[5],[21],[10],[11],[3], [22],[28],[20]. The control problem is to design a proportional and derivative state feedback controller of the form

$$\mathbf{u} = K_1 \mathbf{z} + K_2 \dot{\mathbf{z}} + \mathbf{r}, \tag{2}$$

where $K_1, K_2 \in \mathbb{R}^{m \times n}$ and $\mathbf{r}(t) \in \mathbb{R}^m$, such that the closed loop system

$$J\ddot{\mathbf{z}} - (D + BK_2)\dot{\mathbf{z}} - (C + BK_1)\mathbf{z} = B\mathbf{r} \tag{3}$$

has desired properties. The behaviour of the closed loop system (3) is governed by the eigenstructure of its associated quadratic matrix polynomial

$$P_{cl}(\lambda) \equiv \lambda^2 J - \lambda(D + BK_2) - (C + BK_1). \tag{4}$$

The response of the system can therefore be shaped by selecting the feedback gain matrices K_1 and K_2 to assign the eigenstructure of the quadratic polynomial (4). The control design problem is thus formulated as an inverse quadratic eigenvalue problem. In practice, (if $m > 1$) there is additional freedom in the solution to the problem and it is desirable to choose the feedback to ensure that the eigenstructure of the closed loop system is as *robust*, or insensitive to perturbations in the system matrices J, $D + BK_2$, $C + BK_1$, as possible.

Few computational techniques are available for treating the quadratic eigenstructure assignment problem directly. In [2],[12],[19], methods based on modal decompositions, which require the simultaneous diagonalization of the system matrices, are proposed. This approach is not generally applicable since the open loop system matrices may not always be diagonalizable. In any case, the technique is not numerically reliable because modal decompositions can be highly sensitive to computational errors. Two methods that are numerically reliable are described in [4]. The first of these is a modification of a technique proposed in [13] and the second is a generalization of a feedback stabilization procedure given in [6]. Both of these techniques aim to ensure that the

(augmented) matrix of eigenvectors is well-conditioned for inversion, which is a desirable property of the design. These procedures do not, however, ensure the *robustness* of the closed loop system.

In the majority of methods that have been proposed for solving the robust quadratic eigenvalue assignment problem, the second order control system (1) is rewritten as a first order system and techniques for treating the generalized linear feedback design problem are applied. There are two difficulties in using this approach. The first is that the measure of robustness for the linear problem is not the same as for the quadratic problem, since the allowable perturbations in the linear system are more general than in the quadratic problem. The second difficulty arises because the linear system has double the dimensions of the original quadratic system and, hence, the computational work used to solve the problem is greater than necessary.

In [14] we have developed numerical techniques for maximizing the robustness of the feedback design in linear systems that are subject to *structured* perturbations. We show here that the sensitivity of the eigenvalue problem for the quadratic polynomial is equivalent to that for a generalized linear pencil subject to a specific class of structured perturbations. The robustness of the second-order closed loop system can thus be ensured by solving a generalized linear eigenvalue assignment problem subject to this class of perturbations. We extend the methods derived in [14] to generalized linear systems and show how the special structure of the linear pencil derived from the quadratic polynomial can be exploited to reduce the computational work needed to solve the problem.

We consider here the case where the system matrix J is *nonsingular* and the quadratic polynomial is thus guaranteed to be regular. In a second paper we will consider the case where the system matrix J may be singular and the quadratic polynomial associated with the open loop system may not be regular. The aim of the feedback design is then to guarantee the regularity of the closed loop system as well as to assign the finite eigenvalues of the system robustly.

In the case where J is nonsingular, the quadratic matrix polynomial (4) can be reduced to a monic polynomial by applying the inverse of J from the left. In practice the inversion of J should be avoided to ensure numerical reliability. The nonsingularity of J is assumed here in the theoretical derivation of the robustness measures, but the computational methods derived here do not use this inverse and rely only on numerically stable procedures. We begin by presenting the background and sensitivity theory for the quadratic eigenvalue problem. In Section 3 we establish the relation between the quadratic problem

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and the linear eigenvalue problem subject to structured perturbations. The robust eigenstructure assignment problem is defined and analysed in Section 4, and a numerical method for constructing the feedback controller is described in Section 5. The results are summarized in the final section.

2 Quadratic Eigenvalue Problem

2.1 Preliminary theory

The quadratic matrix polynomial

$$P(\lambda) \equiv \lambda^2 J - \lambda D - C. \tag{5}$$

and the corresponding second order system (1) are said to be *regular* if

$$\det(P(\lambda)) \neq 0 \quad \text{for some } \lambda \in \mathbb{C}. \tag{6}$$

We assume throughout that the matrix J is nonsingular. The polynomial $P(\lambda)$ is thus regular and the system (1) is *solvable* in the sense that it admits a classical twice-differentiable solution $\mathbf{z}(t)$ for all continuous controls $\mathbf{u}(t)$ and *any* initial conditions $\mathbf{z}(0)$, $\dot{\mathbf{z}}(0) \in \mathbb{R}^n$. This solution can be characterized in terms of the eigenstructure of the quadratic polynomial $P(\lambda)$.

For J nonsingular, the generalized eigenvalues of the quadratic polynomial are given by the $2n$ values of $\lambda \in \mathbb{C}$ for which $\det(\lambda^2 J - \lambda D - C) = 0$. The corresponding right and left eigenvectors are defined, respectively, to be nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ satisfying

$$\begin{aligned}
(\lambda^2 J - \lambda D - C)\mathbf{v} &= 0, \\
\mathbf{w}^H(\lambda^2 J - \lambda D - C) &= 0.
\end{aligned} \tag{7}$$

Regularity of the polynomial ensures that there exist full rank matrices $V, W \in \mathbb{C}^{n \times 2n}$ that simultaneously satisfy

$$\begin{aligned}
JV\Lambda^2 - DV\Lambda - CV &= 0, \\
\Lambda^2 W^H J - \Lambda W^H D - W^H C &= 0,
\end{aligned} \tag{8}$$

and

$$VW^H J = 0, \qquad V\Lambda W^H J = I, \tag{9}$$

where $\Lambda \in \mathbb{C}^{2n \times 2n}$ is in Jordan canonical form with the eigenvalues of $P(\lambda)$ on the diagonal. The columns of V and W comprise, respectively, the right

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and left eigenvectors and principle vectors of the quadratic polynomial. The relations (9) define a specific normalization of these vectors.

We assume that the modal matrix V satisfying the first of (8) is such that $\tilde{V} = [V^T, (V\Lambda)^T]^T$ is nonsingular. Then, in the notation of [7], the matrix V and the Jordan matrix Λ together form a Jordan pair of the polynomial $P(\lambda)$. The matrix $W^H = \tilde{V}^{-1}[0, I]^T J^{-1}$ then satisfies the second of (8) and the relations (9) also hold. The matrices V, Λ, W are known as a Jordan triple of the quadratic polynomial. Conversely, we find that if V, Λ, W satisfy (8) and (9), where Λ is in Jordan form, then V, Λ, W form a Jordan triple and we can establish the following lemma.

Lemma 1 *Let V, W be full rank matrices satisfying (8)–(9), where Λ is in Jordan canonical form. Then the matrix $\tilde{V} = [V^T, (V\Lambda)^T]^T$ is nonsingular and its inverse is given by*

$$\tilde{V}^{-1} = [\Lambda W^H J - W^H D, W^H J]. \tag{10}$$

Proof. If (8) and (9) hold, then the conditions

$$\begin{aligned} V\Lambda W^H J - VW^H D &= I, \\ V(\Lambda^2 W^H J - \Lambda W^H D) = VW^H C &= 0, \end{aligned} \tag{11}$$

also hold. Therefore

$$\begin{bmatrix} V \\ V\Lambda \end{bmatrix} [\Lambda W^H J - W^H D, W^H J] = I_{2n}, \tag{12}$$

which proves the result. □

The solution to the second order system (1) can be written in terms of the Jordan triple V, Λ, W as follows.

Theorem 2 *Let V, W, Λ satisfy (8)–(9) and let $\mathbf{u}(t)$ be a continuous function on the interval $t \in [0, T]$. Then, the solution to the second-order system of differential equations (1) is given explicitly for all $t \in [0, T]$ by*

$$\begin{aligned} \mathbf{z}(t) &= V \exp(\Lambda t)(\Lambda W^H J - W^H D)\mathbf{z}(0) + V \exp(\Lambda t)W^H J\dot{\mathbf{z}}(0) \\ &\quad + \int_0^t V \exp(\Lambda(t-s))W^H B\mathbf{u}(s)ds. \end{aligned} \tag{13}$$

Proof. The proof is by differentiation and direct verification. We let $\mathbf{z}(t)$ be defined by (13) and assume that (9) holds. Then, by Leibnitz's rule, the

continuity of $\mathbf{u}(s)$ and $\exp(\Lambda(t-s))$ for $s,t \in [0,T]$ implies that the first and second derivatives of $\mathbf{z}(t)$ are given by

$$
\begin{aligned}
\dot{\mathbf{z}} &= V\Lambda\exp(\Lambda t)(\Lambda W^H J - W^H D)\mathbf{z}(0) + V\Lambda\exp(\Lambda t)W^H J\dot{\mathbf{z}}(0) \\
&\quad + \int_0^t V\Lambda\exp(\Lambda(t-s))W^H B\mathbf{u}(s)ds, \\
\ddot{\mathbf{z}} &= V\Lambda^2\exp(\Lambda t)(\Lambda W^H J - W^H D)\mathbf{z}(0) + V\Lambda^2\exp(\Lambda t)W^H J\dot{\mathbf{z}}(0) \\
&\quad + \int_0^t V\Lambda^2\exp(\Lambda(t-s))W^H B\mathbf{u}(s)ds + J^{-1}B\mathbf{u}(t).
\end{aligned}
\tag{14}
$$

The relations (9) imply also that the initial conditions on $\mathbf{z}$ and $\dot{\mathbf{z}}$ at $t=0$ are both satisfied. The proof then follows from (8) by direct substitution of (14) into (1). (See also [7], [17].) $\square$

The response of the control system is therefore shaped by the eigenstructure of its corresponding quadratic polynomial, and the robustness of the system design depends on the sensitivity of the eigenstructure to perturbations in the system matrices. In the next sections, measures of the sensitivity and robustness of the system are derived.

2.2 Sensitivity and robustness

In order to measure the sensitivity of an eigenvalue of the quadratic polynomial $P(\lambda)$ to perturbations in its coefficent matrices, we follow the approach of Wilkinson [26]. Without loss of generality (since J is nonsingular), we let $J\delta J, J\delta D, J\delta C \in \mathbb{R}^{n\times n}$ denote the perturbations in the coefficient matrices J, D, C, respectively. We assume that λ is a simple eigenvalue of $P(\lambda)$ with corresponding right and left eigenvectors $\mathbf{v}$ and $\mathbf{w}$ satisfying (7). The *condition number* of λ is then defined to be

$$
c(\lambda) = \lim_{\epsilon\to 0}\sup\left(|\delta\lambda|/\epsilon\right),
\tag{15}
$$

where

$$
\left((\lambda+\delta\lambda)^2(J+J\delta J) - (\lambda+\delta\lambda)(D+J\delta D) - (C+J\delta C)\right)(\mathbf{v}+\delta\mathbf{v}) = 0, \tag{16}
$$

and

$$
\|[\delta J, \delta D, \delta C]\|_2 \le \epsilon.
\tag{17}
$$

It is assumed that ϵ is sufficiently small to ensure that $J(I+\delta J)$ is nonsingular and the perturbed polynomial thus remains regular. (It is assumed implicitly

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in the definition that the perturbations $\delta\lambda, \delta\mathbf{v} \to 0$ as $\epsilon \to 0$. See also [9].)
From this definition we have that

$$|\delta\lambda| \le c(\lambda)\epsilon + O(\epsilon^2), \tag{18}$$

and the condition number $c(\lambda)$ therefore gives a measure of the sensitivity of λ to perturbations of order ϵ in the coefficients of $P(\lambda)$. An explicit form for $c(\lambda)$ can be derived as follows.

Theorem 3 *Let λ be a simple eigenvalue of the quadratic polynomial (5). Then, the condition number $c(\lambda)$ is given by*

$$c(\lambda) = \frac{\alpha \left\|\mathbf{w}^H J\right\|_2 \|\mathbf{v}\|_2}{|\mathbf{w}^H (2\lambda J - D)\mathbf{v}|}, \tag{19}$$

where $\alpha = (|\lambda|^4 + |\lambda|^2 + 1)^{\frac{1}{2}}$.

Proof. By expanding (16), premultiplying by $\mathbf{w}^H$ and applying (7) we obtain

$$
\begin{aligned}
\delta\lambda \mathbf{w}^H (2\lambda J - D)\mathbf{v} &= -\mathbf{w}^H J(\lambda^2 \delta J - \lambda \delta D - \delta C)\mathbf{v} + O(\epsilon^2) \\
&= -(\mathbf{w}^H J)[\delta J, \delta D, \delta C] \begin{bmatrix} \lambda^2 \mathbf{v} \\ -\lambda \mathbf{v} \\ -\mathbf{v} \end{bmatrix} + O(\epsilon^2).
\end{aligned}
\tag{20}
$$

The assumption that λ is a simple eigenvalue implies that $\mathbf{w}^H (2\lambda J - D)\mathbf{v} \neq 0$, and hence an upper bound on the first order perturbation in λ is given by

$$|\delta\lambda| \le \frac{\alpha \left\|\mathbf{w}^H J\right\|_2 \|\mathbf{v}\|_2}{|\mathbf{w}^H (2\lambda J - D)\mathbf{v}|} \left\|[\delta J, \delta D, \delta C]\right\|_2 + O(\epsilon^2). \tag{21}$$

To show that this upper bound is attained we let $T = (\epsilon/\alpha)J^T \mathbf{w}\mathbf{v}^H / \left\|\mathbf{w}^H J\right\|_2 \|\mathbf{v}\|_2$, and take $\delta J = \bar{\lambda}^2 T$, $\delta D = -\bar{\lambda} T$ and $\delta C = -T$. Then

$$\left\|[\delta J, \delta D, \delta C]\right\|_2 = \epsilon \tag{22}$$

and, since

$$|\mathbf{w}^H J(\lambda^2 \delta J - \lambda \delta D - \delta C)\mathbf{v}| = \epsilon\alpha \left\|\mathbf{w}^H J\right\|_2 \|\mathbf{v}\|_2, \tag{23}$$

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we obtain equality in (21) for these choices of the perturbations. Dividing (21) by ϵ and taking the limit as $\epsilon \to 0$ then completes the proof. $\square$

The condition number $c(\lambda)$ given by (19) measures the sensitivity of the eigenvalue λ to perturbations in $P(\lambda)$ in an absolute sense. For a nonzero eigenvalue, a measure of the relative sensitivity is given by the condition number $\kappa(\lambda)$ defined, as in [25], to be

$$\kappa(\lambda) = \lim_{\epsilon \to 0} \sup \left(|\delta\lambda| / (\epsilon|\lambda|) \right). \tag{24}$$

With this definition we find that

$$\kappa(\lambda) = c(\lambda)/|\lambda| = \frac{\alpha \left\| \mathbf{w}^H J \right\|_2 \|\mathbf{v}\|_2}{|\lambda| |\mathbf{w}^H (2\lambda J - D)\mathbf{v}|}. \tag{25}$$

This expression is similar to the result derived in [25]. The difference is due to the definition of the perturbations and the form of the bound on $\delta J, \delta D, \delta C$. The particular form chosen here ensures that the same condition number is derived if the polynomial is first reduced to monic form. It also allows the relations between the quadratic and linear cases to be established directly, as shown in Section 3. More importantly, this formulation leads to a numerical procedure for solving the robust eigenstructure assignment problem that does not require the inversion of the matrix J.

To measure the robustness of the second-order system (1), we need an indicator of the overall sensitivity of the eigenvalues of the corresponding quadratic polynomial (5). The condition number (19) gives a proportional measure of the sensitivity of a simple eigenvalue to perturbations of order ϵ in the coefficient matrices. For a nondefective eigenvalue λ of multiplicity p, the condition numbers (19) are also well-defined for a particular choice of the basis eigenvectors $\{\mathbf{v}_j\}_1^p, \{\mathbf{w}_j\}_1^p$ spanning the corresponding right and left invariant subspaces. Provided that these bases are biorthogonal with respect to the matrix $2\lambda J - D$, then an equivalent proportional measure of the sensitivity of the eigenvalue is given by the square root of the sum of the squares of all the associated condition numbers. If the system has a defective multiple eigenvalue, then the sensitivity of *some* eigenvalue to perturbations of order ϵ is expected to be larger by at least an order of magnitude in ϵ. Therefore, systems that have defective eigenvalues are necessarily less robust than nondefective systems.

As a global measure of robustness we thus take

$$\nu^2 = \sum_{j=1}^{2n} \omega_j^2 c(\lambda_j)^2, \tag{26}$$

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where the eigenvalues $\{\lambda_j\}_1^n$ of the system are assumed to be nondefective and the positive weights ω_j, $j = 1, \ldots, 2n$, satisfy $\sum_{j=1}^{2n} \omega_j^2 = 1$, with $\omega_j = \omega_k$ if $\lambda_j = \lambda_k$. (See also [16], [15].) If the right and left eigenvectors $\mathbf{v}_j$, $\mathbf{w}_j$, corresponding to λ_j, are normalized such that

$$(|\lambda_j|^4 + |\lambda_j|^2 + 1)^{\frac{1}{2}} \|\mathbf{v}_j\|_2 = 1, \quad |\mathbf{w}_j^H (2\lambda_j J - D)\mathbf{v}_j| = 1, \quad j = 1, \ldots, 2n, \quad (27)$$

then the robustness measure ν^2 can be written

$$\nu^2 = \sum_{j=1}^{2n} \omega_j^2 \left\|\mathbf{w}_j^H J\right\|_2^2 = \left\|D_\omega W^H J\right\|_F^2, \tag{28}$$

where $D_\omega = \text{diag}\{\omega_1, \ldots, \omega_{2n}\}$. (Here $\|(\cdot)\|_F$ denotes the Frobenius matrix norm.) The normalization (27) is consistent with (9) and is selected to enable the relationships with the linear eigenvalue problem to be established.

2.3   Monic polynomial

Of practical interest is the case where $P(\lambda)$ is a monic polynomial with leading coefficient matrix $J = I$. It is assumed that this leading coefficient matrix is not subject to perturbations. We consider specifically the monic quadratic polynomial

$$\hat{P}(\lambda) \equiv \lambda^2 I - \lambda A_2 - A_1, \tag{29}$$

corresponding to the second order system (1), where $J = I$, $D = A_2$, $C = A_1$.

A measure of the sensitivity of a simple eigenvalue λ of the monic polynomial (29) to perturbations $\delta A_1, \delta A_2$ in its coefficient matrices A_1, A_2, respectively, is given by the condition number $c(\lambda)$ defined in (15). The right and left eigenvectors corresponding to λ are again denoted by $\mathbf{v}, \mathbf{w}$ and the first order perturbation $\delta\lambda$ now satisfies

$$\left((\lambda + \delta\lambda)^2 I - (\lambda + \delta\lambda)(A_2 + \delta A_2) - (A_1 + \delta A_1)\right)(\mathbf{v} + \delta\mathbf{v}) = 0, \tag{30}$$

where

$$\|[\delta A_1, \delta A_2]\|_2 \le \epsilon. \tag{31}$$

An explicit form for $c(\lambda)$ in this case can be derived as follows.

Theorem 4 *Let λ be a simple eigenvalue of the monic polynomial (29). Then, the condition number $c(\lambda)$ is given by*

$$c(\lambda) = \frac{\hat{\alpha} \left\|\mathbf{w}^H\right\|_2 \|\mathbf{v}\|_2}{|\mathbf{w}^H (2\lambda I - A_2)\mathbf{v}|}, \tag{32}$$

where $\hat{\alpha} = (|\lambda|^2 + 1)^{\frac{1}{2}}$.

Proof. By expanding (30), premultiplying by $\mathbf{w}^H$, and using the assumption that λ is a simple eigenvalue, we can show, by similar arguments to those in Theorem 3, that an upper bound on the first order perturbation in λ is given by

$$|\delta\lambda| \leq \frac{\hat{\alpha} \left\| \mathbf{w}^H \right\|_2 \|\mathbf{v}\|_2}{|\mathbf{w}^H(2\lambda I - A_2)\mathbf{v}|} \|[\delta A_1, \delta A_2]\|_2 + O(\epsilon^2). \tag{33}$$

This upper bound is attained for the perturbations $\delta A_1 = T$ and $\delta A_2 = \bar{\lambda}T$, where $T = (\epsilon/\hat{\alpha})\mathbf{w}\mathbf{v}^H / \left\| \mathbf{w}^H \right\|_2 \|\mathbf{v}\|_2$, since this choice ensures that

$$\|[\delta A_1, \delta A_2]\|_2 = \epsilon \tag{34}$$

and

$$|\mathbf{w}^H(\lambda\delta A_2 + \delta A_1)\mathbf{v}| = \epsilon\hat{\alpha} \left\| \mathbf{w}^H \right\|_2 \|\mathbf{v}\|_2. \tag{35}$$

Dividing (33) by ϵ and taking the limit as $\epsilon \to 0$ then completes the proof. $\square$

The form of the condition number in the monic case is thus the same as in the generalized case up to a constant factor. The difference is due to the different assumptions on the allowable perturbations.

The condition number (32) gives an absolute measure of the sensitivity of the eigenvalue λ in the monic case. For a nonzero eigenvalue, a measure of the relative sensitivity is given by the condition number $\kappa(\lambda)$ defined in (24). We find now that

$$\kappa(\lambda) = c(\lambda)/|\lambda| = \frac{\hat{\alpha} \left\| \mathbf{w}^H \right\|_2 \|\mathbf{v}\|_2}{|\lambda||\mathbf{w}^H(2\lambda I - A_2)\mathbf{v}|}. \tag{36}$$

The global measure of robustness in the monic case is also taken to be ν^2, defined as in (26). Normalizing the eigenvectors of the polynomial such that

$$(1 + |\lambda_j|^2)^{\frac{1}{2}} \|\mathbf{v}_j\|_2 = 1, \quad |\mathbf{w}_j^H(2\lambda_j I - A_2)\mathbf{v}_j| = 1, \quad j = 1, \ldots, 2n, \tag{37}$$

then gives

$$\nu^2 = \sum_{j=1}^{2n} \omega_j^2 \left\| \mathbf{w}_j^H \right\|_2^2 = \left\| D_\omega W^H \right\|_F^2. \tag{38}$$

In both the generalized and the monic quadratic polynomial cases, the control design problem is to select the feedback gains to assign a given set of $2n$ nondefective eigenvalues to the second-order closed loop system and to minimize its robustness measure ν^2. In Section 3 we show that this problem can be solved by minimizing the robustness of a generalized linear system subject to a restricted set of perturbations.

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3 Generalized Linear Problem

3.1 Transformation of the system

The inverse quadratic eigenvalue problem is commonly treated by transforming the second-order control system (1) into a generalized linear state-space, or *descriptor*, system of the form

$$E\dot{\mathbf{x}} = A\mathbf{x} + \tilde{B}\mathbf{u}, \quad \mathbf{x}(0) \text{ given}, \tag{39}$$

where $E, A \in \mathbb{R}^{2n \times 2n}, \tilde{B} \in \mathbb{R}^{2n \times m}$ and $\mathbf{x} = [\ \mathbf{z}^T, \ \dot{\mathbf{z}}^T\ ]^T$. Various transformations can be used to embed the second-order equations into the linear form. We consider the generalized linear system where

$$E = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}. \tag{40}$$

This form is suitable for treating the feedback design problem. Different transformations may be desirable for other purposes (see [25]).

The response of the system (39) is governed by the eigenstructure of the generalized linear matrix pencil

$$L(\lambda) \equiv \lambda E - A. \tag{41}$$

Since J is nonsingular, the linear pencil $L(\lambda)$ is regular in the case where E, A are defined by (40). The system (39) is then uniquely solvable for any continuous control $\mathbf{u}(t)$ and the solution is equivalent to that of the second-order system (1). The solutions to (1) can therefore also be characterized in terms of the eigenstructure of $L(\lambda)$.

The generalized eigenvalues of the linear pencil (41) are given by the $2n$ values of $\lambda \in \mathbb{C}$ for which $\det(\lambda E - A) = 0$. The corresponding right and left eigenvectors are defined, respectively, to be nonzero vectors $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{w}}$ satisfying

$$\begin{aligned} (\lambda E - A)\tilde{\mathbf{v}} &= 0, \\ \tilde{\mathbf{w}}^H(\lambda E - A) &= 0. \end{aligned} \tag{42}$$

Regularity of the pencil ensures that there exist nonsingular matrices $\tilde{V}, \tilde{W} \in \mathbb{C}^{2n \times 2n}$ that simultaneously satisfy

$$\begin{aligned} E\tilde{V}\Lambda - A\tilde{V} &= 0, \\ \Lambda\tilde{W}^H E - \tilde{W}^H A &= 0, \end{aligned} \tag{43}$$

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and

$$\tilde{W}^H E \tilde{V} = I, \tag{44}$$

where $\Lambda \in \mathbb{C}^{2n \times 2n}$ is in Jordan canonical form. The columns of $\tilde{V}$ and $\tilde{W}$ comprise, respectively, the right and left eigenvectors and principle vectors of the linear matrix pencil. The relation (44) defines a normalization of these vectors.

The equivalence between the eigenstructure of the linear matrix pencil (41) with coefficients given by (40) and that of the quadratic matrix polynomial (5) can now be established.

Theorem 5 *Let E, A be given by (40). If $\tilde{V}, \tilde{W}$ are nonsingular matrices satisfying (43)–(44), where Λ is in Jordan canonical form, then*

$$\tilde{V} = \begin{bmatrix} V \\ V\Lambda \end{bmatrix}, \qquad \tilde{W}^H = [\Lambda W^H J - W^H D, W^H], \tag{45}$$

where V, W are full rank matrices satisfying (8)–(9). Conversely, if V, W are full rank matrices satisfying (8)–(9), then $\tilde{V}, \tilde{W}$ given by (45) are nonsingular and satisfy (43)–(44).

Proof. We let $\tilde{V} = [\tilde{V}_1^H, \tilde{V}_2^H]^H$. If $\tilde{V}$ satisfies the first of (43), where E, A are defined as in (40), then

$$\begin{aligned} \tilde{V}_2 &= \tilde{V}_1 \Lambda, \\ C\tilde{V}_1 + D\tilde{V}_2 &= J\tilde{V}_2 \Lambda. \end{aligned} \tag{46}$$

It follows that $\tilde{V}_1 = V$ satisfies the first of (8) and $\tilde{V}_2 = V\Lambda$. Conversely, if V satisfies the first of (8), then

$$A \begin{bmatrix} V \\ V\Lambda \end{bmatrix} = \begin{bmatrix} V\Lambda \\ CV + DV\Lambda \end{bmatrix} = \begin{bmatrix} V\Lambda \\ JV\Lambda^2 \end{bmatrix} = E \begin{bmatrix} V \\ V\Lambda \end{bmatrix} \Lambda, \tag{47}$$

and the first of (43) is satisfied. The relation between $\tilde{W}$ and W is shown similarly. The invertibility of E together with (44) then implies that $E^{-1} = \tilde{V}\tilde{W}^H$ and hence, from (45), V, W must satisfy $VW^H = 0$ and $V\Lambda W^H = J^{-1}$ and (9) must hold. Conversely, if conditions (8)–(9) are satisfied, then, by Lemma 1, $\tilde{V}, \tilde{W}$ are invertible and $\tilde{V}\tilde{W}^H = E^{-1}$, which implies that (44) holds. □

We next relate the sensitivity of the eigenstructure of the quadratic matrix polynomial to that of the linear matrix pencil.

3.2 Sensitivity to structured perturbations

The sensitivity of a simple eigenvalue λ of the linear matrix pencil (41) to arbitrary perturbations in the pencil is known to be directly proportional to the condition number

$$c^L(\lambda) = \left\| \tilde{\mathbf{w}}^H E \right\|_2 \| \tilde{\mathbf{v}} \|_2 / |\tilde{\mathbf{w}}^H E \tilde{\mathbf{v}}|, \qquad (48)$$

where $\tilde{\mathbf{v}}, \tilde{\mathbf{w}}$ are the right and left eigenvectors of the pencil corresponding to λ. (See [26],[9],[23],[24].) In the case where the coefficient matrices of the pencil are given by (40), this condition number is *not* equivalent to the condition number $c(\lambda)$ of the embedded quadratic matrix polynomial derived in Section 2.2. The condition number $c^L(\lambda)$ measures the sensitivity of λ to arbitrary perturbations in all components of the coefficient matrices E, A of the pencil, whereas the condition number $c(\lambda)$ measures the sensitivity of λ only to perturbations in the coefficient matrices J, D, C of the quadratic polynomial.

In order to establish a condition number for the generalized linear eigenproblem that is equivalent to that of the quadratic eigenproblem, we need to find a measure of the sensitivity of an eigenvalue of the pencil (41) to a specific class of *structured* perturbations. We assume again that λ is a simple eigenvalue of $L(\lambda)$ with corresponding right and left eigenvectors $\tilde{\mathbf{v}}, \tilde{\mathbf{w}}$, respectively. We consider perturbations $\delta E, \delta A$ to the coefficient matrices E, A of $L(\lambda)$ of the form

$$\delta E = EF\Delta_E G_1^T, \qquad \delta A = EF\Delta_A G_2^T, \qquad (49)$$

where Δ_E, Δ_A are arbitrary (unknown) disturbance matrices and F, G_1, G_2 are specified matrices that define the structure of the perturbations. The sensitivity of λ to perturbations of the form (49) can then be measured by the condition number $\tilde{c}(\lambda)$, defined as in (15), where the first order perturbation $\delta\lambda$ now satisfies

$$\left((\lambda + \delta\lambda)(E + EF\Delta_E G_1^T) - (A + EF\Delta_A G_2^T) \right) (\tilde{\mathbf{v}} + \delta\tilde{\mathbf{v}}) = 0, \qquad (50)$$

and

$$\| [\Delta_E, \Delta_A] \|_2 \le \epsilon. \qquad (51)$$

It is assumed that ϵ is sufficiently small to ensure that $E(I + F\Delta_E G_1^T)$ is nonsingular and the perturbed linear pencil therefore remains regular. An explicit form for $\tilde{c}(\lambda)$ is given as follows.

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Theorem 6 *Let λ be a simple eigenvalue of the linear matrix pencil (41). Then, the condition number $\tilde{c}(\lambda)$ is given by*

$$\tilde{c}(\lambda) = \frac{\left\|\tilde{\mathbf{w}}^H EF\right\|_2 \left\|G_\lambda^T \tilde{\mathbf{v}}\right\|_2}{|\tilde{\mathbf{w}}^H E\tilde{\mathbf{v}}|}, \tag{52}$$

where $G_\lambda = [\ \lambda G_1, \quad -G_2\ ]$.

Proof. Applying arguments analogous to those in the proofs of Theorems 3 and 4, we find from (50) that

$$\begin{aligned}
|\delta\lambda||\tilde{\mathbf{w}}^H E\tilde{\mathbf{v}}| &= |\tilde{\mathbf{w}}^H[\lambda EF\Delta_E G_1^T - EF\Delta_A G_2^T]\tilde{\mathbf{v}}| + O(\epsilon^2) \\
&= |\tilde{\mathbf{w}}^H EF[\Delta_E, \Delta_A] \begin{bmatrix} \lambda G_1^T \\ -G_2^T \end{bmatrix} \tilde{\mathbf{v}}| + O(\epsilon^2) \tag{53} \\
&\leq \left\|\tilde{\mathbf{w}}^H EF\right\|_2 \left\|G_\lambda^T \tilde{\mathbf{v}}\right\|_2 \epsilon + O(\epsilon^2).
\end{aligned}$$

Regularity of the pencil ensures that $\tilde{\mathbf{w}}^H E\tilde{\mathbf{v}} \neq 0$, and hence an upper bound on the first order perturbation in λ is given by

$$|\delta\lambda| \leq \frac{\left\|\tilde{\mathbf{w}}^H EF\right\|_2 \left\|G_\lambda^T \tilde{\mathbf{v}}\right\|_2}{|\tilde{\mathbf{w}}^H E\tilde{\mathbf{v}}|} \|[\Delta_E, \Delta_A]\|_2 + O(\epsilon^2). \tag{54}$$

Equality in (54) is achieved for the perturbations

$$\Delta_E = \bar{\lambda} E^T F^T \tilde{\mathbf{w}}\tilde{\mathbf{v}}^H G_1 \epsilon/\tau, \qquad \Delta_A = -E^T F^T \tilde{\mathbf{w}}\tilde{\mathbf{v}}^H G_2 \epsilon/\tau, \tag{55}$$

where $\tau = \left\|\tilde{\mathbf{w}}^H EF\right\|_2 \left\|G_\lambda^T \tilde{\mathbf{v}}\right\|_2$. Then $\|[\Delta_E, \Delta_A]\|_2 = \epsilon$ and

$$|\tilde{\mathbf{w}}^H E[\lambda F\Delta_E G_1^T - F\Delta_A G_2^T]\tilde{\mathbf{v}}| = \epsilon \left\|\tilde{\mathbf{w}}^H EF\right\|_2 \left\|G_\lambda^T \tilde{\mathbf{v}}\right\|_2, \tag{56}$$

and the upper bound on $|\delta\lambda|$ is attained. Dividing (54) by ϵ and taking the limit as $\epsilon \to 0$ then proves the result. $\square$

In the case where the quadratic polynomial is embedded in the linear pencil (41) and the coefficients of the pencil are given by (40), the arbitrary perturbations may be taken to be

$$\Delta_E = \delta J, \qquad \Delta_A = [\delta D, \delta C], \tag{57}$$

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and the matrices F, G_λ that structure the perturbations may be defined by

$$F = \begin{bmatrix} 0 \\ I_n \end{bmatrix}, \quad G_1^T = [0, I_n], \quad G_2^T = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}. \tag{58}$$

The admissible structured perturbations (49) then have the forms

$$\delta E = \begin{bmatrix} 0 & 0 \\ 0 & J\delta J \end{bmatrix}, \quad \delta A = \begin{bmatrix} 0 & 0 \\ J\delta C & J\delta D \end{bmatrix}, \tag{59}$$

where

$$\|[\delta J, \delta D, \delta C]\|_2 = \|[\Delta_E, \Delta_A]\|_2 \le \epsilon. \tag{60}$$

The condition number $\tilde{c}(\lambda)$ of the linear pencil, subject to the structured perturbations, can now be shown to equal the condition number $c(\lambda)$ of the quadratic polynomial.

Corollary 7 *Let E, A be defined by (40) and let Δ_E, Δ_A and F, G_λ be defined by (57)–(58). Then, the condition number $\tilde{c}(\lambda)$ satisfies*

$$\tilde{c}(\lambda) = \frac{\alpha \left\| \mathbf{w}^H J \right\|_2 \|\mathbf{v}\|_2}{|\mathbf{w}^H (2\lambda J - D)\mathbf{v}|} \equiv c(\lambda), \tag{61}$$

where $\alpha = (|\lambda|^4 + |\lambda|^2 + 1)^{\frac{1}{2}}$ and $\mathbf{v}, \mathbf{w}$ are the right and left eigenvectors of the quadratic polynomial (5) corresponding to the eigenvalue λ.

Proof. From Theorem 5 it follows that

$$\left\| \tilde{\mathbf{w}}^H EF \right\|_2 = \left\| \mathbf{w}^H J \right\|_2, \quad \left\| G_\lambda^T \tilde{\mathbf{v}} \right\|_2 = \left\| \begin{bmatrix} \lambda G_1^T \tilde{\mathbf{v}} \\ G_2^T \tilde{\mathbf{v}} \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} \lambda^2 \mathbf{v} \\ \lambda \mathbf{v} \\ \mathbf{v} \end{bmatrix} \right\|_2 = \alpha \|\mathbf{v}\|. \tag{62}$$

and

$$|\tilde{\mathbf{w}}^H E\tilde{\mathbf{v}}| = |2\lambda \mathbf{w}^H J\mathbf{v} - \mathbf{w}^H D\mathbf{v}|. \tag{63}$$

Substitution into the definitions of the condition numbers then establishes the result. $\square$

An analogous result can be obtained in the case where the embedded quadratic polynomial is monic. In this case the linear pencil is also monic, with coefficient matrices

$$
E = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ A_1 & A_2 \end{bmatrix}.
\tag{64}
$$

It is assumed that the matrix E remains unperturbed. The arbitrary perturbations are now taken to be $\Delta_E = 0$ and $\Delta_A = [\delta A_1, \delta A_2]$. The structure of the perturbations is defined by $F = [0, I_n]^T$ and $G_\lambda^T \equiv G_2^T = I_{2n}$. The admissible perturbations then satisfy $\|[\delta A_1, \delta A_2]\|_2 \le \epsilon$ and the condition number $\tilde{c}(\lambda)$, given by (52), can be shown to equal the condition number $c(\lambda)$, given by (32).

Corollary 8 *In the case of a monic pencil, with coefficients E, A defined by (64), the condition number $\tilde{c}(\lambda)$ satisfies*

$$
\tilde{c}(\lambda) = \frac{\hat{\alpha} \left\| \mathbf{w}^H \right\|_2 \|\mathbf{v}\|_2}{|\mathbf{w}^H (2\lambda I - A_2)\mathbf{v}|} \equiv c(\lambda),
\tag{65}
$$

where $\hat{\alpha} = (|\lambda|^2 + 1)^{\frac{1}{2}}$ and $\mathbf{v}, \mathbf{w}$ are the right and left eigenvectors of the monic quadratic polynomial (29) corresponding to the eigenvalue λ.

Proof. From Theorem 5 we have

$$
\left\| G_\lambda^T \tilde{\mathbf{v}} \right\|_2 = \left\| I_{2n} \begin{bmatrix} \mathbf{v} \\ \lambda\mathbf{v} \end{bmatrix} \right\|_2 = \hat{\alpha} \|\mathbf{v}\|, \quad |\tilde{\mathbf{w}}^H E \tilde{\mathbf{v}}| = |\mathbf{w}^H (2\lambda I - A_2)\mathbf{v}|.
\tag{66}
$$

The proof then follows as in Corollary 7 with $J = I$, $D = A_2$ and $C = A_1$. $\square$

3.3 Robustness

As an overall measure of the sensitivity of the linear matrix pencil (41) to structured perturbations of the form (49), we take a weighted sum of the squares of the conditions numbers $\tilde{c}(\lambda_j)$, $j = 1, \ldots, 2n$, (see [14], [16], [15]). We assume that the pencil is nondefective, since if any eigenvalue is defective, the order of the perturbation is expected to be magnified in some eigenvalue. In the case of a nondefective multiple eigenvalue, the condition numbers are defined with respect to a particular choice of the basis eigenvectors spanning the corresponding invariant subspaces and biorthogonal with respect to the

matrix E. The right and left eigenvectors $\tilde{\mathbf{v}}_j, \tilde{\mathbf{w}}_j$ associated with each eigenvalue λ_j may also be normalized such that

$$\left\| G_{\lambda_j}^T \tilde{\mathbf{v}}_j \right\|_2 = 1, \qquad |\tilde{\mathbf{w}}_j^H E \tilde{\mathbf{v}}_j| = 1, \qquad \forall j - 1, \ldots, 2n. \tag{67}$$

Then (44) holds and the global robustness measure is given by

$$\tilde{\nu}^2 \equiv \sum_{j=1}^{2n} \omega_j^2 \tilde{c}(\lambda_j)^2 = \sum_{j=1}^{2n} \omega_j^2 \left\| \tilde{\mathbf{w}}_j^H E F \right\|_2^2 = \left\| D_\omega \tilde{W}^H E F \right\|_F^2 = \left\| D_\omega \tilde{V}^{-1} F \right\|_F^2, \tag{68}$$

where $D_\omega = \mathrm{diag}\{\omega_1, \ldots, \omega_{2n}\}$ and ω_j, $j = 1, \ldots, 2n$, are positive weights satisfying $\sum_{j=1}^{2n} \omega_j^2 = 1$, with $\omega_j = \omega_k$ if $\lambda_j = \lambda_k$.

In the case where the coefficients of the linear pencil are given by (40), we can show, using Theorem 5, that the robustness measure (68) is equal to the robustness measure (28) of the embedded quadratic polynomial. As demonstrated in the proof of Corollary 7, the normalizations (67) and (27) are equivalent and, since $\tilde{c}(\lambda) = c(\lambda)$, it follows that

$$\tilde{\nu}^2 = \left\| D_\omega \tilde{W}^H E F \right\|_F^2 = \left\| D_\omega \tilde{V}^{-1}[0, I]^T \right\|_F^2 = \left\| D_\omega W^H J \right\|_F^2 = \nu^2, \tag{69}$$

which proves the result.

The equivalence of the robustness measures can also be established in the case where the quadratic polynomial embedded in the linear pencil is monic and the coefficient matrices of the linear pencil are given by (64). Using the definitions of F, G_λ applicable to the monic case, we find that the normalizations (67) and (37) are equivalent. The equality between the robustness measures (68) and (38) of the linear pencil and the monic quadratic polynomial, respectively, then follows immediately from Corollary 8.

The robust eigenstructure assignment problem for the second-order control system (1) can now be formulated as an equivalent problem for a linear pencil. Numerical methods previously developed in [14] can then be applied directly to find the desired feedback gain matrices. In the next section we reformulate the control problem and establish the theory needed for eigenstructure assignment. In Section 5 we derive a modified numerical procedure for solving the design problem that takes advantage of the special structure of the generalized pencil.

4 Robust Eigenstructure Assignment

4.1 Quadratic control problem

The control design problem for the second-order system (1) is to select feedback matrices K_1, K_2 to ensure that the closed loop system (3) has a desired modal response. As demonstrated in Section 2, the modal behaviour of the closed loop system is characterized by the eigenstructure of its corresponding quadratic matrix polynomial $P_{cl}(\lambda) = \lambda^2 J - \lambda(D + BK_2) - (C + BK_1)$. The primary aim of the controller is therefore to determine feedback gains that assign a given set of eigenvalues to the quadratic polynomial. The inverse quadratic eigenvalue problem is stated explicitly as follows.

Problem 1 *Given real matrices $J, D, C \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, and a set of $2n$ complex numbers $\mathcal{L} = \{\lambda_1, \ldots, \lambda_{2n}\}$, closed under complex conjugation, find real matrices $K_1, K_2 \in \mathbb{R}^{m \times n}$ such that the eigenvalues of $P_{cl}(\lambda)$ are equal to $\lambda_j, \ j = 1, \ldots, 2n$.*

Conditions for the existence of solutions to Problem 1 are known and the following theorem is easily established.

Theorem 9 *Solutions K_1, K_2 to Problem 1 exist for every set $\mathcal{L}$ of self-conjugate complex numbers if and only if the system (1) is completely controllable, that is,*

$$\text{rank}[\lambda^2 J - \lambda D - C, B] = n, \quad \forall \lambda \in \mathbb{C}. \tag{70}$$

If the system is not completely controllable, then solutions exist if and only if the set $\mathcal{L} = \{\mathcal{L}_u, \mathcal{L}_c\}$ contains $\mathcal{L}_u$, the set of all values of λ for which the system (1) is uncontrollable (that is, the set of values of λ for which (70) is not satisfied).

Proof. The proof follows directly from the standard theory for the equivalent generalized linear system (39), characterized by the matix triple $(E, A, \tilde{B})$ defined as in (40). (See also [4], for example.) ☐

In the single input case $(m = 1)$, the solution to Problem 1 is unique and the robustness of the closed loop system cannot be controlled. In the multi-input case $(m > 1)$, there are extra degrees of freedom in the design that can be specified so as to optimize a measure of the robustness of the system. The feedback gains can be parameterized in terms of the eigenvectors of the closed loop system and the eigenvectors corresponding to the desired eigenvalues can then be selected to minimize the sensitivity measure ν^2, defined

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by (26). The degrees of freedom in the feedback matrices are reflected precisely by the degrees of freedom available for assigning the eigenvectors. The robust eigenstructure assignment problem is formulated explicitly as follows.

Problem 2 *Given real matrices J, D, C, B and a set $\mathcal{L}$ as in Problem 1, find real matrices $K_1, K_2 \in \mathbb{R}^{m \times n}$ and matrix $V \in \mathbb{C}^{n \times 2n}$ such that $\tilde{V} = [V^T, (V\Lambda)^T]^T$ is nonsingular,*

$$JV\Lambda^2 - (D + BK_2)V\Lambda - (C + BK_1)V = 0, \quad \Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_{2n}\}, \quad (71)$$

and the measure ν^2, defined as in (26), is minimized.

We remark that the requirement that the matrix Λ is diagonal, together with the invertibility of $\tilde{V}$, ensures that the closed loop system is nondefective. This requirement imposes certain simple restrictions on the multiplicity of the eigenvalues that may be assigned. The condition that $\tilde{V}$ is nonsingular is also needed for a well-posed parameterization of the feedback gains in terms of V. In the next section we derive conditions for the solution of Problem 2.

4.2 Eigenstructure assignment

The objective of the design problem now is to select the modal matrix V of right eigenvectors of the closed loop polynomial $P_{cl}(\lambda)$ to satisfy condition (71) of Problem 2 for some choice of K_1, K_2. We let W denote the corresponding modal matrix of left eigenvectors of the polynomial. We assume without loss of generality that B is of full column rank. No restriction is made on the controllability of the open loop system, but it is assumed that the set of prescribed eigenvalues $\mathcal{L}$ contains each uncontrollable eigenvalue with its full multiplicity. We remark that although the values of the uncontrollable eigenvalues of the system are not affected by the feedback, the corresponding eigenvectors may be modified and the conditioning of these eigenvalues may be improved.

The next theorem provides necessary and sufficient conditions under which a given set of nondefective eigenvalues and corresponding eigenvectors can be assigned.

Theorem 10 *Let $V \in \mathbb{C}^{n \times 2n}$ be such that $\tilde{V} = [V^T, (V\Lambda)^T]^T$ is nonsingular, where $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_{2n}\}$. Then, there exist real matrices K_1, K_2, satisfying condition (71) of Problem 2 if and only if*

$$U_1^T(JV\Lambda^2 - DV\Lambda - CV) = 0, \quad (72)$$

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where

$$B = [U_0, U_1] \begin{bmatrix} Z \\ 0 \end{bmatrix}, \qquad (73)$$

with $U = [U_0, U_1]$ orthogonal and Z nonsinsgular. The matrices K_1, K_2 are given explicitly by

$$[K_1, K_2] = Z^{-1} U_0^T (JV\Lambda^2 - DV\Lambda - CV)\tilde{V}^{-1}. \qquad (74)$$

Proof. The assumption that B is full rank implies the existence of the decomposition (73). Condition (71) then holds if and only if the feedback matrices K_1, K_2 satisfy

$$B[K_1, K_2] \begin{bmatrix} V \\ V\Lambda \end{bmatrix} = (JV\Lambda^2 - DV\Lambda - CV). \qquad (75)$$

Premultiplication by U^T gives

$$\begin{aligned}
Z[K_1, K_2]\tilde{V} &= U_0^T(JV\Lambda^2 - DV\Lambda - CV), \\
0 &= U_1^T(JV\Lambda^2 - DV\Lambda - CV).
\end{aligned} \qquad (76)$$

If condition (71) is satisfied, then (72) and (74) follow directly, since $\tilde{V}$ is invertible by assumption. Conversely, if (72) is satisfied and $\tilde{V}$ is nonsingular, then K_1, K_2 given by (74) exist and satisfy (71). (See also [13], [4].) $\square$

An immediate consequence of Theorem 10 is the following.

Corollary 11 *The right eigenvector $\mathbf{v}_j$ of $P_{cl}(\lambda)$ corresponding to the prescribed eigenvalue $\lambda_j \in \mathcal{L}$ must belong to the space*

$$\mathcal{S}_j = \mathcal{N}\{U_1^T(\lambda_j^2 J - \lambda_j D - C)\}, \qquad (77)$$

where $\mathcal{N}\{\cdot\}$ denotes right nullspace. The dimension of $\mathcal{S}_j$ is given by

$$\dim(\mathcal{S}_j) = m + k_{\lambda_j}, \qquad (78)$$

where $k_{\lambda_j} = \dim(\mathcal{N}\{[B, \lambda_j^2 J - \lambda_j D - C]^T\})$.

Proof. From (72) we obtain immediately that

$$U_1^T(\lambda_j^2 J - \lambda_j D - C)\mathbf{v}_j = 0, \qquad (79)$$

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and therefore $\mathbf{v_j} \in \mathcal{S}_j$, $j = 1, \ldots, 2n$, is necessary. Using (72)and (73) we find that

$$U^T[B, \lambda_j^2 J - \lambda_j D - C] = \begin{bmatrix} Z & U_0^T(\lambda_j^2 J - \lambda_j D - C) \\ 0 & U_1^T(\lambda_j^2 J - \lambda_j D - C) \end{bmatrix}. \qquad (80)$$

From the definition of k_{λ_j}, we find also that $\mathrm{rank}(U^T[B, \lambda_j^2 J - \lambda_j D - C]) = n - k_{\lambda_j}$. Since matrix Z is square ($m \times m$) and invertible, we then have $\mathrm{rank}(U_1^T(\lambda_j^2 J - \lambda_j D - C)) = n - m - k_{\lambda_j}$, from which (78) readily follows.
□

From Corollary 11 we can now deduce restrictions on the set $\mathcal{L}$ of eigenvalues that can be assigned. If the system (1) is completely controllable, then the dimension k_λ is zero for all λ. For the closed loop polynomial to be non-defective, the maximum multiplicity of any eigenvalue λ_j that can be assigned is then equal to $\dim(\mathcal{S}_j) = m$. If the system is not completely controllable and $\lambda_j \in \mathcal{S}$ is an uncontrollable eigenvalue, then there exists a set of at least k_{λ_j} independent (left) eigenvectors of the polynomial $P_{cl}(\lambda)$ for every choice of K_1, K_2. The eigenvalue λ_j must, therefore, be assigned with multiplicity at least k_{λ_j} and at most $\dim(\mathcal{S}_j) = m + k_{\lambda_j}$.

As a consequence of Theorem 10 we can also derive explicit expressions for the feedback matrices directly in terms of the right and left modal matrices V, W of the closed loop polynomial.

Corollary 12 *Let V be such that $\tilde{V} = [V^T, (V\Lambda)^T]^T$ is nonsingular and condition (72) of Theorem 10 is satisfied and let $W^H J = \tilde{V}^{-1}[0, I]^T$. Then the feedback matrices K_1, K_2 satisfying condition (71) of Problem 2 are given explicitly by*

$$\begin{aligned} K_1 &= Z^{-1}U_0^T(JV\Lambda^3 W^H J - J(V\Lambda^2 W^H J)^2 - C), \\ K_2 &= Z^{-1}U_0^T(JV\Lambda^2 W^H J - D). \end{aligned} \qquad (81)$$

Proof. By definition, the matrices V, Λ, W form a Jordan triple of the closed loop polynomial $P_{cl}(\lambda)$ for some K_1, K_2. Then conditions (9) hold and W^H satisfies

$$\Lambda^2 W^H J - \Lambda W^H D - W^H C = \Lambda W^H B K_2 + W^H B K_1. \qquad (82)$$

Premultiplying by $Z^{-1}U_0^T JV$, using $Z^{-1}U_0^T B = I$ and applying (9) then gives the result for K_2. Similarly, premultiplying (82) by $Z^{-1}U_0^T JV\Lambda$, applying (9) and substituting for K_2 then gives the result for K_1. (Alternatively, (81) can

be established using the definition of $\tilde{V}^{-1}$ given by Lemma 1 in the closed loop case.) □

The solution to Problem 2 can now be found by selecting the columns $\mathbf{v}_j$ of V from the subspaces $\mathcal{S}_j$, $j = 1, \ldots, 2n$, such that the matrix $\tilde{V} = [V^T, (V\Lambda)^T]^T$ is nonsingular and the robustness measure ν^2 is minimized. The required feedback matrices K_1, K_2 can then be constructed directly from (74) or (81). In the next section we show that this solution can also be obtained by solving the eigenstructure assignment problem for the corresponding generalized linear control system. Methods previously developed for optimizing the robustness of the linear system subject to structured perturbations are then adapted to solve the quadratic control design problem.

4.3 Reformulation of the control problem

In order to solve the control design problem, Problem 1, it is common practice to transform the second-order control system (1) into a generalized linear state-space (descriptor) system of the form (39), where the coefficients $E, A, \tilde{B}$ are given by (40). The matrix $\tilde{B}$ is assumed, without loss of generality, to be of full column rank.

The control problem is now to synthesize a proportional state feedback controller of the form

$$\mathbf{u} = \tilde{K}\mathbf{x} + \mathbf{r} \tag{83}$$

where $\tilde{K} \in \mathbb{R}^{m \times 2n}$, such that the closed loop system

$$E\dot{\mathbf{x}} = (A + \tilde{B}\tilde{K})\mathbf{x} + \tilde{B}\mathbf{r} \tag{84}$$

has desired properties. Specifically, the aim is to select real matrix $\tilde{K}$ such that the $2n$ eigenvalues of the linear matrix pencil

$$L_{cl}(\lambda) \equiv \lambda E - (A + \tilde{B}\tilde{K}) \tag{85}$$

corresponding to the closed loop system (84), are equal to $\lambda_j \in \mathcal{L}$, where $\mathcal{L} = \{\lambda_1, \ldots, \lambda_{2n}\}$ is a specified self-conjugate set of complex numbers. In the case where the system coefficients are given by (40) and $\tilde{K} = [K_1, K_2]$, then the closed loop pencil has the form

$$L_{cl}(\lambda) = \lambda \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} - \begin{bmatrix} 0 & I \\ C + BK_1 & D + BK_2 \end{bmatrix}, \tag{86}$$

and the solution to the generalized linear inverse eigenvalue problem gives the solution to Problem 1 immediately.

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The linear inverse eigenvalue problem has been studied widely and conditions for the existence of solutions are well-known [27]. The eigenvalues of the closed loop pencil $L_{cl}(\lambda)$, given by (85), can be assigned arbitrarily if and only if the system (39) is completely controllable, that is, if and only if $\text{rank}([\tilde{B}, \lambda E - A]) = 2n$ for all $\lambda \in \mathbb{C}$. If the system is not completely controllable, then the prescribed set $\mathcal{L}$ of eigenvalues must contain each value of λ for which the system is uncontrollable, with its full multiplicity. In the case where the coefficients $E, A, \tilde{B}$ of the system are given by (40), the conditions for the existence of solutions are precisely equivalent to those of Theorem 9 for the embedded quadratic polynomial.

The robust eigenstructure assignment problem for the generalized linear system (39) has also been investigated thoroughly [16], [15], [14]. The objective is to find a nonsingular matrix $\tilde{V}$ comprising the right eigenvectors of the closed loop pencil $L_{cl}(\lambda)$ for some feedback $\tilde{K}$ such that the robustness of the closed loop system is optimized. Specifically the aim now is to minimize the sensitivity of the assigned eigenvalues to *structured* perturbations of the form (49). The robustness measure is thus given by $\tilde{\nu}^2$, defined as in (68). For the system (39)–(40), this measure is equal to the robustness measure ν^2 of the embedded second-order system, as shown in Section 3. The solution to the linear robust eigenstructure problem therefore gives the solution $V = [I, 0]\tilde{V}$ and $[K_1, K_2] = \tilde{K}$ to Problem 2 directly.

We remark that the robustness measure for the linear system is commonly taken to be the sum of the squares of the condition numbers $c^L(\lambda)$, defined as in (48). This measure gives the sensitivity of the closed loop eigenvalues to perturbations in all elements of $E, A + \tilde{B}\tilde{K}$. Its minimal value varies with the form of linear embedding used and it is not a true measure of the robustness of the quadratic polynomial. In order for the linear and quadratic inverse problems to be equivalent, it is necessary to apply the measure of robustness for the linear system with respect to the structured perturbations. The generalized linear eigenstructure problem is thus formulated explicitly as follows.

Problem 3 *Given real matrices $E, A \in \mathbb{R}^{2n \times 2n}, \tilde{B} \in \mathbb{R}^{2n \times m}$, a set of $2n$ complex numbers $\mathcal{L} = \{\lambda_1, \ldots, \lambda_{2n}\}$, closed under complex conjugation, and real matrices $F \in \mathbb{R}^{2n \times m_F}, G_{\lambda_j} \in \mathbb{R}^{2n \times m_G}$, $j = 1, \ldots, 2n$, find real matrix $\tilde{K} \in \mathbb{R}^{m \times 2n}$ and nonsingular matrix $\tilde{V} \in \mathbb{C}^{2n \times 2n}$ such that*

$$E\tilde{V}\Lambda - (A + \tilde{B}\tilde{K})\tilde{V} = 0, \quad \Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_{2n}\}, \quad (87)$$

and $\tilde{\nu}^2 \equiv \left\| D_\omega \tilde{V}^{-1} F \right\|_F^2$ is minimized, subject to $\left\| G_{\lambda_j}^T \tilde{V} \mathbf{e}_j \right\|_2 = 1$, $j = 1, \ldots, 2n$. (Here $\mathbf{e}_j$ denotes the jth unit vector.)

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Conditions under which a given set of nondefective eigenvalues and eigenvectors can be assigned to the linear system are given in the monic case in [16]. These conditions can be extended to the generalized (nonsingular) case with minor modifications and the following results can be established by similar arguments to those used in [16] and in the proof of Theorem 10.

Theorem 13 *Let $\tilde{V} \in \mathbb{C}^{2n \times 2n}$ be nonsingular. Then, there exists real matrix $\tilde{K}$ satisfying condition (87) of Problem 3 if and only if*

$$\tilde{U}_1^T (E\tilde{V}\Lambda - A\tilde{V}) = 0, \tag{88}$$

where

$$\tilde{B} = [\tilde{U}_0, \tilde{U}_1] \begin{bmatrix} \tilde{Z} \\ 0 \end{bmatrix} \tag{89}$$

with $\tilde{U} = [\tilde{U}_0, \tilde{U}_1]$ orthogonal and $\tilde{Z}$ nonsingular. The matrix $\tilde{K}$ is given explicitly by

$$\tilde{K} = \tilde{Z}^{-1}\tilde{U}_0^T (E\tilde{V}\Lambda - A\tilde{V})\tilde{V}^{-1}. \tag{90}$$

Proof. See [16]. $\square$

In the case where the coefficients of the control system are defined by (40), the decomposition of $\tilde{B}$ can be written in terms of the decomposition (73) of B. Using the orthogonal matrix $U = [U_0, U_1]$ from (73), we find that the decomposition (89) is given by

$$\tilde{U}_0 = \begin{bmatrix} 0 \\ U_0 \end{bmatrix}, \quad \tilde{U}_1 = \begin{bmatrix} 0 & I \\ U_1 & 0 \end{bmatrix}, \quad \tilde{Z} = Z. \tag{91}$$

The following corollary is then a direct consequence of Theorem 13.

Corollary 14 *Let $E, A, \tilde{B}$ be defined by (40). Then the right eigenvector $\tilde{\mathbf{v}}_j$ of $L_{cl}(\lambda)$ corresponding to the prescribed eigenvalue $\lambda_j \in \mathcal{L}$ must belong to the space*

$$\tilde{\mathcal{S}}_j = \mathcal{N}\{\tilde{U}_1^T (\lambda_j E - A)\}, \tag{92}$$

and must satisfy $\tilde{\mathbf{v}}_j = [I, \lambda_j I]^T \mathbf{v}_j$ with $\mathbf{v}_j \in \mathcal{S}_j$, where $\mathcal{S}_j$ is defined by (77).

Proof. From (88) we immediately obtain

$$\tilde{U}_1^T (\lambda_j E - A)\tilde{\mathbf{v}}_j = 0, \tag{93}$$

and therefore $\tilde{\mathbf{v}}_j \in \tilde{\mathcal{S}}_j, j = 1, \ldots, 2n$, is necessary. Using (40) and (91) in (93) then gives

$$\begin{bmatrix} -U_1^T C & U_1^T(\lambda_j J - D) \\ \lambda_j I & -I \end{bmatrix} \tilde{\mathbf{v}}_j = 0. \tag{94}$$

The second of these relations implies that $\tilde{\mathbf{v}}_j = [\mathbf{v}_j^T, (\lambda_j \mathbf{v}_j)^T]^T$ and the first establishes that $\mathbf{v}_j \in \mathcal{S}_j$ is necessary. $\square$

Finally, from the result (90) of Theorem 13 and from (91) we may establish a direct relation between the solutions to the linear and quadratic feedback design problems.

Corollary 15 *Let $E, A, \tilde{B}$ be defined by (40). Let $\tilde{V}$ be a nonsingular matrix satisfying condition (88) of Theorem 13 and let $V = [I, 0]\tilde{V}$, $W^H J = \tilde{V}^{-1}[0, I]^T$. Then, the feedback matrix $\tilde{K}$ satisfying condition (87) of Problem 3 is equal to $\tilde{K} = [K_1, K_2]$, where K_1, K_2 are defined by (74), or equivalently, by (81).*

Proof. Substituting (40) and (91) into (90) gives the result immediately. $\square$

In summary, the solution to Problem 3 can then be found by selecting the columns $\tilde{\mathbf{v}}_j$ of $\tilde{V}$ from the subspaces $\tilde{\mathcal{S}}_j$, $j = 1, \ldots, 2n$, such that the matrix $\tilde{V}$ is nonsingular and the robustness measure $\tilde{\nu}^2 \equiv \nu^2$ is minimized, subject to the constraints $\left\| G_{\lambda_j}^T \tilde{\mathbf{v}}_j \right\|_2 = 1$. The required feedback matrix $\tilde{K}$ can then be constructed from (90). If $E, A, \tilde{B}$ are given by (40) and $F, G_{\lambda_j} = [\lambda_j G_1, G_2]$, $j = 1, \ldots, 2n$, are determined by (58), then the solution to Problem 3 immediately gives the solution $V = [I, 0]\tilde{V}$, $[K_1, K_2] = \tilde{K}$ to the quadratic robust eigenstructure assignment problem, Problem 2.

5 Numerical Algorithm

Previously, in [14], we have developed a numerical algorithm for solving the linear robust eigenstructure assignment problem subject to structured perturbations. In the monic case this method can be applied directly to solve Problem 3. The algorithm is easily adapted to treat the generalized case. The method does not, however, take direct advantage of the special structure of the linear pencil in the case where the linear system represents an embedded quadratic system.

We now present a modified form of the algorithm that can be applied to solve the robust quadratic eigenstructure problem, Problem 2.

5.1 Basic steps

The basic steps of the algorithm are first described. Details of the implementation are then discussed.

Algorithm 1

INPUT: *real matrices $J, D, C \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, a set of $2n$ complex numbers $\mathcal{L} = \{\lambda_1, \ldots, \lambda_{2n}\}$ and a diagonal matrix $D_\omega = \text{diag}\{\omega_1, \ldots, \omega_{2n}\}$, where ω_j, $j = 1, \ldots, 2n$, are real positive weights satisfying $\sum_{j=1}^{2n} \omega_j^2 = 1$, with $\omega_j = \omega_k$ if $\lambda_j = \lambda_k$.*

Step 1. *Find the decomposition (73) of B and an orthonormal basis, comprised by the columns of the matrix S_j, for the subspaces $\mathcal{S}_j$, $j = 1, \ldots, 2n$, defined in (77).*

Step 2. *Select an initial matrix $V = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2n}]$ such that $\mathbf{v}_j \in \mathcal{S}_j$, $\alpha_j \|\mathbf{v}_j\|_2 = 1$, and $\tilde{V} = [V^T, (V\Lambda)^T]^T$ is nonsingular, where $\Lambda = \text{diag}\{\lambda_j, \, j = 1, \ldots, 2n\}$ and $\alpha_j = (|\lambda_j|^4 + |\lambda_j|^2 + 1)^{\frac{1}{2}}$.*

Step 3. *For $j = 1, 2, \ldots, 2n$ do*

Step 3.1 *Find vector $\hat{\mathbf{v}}_j$ that minimizes*

$$\nu^2 = \left\| D_\omega W^H J \right\|_F^2 \equiv \left\| D_\omega \begin{bmatrix} V \\ V\Lambda \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \right\|_F^2$$

over all $\mathbf{v}_j \in \mathcal{S}_j$, subject to $\alpha_j \|\mathbf{v}_j\|_2 = 1$ and $\mathbf{v}_i$ fixed for all $i \neq j$.

Step 3.2. *Form updated matrices $V = [\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \hat{\mathbf{v}}_j, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_{2n}]$, and $\tilde{V} = [V^T, (V\Lambda)^T]^T$ and CONTINUE.*

Step 4. *Repeat Step 3 until ν^2 has 'converged'.*

Step 5. *Construct feedback matrices K_1, K_2 by solving*

$$[K_1, K_2]\tilde{V} = Z^{-1} U_0^T (JV\Lambda^2 - DV\Lambda - CV). \qquad \square$$

We remark that the decomposition of B in Step 1 can be found either by the QR or the SVD method (see [8]). The matrix S_j can be found from the QR decomposition of $(U_1^T(\lambda_j^2 J - \lambda_j D - C))^T$.

If the system (1) is completely controllable and the prescribed eigenvalues are distinct, then an initial matrix V satisfying the requirements in Step 2 can always be selected. (Under mild restrictions, this also holds for uncontrollable systems and/or for prescribed multiple eigenvalues.) To obtain the initial matrix V it is generally sufficient to select *random* vectors from each subspace

$\mathcal{S}_j$. The conditioning of the initial matrix $\tilde{V}$ is not significant and it may be very close to singular without detriment.

The key step of the algorithm is Step 3. Details of the procedure used for updating the eigenvectors in Step 3.1 are discussed in the next section. If the initial matrix $\tilde{V}$ is nonsingular, then each subsequent matrix $\tilde{V}$ generated in this step is guaranteed also to be nonsingular.

The problem of computing the feedback matrices in Step 5 from the constructed matrix $\tilde{V}$ is well-conditioned if $\tilde{V}$ is well-conditioned for inversion. Since the aim of the procedure is essentially to orthogonalize $\tilde{V}$ with respect to $[0, I]^T$, $\tilde{V}$ is expected to be reasonably well-conditioned. Additional degrees of freedom in $\tilde{V}$ may exist, however, and these are then selected *explicitly* in Step 3.1 to make $\tilde{V}$ as well-conditioned as possible. If the constructed matrix $\tilde{V}$ is, nevertheless, very badly conditioned, then the closed loop system will necessarily be very sensitive to perturbations, regardless of the accuracy of the computed feedback gains. It is then recommended that the set of prescribed eigenvalues should be altered, allowing a less sensitive closed loop system to be derived.

As an alternative to the procedure in Step 5, the matrices K_1, K_2 could be determined by solving for $W^H J$ from $\tilde{V}(W^H J) = [0, I]^T$, and then substituting directly into (81). Analysis suggests, however, that this procedure will be less efficient and less accurate than that proposed in Step 5. The solution for $W^H J$ requires the inversion of $\tilde{V}$ into n right-hand-side vectors, whereas the solution in Step 5 requires the inversion of $\tilde{V}$ into only $m \leq n$ right-hand sides. Moreover, forming the product of the computed $W^H J$ with the other factors in (81), which already contain numerical errors, is likely to magnify the computational errors introduced into the feedback matrices and hence to give less accurate solutions.

5.2 Updating the eigenvectors

The computation of the update to the vector $\mathbf{v}_j$ in Step 3.1 of the algorithm is accomplished explicitly. In essence, this step aims to orthogonalize the vectors $\tilde{\mathbf{v}}_j = [I, \lambda_j I]^T \mathbf{v}_j$, $j = 1, \ldots, 2n$, with respect to the matrix $[0, I]^T$, subject to the constraints. In the first phase, orthogonal bases Q and $\mathbf{q}$ are found for the space spanned by the fixed vectors $\tilde{\mathbf{v}}_i$, $i \neq j$, and its orthogonal complement, respectively, and the measure ν^2 is expressed in terms of these bases. Next, the required vector is scaled to have a fixed normalization and the direction of the minimizing vector in the required subspace is found by solving a least squares problem. The optimal normalization is then determined to satisfy

the constraint. These steps follow the Algorithm of [14], but are modified to produce the vector $\mathbf{v}_j$ as efficiently as possible. The technical details are as follows.

We denote $\tilde{V}_j = [\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_{j-1}, \tilde{\mathbf{v}}_{j+1}, \ldots, \tilde{\mathbf{v}}_{2n}]$ and let the (complex) QR decomposition of $\tilde{V}_j$ be given by

$$\tilde{V}_j = [Q, \mathbf{q}] \begin{bmatrix} R \\ \mathbf{0}^T \end{bmatrix}, \tag{95}$$

where $[Q, \mathbf{q}]$ is orthogonal and R is upper triangular and nonsingular. We write $\mathbf{v}_j = S_j \boldsymbol{\eta} \in \mathcal{S}_j$. Then we obtain $\nu^2 = \left\| D_\omega \tilde{V}^{-1}[0, I]^T \right\|_F^2 = \|Y\|_F^2$, where

$$
\begin{aligned}
Y &= D_\omega \left[\tilde{V}_j, \begin{bmatrix} I \\ \lambda_j I \end{bmatrix} S_j \right]^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \\
&= D_\omega \begin{bmatrix} R^{-1} & -\rho R^{-1} Q^H [I, \lambda_j I]^T S_j \boldsymbol{\eta} \\ 0 & \rho \end{bmatrix} \begin{bmatrix} Q_2^H \\ \mathbf{q}_2^H \end{bmatrix} \\
&= D_\omega \begin{bmatrix} R^{-1} Q_2^H - R^{-1}(Q_1^H + \lambda_j Q_2^H) S_j \boldsymbol{\eta} \rho \mathbf{q}_2^H \\ \rho \mathbf{q}_2^H \end{bmatrix},
\end{aligned}
\tag{96}
$$

and $\rho = \left(\mathbf{q}^H [I, \lambda_j I]^T S_j \boldsymbol{\eta} \right)^{-1}$, $Q^H = [Q_1^H, Q_2^H]$, $\mathbf{q}^H = [\mathbf{q}_1^H, \mathbf{q}_2^H]$.

If $\mathbf{q}_2 \neq 0$, then using $\alpha_j \|\mathbf{v}_j\|_2 = \|\alpha_j \boldsymbol{\eta}\|_2 = 1$ and applying Lemma 2 of [14], for example, we can show that

$$
\delta^2 \|Y\|_F^2 = \left\| \delta^2 \begin{bmatrix} D_{\hat{\omega}} R^{-1}(Q_1^H + \lambda_j Q_2^H) S_j \\ \alpha_j I_m \end{bmatrix} (\rho \boldsymbol{\eta}) - \begin{bmatrix} D_{\hat{\omega}} R^{-1} Q_2^H \mathbf{q}_2 \\ 0 \end{bmatrix} \right\|_F^2 + c,
\tag{97}
$$

where $\delta^2 = \mathbf{q}_2^H \mathbf{q}_2$, $D_{\hat{\omega}} = \mathrm{diag}\{\omega_1, \ldots, \omega_{2n-1}\}$, and c is a constant independent of $\boldsymbol{\eta}$.

The problem now is to minimize $\|Y\|_F^2$ over all $\boldsymbol{\eta} \in \mathbb{C}^m$. In order to reduce this nonlinear minimization problem to a linear least-squares problem, we fix the normalization of the vector $\rho \boldsymbol{\eta}$. We find the Householder transformation P such that

$$(\mathbf{q}_1^H + \lambda_j \mathbf{q}_2^H) S_j P = \sigma \mathbf{e}_m^T, \tag{98}$$

where $\mathbf{e}_m$ is the mth unit vector. From the definition of ρ, we then have

$$1 = \mathbf{q}^H [I, \lambda_j I]^T S_j \rho \boldsymbol{\eta} = (\mathbf{q}_1^H + \lambda_j \mathbf{q}_2^H) S_j P P^H \rho \boldsymbol{\eta} = \sigma \mathbf{e}_m^T P^H \rho \boldsymbol{\eta}. \tag{99}$$

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We may therefore define $\hat{\eta}$ to be such that $[\hat{\eta}^H, 1]^H = \sigma P^H \rho \eta$.

Writing $P = [P_1, \mathbf{p}]$ then gives $\sigma P P^H \rho \eta = P[\hat{\eta}^H, 1]^H = P_1 \hat{\eta} + \mathbf{p}$ and the minimization problem becomes

$$\min_{\hat{\eta}} \left\| \delta^2 \begin{bmatrix} D_{\hat{\omega}} H \\ \alpha_j I \end{bmatrix} P_1 \hat{\eta} + \begin{bmatrix} D_{\hat{\omega}}(H\mathbf{p} - \sigma \mathbf{h}) \\ \alpha_j \mathbf{p} \end{bmatrix} \right\|_F^2, \tag{100}$$

where $H = R^{-1}(Q_1^H + \lambda_j Q_2^H)S_j$, $\mathbf{h} = R^{-1} Q_2^H \mathbf{q}_2$. This is a standard linear least-squares problem that can be solved by the QR (or SVD) method.

Finally we restore the scaling of the optimal vector to satisfy the constraints. Since P is orthogonal and the columns of S_j form a set of orthonormal vectors, the required update is given by

$$\hat{\mathbf{v}}_j = S_j P \begin{bmatrix} \hat{\eta} \\ 1 \end{bmatrix} \Big/ \left\| \alpha_j \begin{bmatrix} \hat{\eta} \\ 1 \end{bmatrix} \right\|_2. \tag{101}$$

In the special case where $\mathbf{q}_2 = 0$ (or is very small), then $\|Y\|_F$ is constant (almost), independent of η. In this case the new vector $\hat{\mathbf{v}}_j$ could be selected to be *any* vector in S_j. In order to maximize the orthogonality of $\tilde{V}$, however, the new vector is chosen such that $[I, \lambda_j I]^T \hat{\mathbf{v}}_j$ equals the closest vector to $\mathbf{q}$ in the allowable subspace, given by the projection of $\mathbf{q}$ into $[I, \lambda_j I]^T S_j$. The required update is then

$$\hat{\mathbf{v}}_j = S_j S_j^H (\mathbf{q}_1 + \bar{\lambda}_j \mathbf{q}_2) \Big/ \left\| \alpha_j S_j^H (\mathbf{q}_1 + \bar{\lambda}_j \mathbf{q}_2) \right\|_2. \tag{102}$$

The new updated matrix $\tilde{V}$ generated by this procedure must be nonsingular. Since the original matrix was nonsingular, the definition of $\mathbf{q}$ implies that $\mathbf{q}^H[I, \lambda_j I]^T S_j \neq 0$ and $\sigma \neq 0$. Hence $\mathbf{q}^H[I, \lambda_j I]^T \hat{\mathbf{v}}_j \neq 0$ and the vector $[I, \lambda_j I]^T \hat{\mathbf{v}}_j$ has a component in the direction orthogonal to all the other columns of $\tilde{V}$. The columns of the updated matrix $\tilde{V}$ must therefore all be linearly independent, which establishes the result.

We may summarize the update step of the algorithm as follows.

Algorithm 1, Step 3.1

INPUT: tol

Step 3.1.1. *Form matrix $\tilde{V}_j$ and find its QR decomposition (95) to determine* $Q = [Q_1^H, Q_2^H]^H, \mathbf{q} = [q_1^H, q_2^H]^H$ *and* R. *Form* $\delta^2 = \mathbf{q}_2^H \mathbf{q}_2$.

Step 3.1.2. *If* $|\delta^2| > $ tol, *form* $(\mathbf{q}_1^H + \lambda_j \mathbf{q}_2^H)S_j$ *and find the Householder matrix P statisfying (98). Solve* $R[H, \mathbf{h}] = [(Q_1^H + \lambda_j Q_2^H)S_j, Q_2^H \mathbf{q}_2]$ *for* $H, \mathbf{h}$ *by back-substitution and solve the least-squares problem (100) for $\hat{\eta}$.*

Step 3.1.3. *If $|\delta^2| >$ tol, define the update $\mathbf{v}_j$ by (101); else define $\mathbf{v}_j$ by (102).* □

In the case where $\hat{\mathbf{v}}_j$ corresponds to a real eigenvalue λ_j, the method generates a real update. In the case λ_j is complex, a complex eigenvector is generated and, in order to ensure that the computed feedback matrices are real, the updated eigenvector corresponding to the conjugate eigenvalue $\bar{\lambda}_j$ must taken to be the conjugate vector $\bar{\hat{\mathbf{v}}}_j$. In practice, complex arithmetic can be avoided by generating the real and imaginary parts of $\hat{\mathbf{v}}_j$ independently. The optimization is no longer precise, however, and a reduction in ν^2 cannot be guaranteed at every iteration step. Experience indicates that this is not a drawback and rapid overall convergence is obtained in practice.

We remark that the QR decomposition of $\tilde{V}_j$ can be found by inexpensive updating techniques from the QR decomposition of $\tilde{V}_{j-1}$. The solution of the least-squares problem (100) requires the decomposition of a matrix of order $m - 1$, which may be small even where the order $2n$ of the full system is large. The procedure is then relatively efficient. Each update requires $\mathcal{O}(4n^2m) + \mathcal{O}(2nm^2)$ operations. Practical experiments have shown that the reduction of the minimization problem to a sequence of linear least-squares problems is generally more efficient than global nonlinear optimization techniques for objective functions of this form [18]. Further work on the procedure for maximizing robustness would, however, be useful.

6 Conclusions

We have investigated here the problem of robust eigenstructure assignment by state feedback in a second-order control system. The response of the system is determined by the eigenstructure of the associated quadratic matrix polynomial and the aim of the controller design is to assign specified eigenvalues to the closed loop system polynomial.

In the first sections of the paper we derive sensitivity measures, or condition numbers, for the eigenvalues of the quadratic matrix polynomial and define a measure of the robustness of the corresponding system. In practice the second-order system is commonly embedded in a generalized linear first-order control system. The standard measure of sensitivity, or robustness, of the corresponding generalized linear matrix pencil is not equivalent to that of the embedded quadratic polynomial. We show, however, that an equivalent robustness measure for the linear pencil can be established by considering

its sensitivity to *structured* perturbations. We derive condition numbers for the eigenvalues of the generalized linear pencil subject to perturbations with specified structure and show that these condition numbers are equal to the sensitivity measures for the embedded quadratic polynomial. We show also that the robustness measures based on these condition numbers are equal.

In the remaining sections of the paper we review and extend the theory of eigenstructure assignment in second-order control systems. We show that the solution of the robust eigenstructure assignment problem for the second-order system can be achieved by solving the generalized linear problem subject to structured perturbations. Reliable and efficient numerical methods for determining the required feedback matrices are then developed, based on methods previously devised for solving the structured linear problem.

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