On Best Piecewise Linear $L_2$ Fits with Adjustable Nodes

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Department of Mathematics
University of Reading
P O Box 220
Reading

†Department of Mathematics, University of California, Berkeley, U.S.A.

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Abstract

In this report a simple procedure is used to determine the best continuous piecewise linear $L_2$ fit to a convex function of a single variable with adjustable nodes. An extension gives a very good continuous piecewise linear $L_2$ fit to non-convex functions, again with adjustable nodes.
§1. Theory

Let $f(x)$ be a given function of $x$ and denote by $u_k(x)$ the best linear $L_2$ fit to $f(x)$ in the interval $(x_{k-1}, x_k)$. Then

$$\delta \int_{x_{k-1}}^{x_k} \left( f(x) - u_k(x) \right)^2 \, dx = 0 \quad u_k \in S_k \quad (1)$$

or

$$\delta \int_{x_{k-1}}^{x_k} \left( f(x) - u_k(x) \right) \delta u_k(x) \, dx = 0 \quad \delta u_k(x) \in S_k \quad (2)$$

where $S_k$ is the family of straight lines on the interval $(x_{k-1}, x_k)$. For an interval $(x_0, x_{n+1})$ which is the union of intervals $(x_{k-1}, x_k)$, $(k=1, n+1)$, the best $L_2$ fit to $f(x)$ amongst piecewise linear functions discontinuous at $x_k$, $(k=1, n)$, is also given by (1) and (2), $(k=1, n+1)$, since the problems decouple.

Now consider the problem of determining the best $L_2$ fit $u(x)$ to $f(x)$ amongst all discontinuous piecewise linear functions on the fixed interval $(x_0, x_{n+1})$ on a variable partition $(x_1, x_2, \ldots, x_k, \ldots, x_n)$ of the interval. Then

$$\delta \int_{x_0}^{x_{n+1}} \left( f(x) - u(x) \right)^2 \, dx = \delta \sum_{k=1}^{n+1} \int_{x_{k-1}}^{x_k} \left( f(x) - u(x) \right)^2 \, dx = 0 \quad (3)$$

where the $x_k$, $(k=1, n)$, are also varied. It is convenient to introduce
here a new independent variable $\xi$ which remains fixed, while $x$ joins $u$ as a dependent variable, both now depending on $\xi$ and denoted by $\hat{x}$ and $\hat{u}$. Then (3) becomes

$$\delta \sum_{k=1}^{n+1} \int_{x_{k-1}}^{x_k} \left\{ f(\hat{x}(\xi)) - \hat{u}(\xi) \right\}^2 \frac{d\hat{x}}{d\xi} \, dx = 0$$

(4)

with $\hat{u}(\xi) = \hat{u}(x(\xi))$.

Taking the variations of the integral in (4) gives

$$\left\{ 2 \left\{ f(\hat{x}(\xi)) - \hat{u}(\xi) \right\} \left\{ f'(\hat{x}(\xi)) \frac{d\hat{x}}{d\xi} \right\} \delta \hat{x} - \delta \hat{u}(\xi) \right\} d\xi + \left\{ f(\hat{x}(\xi)) - \hat{u}(\xi) \right\}^2 \frac{d}{d\xi} (\delta \hat{x}) d\xi.$$  

(5)

Integrating the last term by parts leads to

$$- \left\{ 2 \left\{ f(\hat{x}(\xi)) - \hat{u}(\xi) \right\} \left\{ f'(\hat{x}(\xi)) \frac{d\hat{x}}{d\xi} - \frac{du}{d\xi} \right\} \delta \hat{x} \right\} d\xi$$

$$+ \sum_{k=1}^{n+1} \left\{ \left( f(\hat{x}(\xi)) - \hat{u}(\xi) \right)_{k-1}^k \delta \hat{x}_{k-1} + f(\hat{x}(\xi)) - \hat{u}(\xi) \right\} \delta \hat{x}_k \right\}.$$  

(6)

Collecting terms and returning to the $x,u$ notation, (4) yields

$$\sum_{k=1}^{n+1} \int_{x_{k-1}}^{x_k} 2\left( f(x) - u(x) \right) \left\{ \delta u - u \frac{\delta x}{x} \right\} dx + \sum_{k=1}^{n} \left[ f(x) - u(x) \right] \delta x_k = 0$$

(7)

where the square bracket notation $[\cdot]_k$ denotes the jump in the quantity.
at the node \( k \).

With \( \delta x = 0 \) this leads back to (2) and equations for the best piecewise linear discontinuous \( L_2 \) fit to \( f(x) \). The full conditions are however

\[
\int_{x_{k-1}}^{x_k} \left\{ f(x) - u(x) \right\} \delta u \, dx = 0 \tag{8}
\]

\[
\int_{x_{k-1}}^{x_k} 2 \left\{ f(x) - u(x) \right\} (-u_x) \delta x_k \, dx + \left[ (f(x_k) - u(x_k))^2 \right]_{x_k} \delta x_k = 0 \quad \forall k \tag{9}
\]

With \( \delta u \) in the space of piecewise linear discontinuous functions the orthogonality condition (8) is equivalent [1] to the conditions

\[
\int_{x_{k-1}}^{x_k} \left\{ f(x) - u(x) \right\} \phi_{k1} \, dx = 0 \tag{10}
\]

\[
\int_{x_{k-1}}^{x_k} \left\{ f(x) - u(x) \right\} \phi_{k2} \, dx = 0 \tag{11}
\]

where \( \phi_{k1}, \phi_{k2} \) are the half linear basis functions in element \( k \) (see fig. 1). On the other hand, since \( \delta x \) lies in the space of piecewise linear continuous functions, we may set \( \delta x = a_k \), \( \delta u = u_x \delta x \) in (7) to obtain

\[
\left[ (f(x_k) - u(x_k))^2 \right]_{x_k} = 0 \tag{12}
\]
Using $L, R$ for left and right values at the (variable) node $k$, it follows from (13) that either

$$f - u_L = f - u_R \Rightarrow u_L = u_R$$  \hfill (13)

and $u$ is continuous at the new position of node $k$, or that

$$(f - u_L) = f - u_R \Rightarrow u_L + u_R = 2f$$  \hfill (14)

there.

Now it is known [2],[3] that for convex functions $f(x)$ the best $L_2$ fit amongst discontinuous piecewise linear functions is continuous, which clearly corresponds to (13). The case leading to (14) cannot therefore correspond to convexity in $f(x)$ and may apply only at inflection points.

It follows that the solution of the problem (10),(11),(12) is the set of best linear fits in separate elements which have the continuity property (13) or the averaging property (14), the former in the presence of convexity of $f(x)$.

§2. The Algorithm

The algorithm used here to find the best $L_2$ fit with variable nodes is in two stages (carried out repeatedly until convergence), corresponding to the choices of variations referred to in §1 above.

Stage (1) $\delta x = 0$, $\delta u = \phi_{k1}$ or $\phi_{k2}$ ($k=1,2,\ldots,n+1$)  \hfill (15)
This stage of the algorithm corresponds to the best L_2 fit amongst linear functions discontinuous at prescribed nodes, as in (1),(2).

Stage (ii) \[ \delta x = \alpha_k, \quad \delta u \big|_x \delta x = 0 \quad (k=1,2,\ldots, n+1) \] (16)

This stage corresponds to finding \( x_k \) such that (12) holds, with variations of \( x,u \) restricted to points lying on the piecewise linear approximation (possibly linearly extrapolated) in element \( k \).

The algorithm is analogous to minimising a quadratic function \( f(x,y) \) using two search directions \( v_1 \) and \( v_2 \) spanning the plane. Starting from some initial guess we may alternately minimise \( f \) in the directions \( v_1 \) and \( v_2 \). Similarly, to find the best \( L_2 \) fit we may begin with an initial guess \( \{x_k\},\{u_k\}_L,\{u_k\}_R \). Stage (i) is to find the minimum in the linear manifold specified by the variations given in (15) and so solve (10)-(11) for new \( \{x_k\},\{u_k\}_L,\{u_k\}_R \) with the \( x_k \) fixed. Stage (ii) is to find the minimum in the linear manifold specified by the variations given in (16) and so solve (12) for new \( \{x_k\},\{u_k\}_L,\{u_k\}_R \) by the implementation of (13),(14), more fully described below.

For regions in which \( f(x) \) is convex the solution for \( x_k \) is provided by (13), i.e. the intersections of lines in adjacent elements (see fig. 2). In this case \( f(x_k) - u(x_k) \) is of the same sign when approached from left or right. Where \( f(x) \) has an inflection point the intersection construction may fail and need to be replaced by the averaging construction (14). This will occur when values of \( f(x_k) - u(x_k) \) are of opposite sign when approached from left or right, as in fig. 3. Note that the calculation of \( x_k \) from (14) is implicit.
since \( f \) depends on \( x_k \) and \( u_L, u_R \) are new values, but the main iteration may be used to move towards the converged \( x_k \) by simply using the previous \( x_k \) and \( u \) values.

If \( f(x) \) is convex the result of the converged iteration (stage (i) – stage (ii) – repeated) is the grid with the best continuous \( L_2 \) fit using piecewise linear approximation. If \( f(x) \) is not convex there will in general be discontinuities in the fitted function but only at inflection points. It is simple to replace such a discontinuity locally by a continuous approximation (by say simply averaging the nodal values – in which case the result is the function value). This is of course at the expense of slightly moving away from the best fit minimisation at isolated points; the resulting approximation may however be used as an initialisation for more thorough algorithms [3].

The \( L_2 \) error of the fit described here is never worse than the error of the interpolant \( u_I \) which is well known [4] to satisfy

\[
\left\| u_I - f \right\|_2 \leq \frac{n-2}{6} \left\| f'' \right\|_2
\]

on \((0,1)\). (See also Appendix).

53 Results

We show results for five examples.

(a) \( e^{-20(1-x)} \) \hspace{1cm} \( 0 \leq x \leq 1 \) \hspace{1cm} 11 interior nodes

(b) \( \tanh(20(x-0.5)) \) \hspace{1cm} \( 0 \leq x \leq 1 \) \hspace{1cm} 11 interior nodes

(c) \( \sin 2\pi x \) \hspace{1cm} \( 0 \leq x \leq 1 \) \hspace{1cm} 11 interior nodes

(d) \( \sin 2\pi x \) \hspace{1cm} \( 0 \leq x \leq 1 \) \hspace{1cm} 10 interior nodes

(e) \[
\begin{cases}
  e^x & 0 \leq x \leq 0.5 \\
  (1.5-x) e^x & 0.5 \leq x \leq 1
\end{cases}
\] \hspace{1cm} 11 interior nodes
In each case the initial grid is equally spaced. Examples (c) and (d) distinguish between the constructions (13) \& (14) (see figs. 2 and 3).

In each example the trajectories of the nodes towards the final positions are shown together with the function and the fit obtained. The process is said to have converged when the $\ell_\infty$ norm of the nodal position updates is less than $10^{-4}$. The number of iterations appears on the ordinate axis of the trajectories.
§4. References


Appendix A

In this appendix, following [5], we give an asymptotic equidistribution result for the convex case. From (11) and (12) it follows that \( u - f \) vanishes at at least two points in each element, \( s_k \) and \( t_k \) say. Hence \( u' - f' \) vanishes at at least one point in each element, \( r_k \) say. Then, since \( u'' = 0 \),

\[
\int_{r_k}^{X} f'' \, d\xi = \int_{r_k}^{X} (f'' - u'') \, d\xi = f'(x) - u'(x) \quad (A1)
\]

and

\[
\int_{s_k \text{ or } t_k}^{X} (f' - u') \, d\eta = f(x) - u(x) \ . \quad (A2)
\]

Hence

\[
\int_{x_{k-1}}^{x_k} (f-u)^2 \, dx = \int_{x_{k-1}}^{x_k} \left\{ \int_{s_k \text{ or } t_k}^{X} \int_{r_k}^{X} f''(\xi) \, d\xi \right\}^2 \, dx \quad (A3)
\]

\[
\leq \int_{x_{k-1}}^{x_k} \left\{ (x_k - x_{k-1})^2 \, f''_{\text{max},k} \right\}^2 \, dx \quad (A4)
\]

where \( f''_{\text{max},k} \) is the maximum norm of \( f'' \) in element \( k \).

Now, if \( E(x) \) is an equidistributing function

\[
(x_k - x_{k-1}) \, E(\theta_k) = \text{a constant, } C \ , \quad (A5)
\]

where \( x_{k-1} < \theta_k < x_k \), and we have

\[
\int_{x_{k-1}}^{x_k} (f-u)^2 \, dx \leq C^4 \int_{x_{k-1}}^{x_k} \left\{ E(\theta_k) \right\}^{-4} \left\{ f''_{\text{max},k} \right\}^2 \, dx \quad (A6)
\]
so that

\[
\int_{x_0}^{x_n} (f-u)^2 \, dx \leq C^4 \sum_{k=1}^{\infty} \int_{x_{k-1}}^{x_k} \left( E(\theta_k) \right)^{-4} \left( f_{\text{max},k}'' \right)^2 \, dx.
\] (A7)

Finally, as in [5], we approximate the right hand side of (A7) by the integral

\[
C^4 \int_{x_0}^{x_n} \left( E(x) \right)^{-4} \left( f_{\text{max},k}'' \right)^2 \, dx.
\] (A.8)

and minimise over functions \( E(x) \), yielding

\[
\frac{d}{dx} \left[ \left( E(x) \right)^{-5} \left( f''(x) \right)^2 \right] = 0
\] (A9)

or

\[
E(x) = \left( f''(x) \right)^{2/s}
\] (A.10)

which may be regarded as the asymptotically equidistributed function.
Appendix B

In this appendix we extend the result in the main body of the report to general extremals.

For the problem of finding the extremal of the integral

\[ \int F(x,u) dx \]  

(B1)

over piecewise linear discontinuous functions \( u(x) \) with variable nodes, we follow the same procedure as in §1, obtaining

\[ \int_{x_{k-1}}^{x_k} F_u(x,u) \delta u \, dx = 0 \]  

(B2)

\[ \int_{x_{k-1}}^{x_k} F_u(x,u)(-u_x) \delta u \, dx + \left[ F(x,u) \right]_k \delta x_k = 0 \quad \forall k \]  

(B3)

in place of (8) and (9). Then (10), (11) and (12) become

\[ \int_{x_{k-1}}^{x_k} F_u(x,u) \phi_{k,i} \, dx = 0 \quad i = 1,2 \]  

(B4)

\[ \left[ F(x,u) \right]_k = 0 . \]  

(B5)

The corresponding algorithm is to solve (B4) for \( u \) in each element with fixed \( x_k \) (stage (i)) and then to solve (B5) for the \( x_k \) with \( u \) restricted to the stage (i) solution, possibly extrapolated (stage (ii)). Both problems are nonlinear and may or may not have
unique solutions. An example in which

\[ F(x,u) = Q(x).u + p(u) \]  \hspace{1cm} (B6)

where \( Q(x) \) is the given mass flow in a nozzle and \( u, p(u) \) are the velocity, pressure has been treated in [6].
Fig. 1a  \( \alpha_k \)

Fig. 1b  \( \phi_{kl} \phi_{kl} \)

Fig. 2

Fig. 3
\[ e^{-20(1-x)} \]

Files used: exp(1-15)

Frames: 14
Comp: 1
Clipping Values
x: 0.00 to 1.00*
y: -0.000 to 0.996*

Date: 900521
Time: 130453
User: mensbains
Current Plot Data
Format: 4 size: 7
Files used:
exp(1-15)

frames: 1-13
comp: 1
Clipping Values
x: 0.00 to 1.00*
y: 0.0 to 80.0*

Date: 900518
Time: 164207
User: smsbains
Current Plot Data
format: 4 size: 7

tanh 20 (x-0.5)

trajectories
\text{tanh} \ 20(x - 0.5)
$$\tanh 20(x-0.5)$$

fit
Files used: exp(1-15)

frames: 1-13
comp: 1
Clipping Values
x: 0.00 to 1.00*
y: 0.0 to 19.0*

$\sin 2\pi x$

trajectories

Date: 090521
Time: 130655
User: smshains
Current Plot Data
format: 4 size: 7
\[ \sin 2\pi x \]

function
Files used: exp(1-15)

\[ \sin 2\pi x \]

frames: 14
comp: 1
Clipping Values
x: 0.00 to 1.00*
y: -1.02 to 1.02*

Date: 900514
Time: 094516
User: smsbains
Current Plot Data
format: 4 size: 7
Files used:
exp(1-14)

\[ \sin 2\pi x \]

frames: 1-12
comp: 1
Clipping Values
x: 0.00 to 1.00*
y: 0.0 to 46.0*

Date: 900521
Time: 130847
User: smshains
Current Plot Data
format: 4 size: 7
Files used:
exp(1-14)

frames: 14
comp: 1
Clipping Values
ox: 0.00 to 1.00*
y: -1.00 to 1.00*

\[ \sin \frac{2\pi x}{\text{function}} \]
$\sin 2\pi x$

files used: exp(1-14)

frames: 13
comp: 1
Clipping Values
x: 0.00 to 1.00*
y: -1.02 to 1.02*

Date: 900521
Time: 130908
User: smsbains
Current Plot Data
format: 4 size: 7
Files used: exp(1-15)

\[
\begin{align*}
e^x & \quad 0 \leq x \leq 0.5 \\
(1.5 - x)e^x & \quad 0 < x < 1
\end{align*}
\]

frames: 1-13
comp: 1
Clipping Values
x: 0.00 to 1.00*
y: 0.0 to 53.0*

Date: 900521
Time: 125456
User: smsbains
Current Plot Data
format: 4 size: 7
\[ f(x) = \begin{cases} e^x & 0 \leq x \leq 0.5 \\ (1.5-x)e^x & 0.5 < x \leq 1 \end{cases} \]
\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{e^x}{(1.5-x)e^x} & 0 \leq x < 0.5 \\
\frac{1}{(1.5-x)e^x} & 0.5 \leq x \leq 1
\end{array} \right.
\end{align*}
\]