A UNIFIED SET OF SINGLE-STEP ALGORITHMS PART 4 : BACKWARD ERROR ANALYSIS APPLIED TO THE SOLUTION OF THE DYNAMIC VIBRATION EQUATION

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SUMMARY

This paper applies the method of backward error analysis to the numerical solution of the dynamic vibration equation by the single step algorithms proposed in references [1,2,3] and obtains some general rules to help with the choice of parameters.

INTRODUCTION

The general idea of backward error analysis applied to the numerical solution of the scalar dynamic equation

$$m\ddot{x} + \mu\dot{x} + kx = f(t) \tag{1}$$

is to say that our <u>approximate</u> solution of the equation (1) is the <u>exact</u> solution of a slightly perturbed form of equation (1):

$$m\ddot{x} + (\mu + \delta\mu)\dot{x} + (k+\delta k)x = f(t) \tag{2}$$

for small values of a parameter which is proportional to Δt, the time step. We suppose that the mass m stays the same and the forcing function is represented adequately. Thus we could say that our numerical solution is the exact solution for a system with slightly different damping and stiffness. It is more relevant in this context however instead to use the fact that we know the exact solution of equation (1) represents a damped oscillation with terms of the form

$$\exp(vt), \quad \text{where } v = \frac{-\mu}{2m} + i\overline{w} \tag{3}$$

We suppose here that the damping is less than critical i.e. $\mu = 2\nu\sqrt{km}$, $0 \leqq \nu < 1$ where ν is the fraction of critical damping. Then, if $\cos\alpha = \nu$, we have

$$v = \sqrt{\frac{k}{m}} \exp\,[i(\pi-\alpha)] \quad \text{and} \quad \overline{w} = \sqrt{\frac{k}{m}} \sin\alpha \tag{4}$$

It is thus convenient to assume that our numerical solution represents

$$\exp\left\{\left[-\left(\frac{\mu}{2m} + \delta\right) + i(\overline{w} + \varepsilon)\right]\Delta t\right\} \tag{5}$$

so that we can regard our numerical solution as having the effect of

(a) increasing the damping by a quantity δ

(b) increasing the frequency of the oscillation by $\frac{\varepsilon}{2\pi}$

or giving a fractional period increase of $\left(1 + \frac{\varepsilon}{\bar{w}}\right)^{-1} - 1 \approx -\frac{\varepsilon}{\bar{w}}$ (6)

for small values of this quantity.

We are applying this method of backward error analysis here to the single step methods SSp2 described in references [1], [2]. The motivation is that each of these SSp2 methods is expressed in terms of p parameters $\theta_1, \ldots, \theta_p$ and we need a strategy to decide how to choose these parameters. This paper is investigating what useful information we can get out of the backward error analysis approach.

The accuracy and stability qualities of these single-step methods (including also the Generalised Newmark "β-m" methods given in reference [3]) can be studied from the characteristic polynomials of the equivalent p-step methods which have the form

$$\sum_{j=0}^{p} \alpha_j r^j + 2\nu\sqrt{\frac{k}{m}}\,\Delta t \sum_{j=0}^{p} \gamma_j r^j + \frac{k}{m}\,\Delta t^2 \sum_{j=0}^{p} \beta_j r^j = 0 \qquad (7)$$

Gladwell and Thomas [4] pointed out that for $p = 4$ these equivalent 4-step methods are automatically 3rd order accurate and hence by Dahlquist's theorem [5] have no possibility of unconditional stability. In this paper we concentrate on the backward error analysis of SS22 and SS32 methods.

Substituting $r = \exp(\tilde{v}\Delta t)$ into equation (7) and expanding we have

$$\sum_{j=0}^{p} \alpha_j + \tilde{v}\Delta t \sum_{j=0}^{p} j\alpha_j + 2\nu\sqrt{\frac{k}{m}}\,\Delta t \sum_{j=0}^{p} \gamma_j$$

$$+ \tilde{v}^2 \frac{\Delta t^2}{2!} \sum_{j=0}^{p} j^2\alpha_j + 2\nu\sqrt{\frac{k}{m}}\,\Delta t^2 \tilde{v} \sum_{j=0}^{p} j\gamma_j + \frac{k}{m}\,\Delta t^2 \sum_{j=0}^{p} \beta_j$$

$$+ \tilde{v}^3 \frac{\Delta t^3}{3!} \sum_{j=0}^{p} j^3\alpha_j + 2\nu\sqrt{\frac{k}{m}}\,\frac{\Delta t^3}{2!}\tilde{v}^2 \sum_{j=0}^{p} j^2\gamma_j + \frac{k}{m}\,\Delta t^3 \tilde{v} \sum_{j=0}^{p} j\beta_j$$

$$+ \ldots\ldots\ldots = 0 \qquad (8)$$

We know that for these SSp2 methods

$$\left.\begin{aligned} &\sum_{j=0}^{p} \alpha_j = \sum_{j=0}^{p} j\alpha_j = \sum_{j=0}^{p} \gamma_j = 0 \\ \text{and} \\ &\frac{1}{2!} \sum_{j=0}^{p} j^2\alpha_j = \sum_{j=0}^{p} j\gamma_j = \sum_{j=0}^{p} \beta_j = A_2, \text{ say} \end{aligned}\right\} \quad (9)$$

Hence putting

$$B_1 = \sum_{j=0}^{p} j\beta_j, \quad B_2 = \frac{1}{2!} \sum_{j=0}^{p} j^2\gamma_j, \quad B_3 = \frac{1}{3!} \sum j^3\alpha_j$$

$$C_1 = \frac{1}{2!} \sum_{j=0}^{p} j^2\beta_j, \; \ldots\ldots$$

$$D_1 = \frac{1}{3!} \sum j^3\beta_j, \; \ldots\ldots$$

we have

$$\begin{aligned} &A_2\left(\tilde{v}^2 + 2\nu\sqrt{\frac{k}{m}}\,\tilde{v} + \frac{k}{m}\right) \\ &+ \Delta t\left(B_3\tilde{v}^3 + 2B_2\nu\sqrt{\frac{k}{m}}\,\tilde{v}^2 + B_1\frac{k}{m}\,\tilde{v}\right) \\ &+ \Delta t^2\left(C_3\tilde{v}^4 + 2C_2\nu\sqrt{\frac{k}{m}}\,\tilde{v}^3 + C_1\frac{k}{m}\,\tilde{v}^2\right) \\ &+ \Delta t^3\left(D_3\tilde{v}^5 + 2D_2\nu\sqrt{\frac{k}{m}}\,\tilde{v}^4 + D_1\frac{k}{m}\,\tilde{v}^3\right) + \ldots = 0 \end{aligned} \quad (10)$$

where $\tilde{v} = v + (-\delta + i\varepsilon)$, $v = \sqrt{\frac{k}{m}}\, e^{i(\pi-\alpha)}$ (11)

For SS22 we have

$$\alpha_2 = 1, \quad \gamma_2 = \theta_1, \quad \beta_2 = \frac{\theta_2}{2}$$

$$\alpha_1 = -2, \quad \gamma_1 = 1 - 2\theta_1, \quad \beta_1 = \tfrac{1}{2} + \theta_1 - \theta_2$$

$$\alpha_0 = 1, \quad \gamma_0 = \theta_1 - 1, \quad \beta_0 = \tfrac{1}{2} - \theta_1 + \frac{\theta_2}{2}$$

Hence

$$A_2 = 1, \quad B_1 = B_2 = \theta_1 + \frac{1}{2}, \quad B_3 = 1$$

$$C_1 = \frac{1}{4}(2\theta_2 + 2\theta_1 + 1), \quad C_2 = \theta_1 + \frac{1}{6}, \quad C_3 = \frac{7}{12} \quad . \qquad (12)$$

With SS32 it is simpler to work with a,b,c which are such that

$a_1 = \theta_1 - \frac{1}{2}$, $b = \theta_2 - \theta_1 - 1/6$, $c = \theta_3 - 3\theta_2/2 + \frac{1}{4}$.

We then have

$$\alpha_3 = \frac{1+2a}{2}, \quad \alpha_2 = -\frac{(1+6a)}{2}, \quad \alpha_1 = \frac{(6a-1)}{2}, \quad \alpha_0 = \frac{(1-2a)}{2}$$

$$\gamma_3 = \frac{1}{6}(3a+3b+2), \quad \gamma_2 = -\frac{1}{2}(a+3b), \quad \gamma_1 = \frac{1}{2}(-a+3b), \quad \gamma_0 = \frac{1}{6}(3a-3b-2)$$

$$\beta_3 = \frac{1}{24}(6a+6b+4c+3), \quad \beta_2 = \frac{1}{8}(2a-2b-4c+3),$$

$$\beta_1 = \frac{1}{8}(-2a-2b+4c+3), \quad \beta_0 = \frac{1}{24}(-6a+6b-4c+3) \quad .$$

Hence

$$A_2 = 1, \quad B_1 = B_2 = B_3 = a + 3/2$$

$$C_1 = C_2 = \frac{1}{2}(3a+b+3), \quad C_3 = \frac{1}{6}(9a+8)$$

$$D_1 = \frac{1}{24}(34a+18b+4c+27), \quad D_2 = \frac{1}{24}(32a+18b+27) \qquad (13)$$

$$D_3 = \frac{1}{8}(10a+7)$$

We first produce results with $\nu = 0$ in order to show that this approach gives results in agreement with those already published.

SS22, $\nu = 0$

We know from reference [2] that when $\nu = 0$ we have for $\theta_1 \neq \frac{1}{2}$, ε is $O(\Delta t^2)$, δ is $O(\Delta t)$ and for $\theta_1 = \frac{1}{2}$, ε is $O(\Delta t^2)$ and δ is zero (N.B. misprinted in Table II in reference [2]).

Hence to include in the expansions all terms up to and including $O(\Delta t^2)$ we must include the terms in δ^2 and $\Delta t\delta$.

Thus from equation (10) with $\nu = 0$, $\tilde{v} = i\sqrt{\frac{k}{m}} + (-\delta + i\varepsilon)$

we have

$$2i\sqrt{\frac{k}{m}}\,(-\delta + i\varepsilon) + \delta^2$$
$$+ \Delta t\left[(\theta_1 - \tfrac{1}{2})\, i\left(\frac{k}{m}\right)^{3/2} + \left(\frac{5}{2} - \theta_1\right)\frac{k}{m}\,\delta\right]$$
$$+ \Delta t^2\,\frac{k^2}{m^2}\left[\frac{1}{3} - \frac{\theta_2}{2} - \frac{\theta_1}{2}\right] + i\,O(\Delta t^3) + O(\Delta t^4) = 0 \qquad (14)$$

Equating real and imaginary parts in equation (14) and sorting out the terms we eventually arrive at

$$\delta = \frac{1}{2}\frac{k}{m}\,\Delta t\,(\theta_1 - \tfrac{1}{2}) + O(\Delta t^3) \qquad (15)$$

and

$$\frac{-\varepsilon}{w} = \frac{1}{8}\frac{k}{m}\,\Delta t^2\left[\theta_1^2 - 3\theta_1 + 2\theta_2 + \frac{11}{12}\right] + O(\Delta t^4) \qquad (16)$$

For unconditional stability we know from reference [2] that we require $\theta_2 \geq \theta_1 \geq \frac{1}{2}$ and we can see the link between this and the sign of δ in equation (15) as is to be expected.

We also see that the factor on the right hand side of equation (16) is

$$\theta_1^2 - 3\theta_1 + 2\theta_2 + \frac{11}{12} > \theta_1^2 - \theta_1 + \frac{11}{12}$$
$$= (\theta_1 - \tfrac{1}{2})^2 + \frac{2}{3}, \quad \text{for} \quad \theta_2 \geq \theta_1$$

and hence unconditionally stable schemes always have a period increase for values of the time step such that the $O(\Delta t^2)$ term in equation (16) dominates. It is usually considered that there should be at least 20 time steps per period and this is sufficient to make the $O(\Delta t^2)$ term dominant for practical values of θ_1 and θ_2. The lowest period increase is for $\theta_1 = \theta_2 = 0.5$.

For an explicit scheme we take $\theta_2 = 0$, [1], and then

$$-\frac{\varepsilon}{w} = \frac{1}{8}\frac{k}{m}\,\Delta t^2\left[\theta_1^2 - 3\theta_1 + \frac{11}{12}\right] + O(\Delta t^4) \qquad (17)$$

and hence there is a period decrease if $0.34 < \theta_1 < 2.66$, or, together with a non-negative δ , this gives $0.5 \leq \theta_1 < 2.6$.

We use as test problem (1) equation (1) with $k = m = 1$, $\mu = 0$ and $f \equiv 0$.

Figure 1 shows 2U1 with $\theta_1 = \theta_2 = 0.5$ which gives zero numerical damping and the minimum period increase for unconditionally stable schemes, and 2E1 with $\theta_1 = 0.5$, $\theta_2 = 0$, again with zero damping and now with period decrease.

SS32 with $\nu = 0$

Now from equation (10) using the coefficients given by equations (13) and preserving all terms up to $O(\Delta t^3)$ we obtain

$$-\frac{\varepsilon}{w} = \frac{\Delta t^2}{12}\,\frac{k}{m}\,(3b + 1) + O(\Delta t^4) \tag{18}$$

$$\delta = \frac{\Delta t^3}{12}\,\frac{k^2}{m^2}\,(3ab - c) + O(\Delta t^5) \tag{19}$$

We know from reference [2] that for unconditional stability we require $a > 0$, $b \geq 0$, $c \geq 0$ and $3ab - c \geq 0$. Hence again we see the link between the sign of δ as given by equation (19) and the stability condition. Also we again see that with $b \geq 0$ there is a period increase for all unconditionally stable schemes, the most accurate period being for $b = 0$ which also gives $\delta \equiv 0$.

A well-known method included in SS32 is Wilson-θ where $\theta_1 = \theta$, $\theta_2 = \theta^2$, $\theta_3 = \theta^3$. For $\theta = 1.4$ we have

$$\delta = 0.084\,\frac{k^2}{w^2}\,\Delta t^3 \quad \text{and} \quad -\frac{\varepsilon}{w} = 0.182\,\frac{k}{m}\,\Delta t^2 \tag{20}$$

We also obtain the single step equivalent of Houbolt by taking $\theta_1 = 2$, $\theta_2 = 11/3$, $\theta_3 = 6$ which give

$$\delta = 0.5\,\frac{k^2}{m^2}\,\Delta t^3 \quad \text{and} \quad -\frac{\varepsilon}{w} = 0.45\,\frac{k}{m}\,\Delta t^2 \quad . \tag{21}$$

Figure 2 shows Wilson $\theta = 1.4$ and Houbolt applied to test problem (1) together with another example:

3U1 : $\theta_1 = 0.5$, $\theta_2 = 2/3$, $\theta_3 = 0.75$. This is a degenerate case as it gives $a = 0$ as well as $b = c = 0$; thus δ is zero and

$$-\frac{\varepsilon}{w} = \frac{1}{12}\frac{k}{m}\Delta t^2 \qquad (22)$$

is the same minimum period increase with unconditional stability as with 2U1. The stability polynomial for 3U1 is the same as that for 2U1 with the extra factor $(r+1)$ so it is better to use 2U1.

For explicit methods we take $\theta_3 = 0$; for example:

3E1 : $\theta_1 = 0.5$, $\theta_2 = 1/6$, $\theta_3 = 0$ i.e. $a = c = 0$, $b = 0.5$. This gives the same zero δ and the same period decrease as in 2E1

3E2 : $\theta_1 = 13/6$, $\theta_2 = 2$, $\theta_3 = 0$ i.e. $a = 5/3$, $b = -1/3$, $c = -11/4$. This gives $\delta = 0.09\frac{k^2}{m^2}\Delta t^3$ which is between Wilson $\theta = 1.4$ and Houbolt and $-\frac{\varepsilon}{w} = 0.15\frac{k^2}{m^2}\Delta t^4$ which means increased accuracy for $\frac{\Delta t}{T} < 0.08$ approximately. Figure 3 shows 3E1 and 3E2 applied to test problem 1.

Figure 4 shows the % period increase or decrease plotted against $\Delta t/T$ for the examples of SS22 and SS32 mentioned above.

Various references, for example [3] and [6], also show results which conform with these predictions.

SS22 $\nu \neq 0$

We know from reference 2 that in general now ε and δ are $O(\Delta t)$ but when $\nu = 0$ the $O(\Delta t)$ term in the expansion for ε will vanish and when $\theta_1 = \frac{1}{2}$ both ε and δ are $O(\Delta t^2)$. Hence we now include in our expansion terms with δ^2, $\delta\varepsilon, \varepsilon^2$, $\Delta t\varepsilon$ and $\Delta t\delta$ in order to include all $O(\Delta t^2)$ terms.

From equation (10) we now have, putting $s = \sqrt{1-\nu^2}$,

$$\begin{aligned} 2\sqrt{\frac{k}{m}}\, s(\varepsilon+i\delta) &= \delta^2 - 2i\delta\varepsilon - \varepsilon^2 \\ &+ \Delta t\left(\frac{k}{m}\right)^{3/2}\left[e^{3i(\pi-\alpha)} + (\theta_1+\tfrac{1}{2})(2\nu e^{2i(\pi-\alpha)} + e^{i(\pi-\alpha)}\right] \\ &+ \Delta t\,\frac{k}{m}\left[3e^{2i(\pi-\alpha)} + (\theta_1+\tfrac{1}{2})(4\nu e^{i(\pi-\alpha)}+1)\right](-\delta+i\varepsilon) \\ &+ \Delta t^2\,\frac{k^2}{m^2}\left[C_3 e^{4i(\pi-\alpha)} + 2\nu C_2 e^{3i(\pi-\alpha)} + C_1 e^{2i(\pi-\alpha)}\right] \\ &+ O(\Delta t^3) \end{aligned} \qquad (24)$$

Equating real and imaginary parts in equation (24) and sorting out the algebra gives:

$$\delta = \frac{\Delta t}{2}\ \frac{k}{m}\ (\theta_1 - \tfrac{1}{2})(1 - 4\nu^2) + B\Delta t^2 + O(\Delta t^3) \qquad (25)$$

where

$$B = \left(\frac{k}{m}\right)^{3/2} \nu \left\{\nu^2\left(\theta_1^2 + \frac{1}{2}\theta_1 - \frac{1}{6}\right) - \frac{\theta_1^2}{4} + \frac{3\theta_1}{8} - \frac{\theta_2}{2} - \frac{1}{8}\right\} \qquad (26)$$

and

$$\varepsilon = \frac{\Delta t^2}{2}\ \frac{k}{m}\ \frac{\nu}{\sqrt{1-\nu^2}}\ (\theta_1 - \tfrac{1}{2})(4\nu^2 - 3) + D\Delta t^2 + O(\Delta t^3) \qquad (27)$$

We can see that for "small" values of ν and Δt and $\theta_1 > \frac{1}{2}$ the formula for δ is dominated by the first term and is positive i.e. the damping is increased. Figure 5 illustrates this for $\theta_1 = 0.6$, $\theta_2 = 0.605$, $\nu = 0.1$. When $\theta_1 = 0.5$ the first term is removed and we have

$$\delta = \left(\frac{k}{m}\right)^{3/2} \Delta t^2\ \frac{\nu}{2}\left[\frac{2\nu^2}{3} - \theta_2\right] \qquad (28)$$

which is positive for the explicit method with $\theta_1 = \frac{1}{2}$, $\theta_2 = 0$, and δ is negative for $\theta_2 > \frac{2\nu^2}{3}$. Figure 6 shows this for $\theta_1 = \theta_2 = 0.5$, $\nu = 0.1$ with test problem 2 where the forcing function f in equation (1) is given by

$$f = 0,\ t \leqq 0;\quad f = +1,\ 0 < t \leqq 25;\quad t > 25 \quad . \qquad (29)$$

Here the position of the first turning point after the start only depends on the damping.

If we write

$$\delta = A\Delta t + B\Delta t^2 + O(\Delta t^3) \qquad (30)$$

and A and B are of opposite signs, then we must have $\Delta t < \left|\frac{A}{B}\right|$ for the first term to dominate. Table 1 shows some values of A, B and $|A/B|$ for $k = m = 1$. Figure 7 shows the negative δ resulting from a too-large value of Δt (= 0.75) with $\theta_1 = 0.6$, $\theta_2 = 1.0$, $\nu = 0.4$.

We now have a formula for the fractional period increase:

$$-\frac{\varepsilon}{\bar{w}} = -\frac{\Delta t}{2}\sqrt{\frac{k}{m}}\ \frac{\nu}{1-\nu^2}\,(\theta_1 - \tfrac{1}{2})(4\nu^2 - 3) - \frac{m}{k}\frac{D}{\sqrt{1-\nu^2}}\ \Delta t^2 + O(\Delta t^3) \tag{31}$$

where

$$\begin{aligned}\left(\frac{m}{k}\right)^{3/2} sD = {} & \frac{1}{8}(\theta_1-\tfrac{1}{2})^2(1-4\nu^2)^2 - \frac{1}{8}\frac{\nu^2}{s^2}(\theta_1-\tfrac{1}{2})^2(4\nu^2-3)^2 \\ & - \frac{1}{4}\left[4\nu^2(1-\theta_1) + \theta_1 - \frac{5}{2}\right](\theta_1-\tfrac{1}{2})(1-4\nu^2) \\ & - \nu^2(\theta_1-1)(\theta_1-\tfrac{1}{2})(4\nu^2-3) \\ & + \tfrac{1}{2}\left[8\nu^4(C_3-C_2) + 2\nu^2(-4C_3 + 3C_2 + C_1) + C_3 - C_1\right]\end{aligned} \tag{32}$$

Writing

$$\varepsilon = C\Delta t + D\Delta t^2 \tag{33}$$

we note that C has a factor s^{-1} and D has a factor s^{-3} for $\theta_1 \neq \frac{1}{2}$ hence we expect that this formula will only be useful for smaller values of ν. We restrict consideration of the period increase or decrease to problems where $\nu < 0.5$. Table 2 gives values of $-C/\bar{w}$ and $-D/\bar{w}$ and $|C/D|$ (where relevant) for $\nu = 0.1$ and two unconditionally stable and two explicit methods. We see that this again predicts a period increase with the unconditionally stable methods. Figure 5 shows the period increase with $\theta_1 = 0.6$, $\theta_2 = 0.605$. When $\theta_1 = \frac{1}{2}$ the equation (31) reduces to

$$-\frac{\varepsilon}{\bar{w}} = -\frac{k}{2m}\,\frac{\Delta t^2}{(1-\nu^2)}\left[-\frac{2}{3}\nu^4 + \left(\frac{1}{3} + \theta_2\right)\nu^2 + \frac{1}{12} - \frac{\theta_2}{2}\right] \tag{34}$$

This formula clearly gives a period increase for $\theta_2 \geq \frac{1}{2}$, and $\nu = 0.1$ (Fig. 6), and a period decrease for the explicit method with $\theta_2 = 0$ for $\nu \leq 0.1$.

SS32 $\nu \neq 0$

From reference [2] we now expect both ε and δ to be $O(\Delta t^2)$ for general $\theta_1, \theta_2, \theta_3$, but we know that δ is $O(\Delta t^3)$ when $\nu = 0$ hence we include the $\Delta t \varepsilon$ term in substituting into equation (10). With the coefficients given by equations (13), equating real and imaginary parts and sorting out we have:

$$-\frac{\varepsilon}{\bar{w}} = \frac{\Delta t^2}{12(1-\nu^2)} \frac{k}{m} (8\nu^4 - 8\nu^2 + 1)(3b + 1) + O(\Delta t^3) \tag{35}$$

where we know the $O(\Delta t^3)$ term is zero when $\nu = 0$, and

$$\delta = \Delta t^2 \left(\frac{k}{m}\right)^{3/2} \nu(2\nu^2 - 1)(b + \frac{1}{3}) + \Delta t^3 \frac{k^2}{m^2} \left[\nu^4(2ab - 3b - 1) + \nu^2(-2ab + \frac{3b}{2} + \frac{c}{3} + \frac{1}{2}) + \frac{1}{12}(3ab - c)\right] + O(\Delta t^4) \tag{36}$$

We see that the formula for $-\varepsilon/\bar{w}$ given by equation (35) has a first term which reduces to the first term in formula (18) for $\nu = 0$. We know from reference [2] that we require $b \geqq 0$ for unconditional stability. Hence we again have a period increase with an unconditionally stable scheme (provided Δt is small enough) and for values of ν such that

$$8\nu^4 - 8\nu^2 + 1 > 0 \tag{37}$$

This is true for $\nu < 0.4$ approx. Figure 8 shows the period increase with Houbolt parameters and $\nu = 0.3$. Figure 9 shows the period decrease with Houbolt and $\nu = 0.5$. The formula (36) for δ has a first term which reduces to zero for $\nu = 0$; for $b \geqq 0$ and $\nu < 0.7$ this term is negative and we know from equation (19) that when $\nu = 0$ the $O(\Delta t^3)$ term is positive with values of the parameters which give unconditional stability. Thus the sign of δ for very small values of ν can change at a value of Δt around the 20 time steps per period mark, e.g. with the Wilson-θ = 1.4 parameters and $\nu = 0.04$, at $\Delta t = 0.35$ approximately. For larger values of

ν the critical time step is larger. Figure 8 shows the negative δ with Houbolt parameters and $\nu = 0.3$. As b is larger for Houbolt (1.5) than for Wilson-θ (0.39) the effect is more obvious.

It is interesting to note that the explicit scheme 3E1 with $\theta_1 = 0.5$, $\theta_2 = 1/6$, $\theta_3 = 0$ is unconditionally unstable for $\nu > 0$ (thus disproving the idea that any scheme which works with $\nu = 0$ is bound to be all right with natural damping included). Figure 10 illustrates this for $\nu = 0.1$. We can see why this is so by looking at the stability conditions for SS32 from reference [7,8]:

$$\left.\begin{aligned} &12ma + \mu\Delta t(6b+1) + 2k\Delta t^2 c > 0 \\ &2m + 2a\mu\Delta t + bk\Delta t^2 \geq 0 \\ &\mu\Delta t + ak\Delta t^2 \geq 0 \\ &k\Delta t^2 \geq 0 \end{aligned}\right\} \quad (38)$$

and

$$12m\mu + 12a\mu^2\Delta t + \mu k\Delta t^2(12a^2-1) + 2k^2\Delta t^3(3ab-c) \geq 0 \quad (39)$$

The 3E1 parameters correspond to $a = c = 0$, $b = -\frac{1}{2}$ and with these it is impossible to satisfy the stability conditions with $\mu \neq 0$.

When $\mu = 0$ inequality (39) reduces to $(3ab - c) \geq 0$; when $\mu \neq 0$ this is only one of the conditions for inequality (39) to be satisfied. When we have an explicit scheme with $\theta_3 = 0$ we have $c = -\frac{3}{2}(a + b + \frac{1}{2})$. We can now choose $b = -\frac{1}{3}$ which we can see from equations (35) and (36) will make both δ and ε equal to $O(\Delta t^3)$. The stability conditions now become:

$$\left.\begin{aligned} &12ma - \mu\Delta t - 3k\Delta t^2(a + \tfrac{1}{6}) > 0 \\ &2m + 2a\mu\Delta t - \tfrac{1}{3}k\Delta t^2 \geq 0 \\ &\mu\Delta t + ak\Delta t^2 \geq 0 \\ &k\Delta t^2 \geq 0 \\ &\text{and} \\ &12m\mu + 12a\mu^2\Delta t + \mu k\Delta t^2(12a^2-1) + \Delta t^3(a+\tfrac{1}{2}) \geq 0 \end{aligned}\right\} \quad (40)$$

We still have to choose the value of a; a simple and obvious choice which works out quite well is to take $a = 1/\sqrt{12} = 0.2887$ which satisfies the last of inequalities (40) and leaves the stability condition to be deduced from the first two. For $m = k = 1$ these give

$$\left.\begin{aligned} &(2 + \sqrt{12})\Delta t^2 + 8\nu\Delta t - 4\sqrt{12} < 0 \\ \text{and} \qquad &\Delta t^2 - \sqrt{12}\nu\Delta t - 6 \leqq 0 \end{aligned}\right\} \qquad (41)$$

For $\nu = 0.1$ these are both satisfied for $\Delta t < 1.5$ and for $\nu = 0.4$ these are both satisfied for $\Delta t < 1.3$. In fact they are satisfied for $\Delta t \leqq 1$ for any value of $\nu < 1$.

We call this method 3E3; it is given by $\theta_1 = 0.7887$, $\theta_2 = 0.6220$ and $\theta_3 = 0$. Figure 11 illustrates 3E 3 applied to test problem 1 with $\nu = 0.1$ and $\Delta t = 1$.

Summary of Results

$\nu = 0$

Of course $\delta \geqq 0$ for stability. Unconditionally stable schemes always have a period increase; the lowest is for the marginal stability case when $\delta = 0$. Explicit schemes have a period decrease for practical values of the parameters.

$\nu \neq 0$

SS22

$0 < \nu < 0.5$ (1) $\theta > 0.5$ (a) $\theta_2 \geqq \theta_1$ unconditionally stable; $\delta = O(\Delta t)$; $\delta > 0$ for smaller Δt, $\delta < 0$ for larger Δt; period increase.

(b) $\theta_2 = 0$; explicit, conditionally stable; $\delta > 0$; period increase for smaller Δt, decrease for larger Δt.

(2) $\theta_1 = 0.5$, (a) $\theta_2 \geqq \theta_1$ unconditionally stable; $\delta = O(\Delta t^2)$; $\delta < 0$; period increase.

(b) $\theta_2 = 0$; explicit; conditionally stable, $\delta > 0$; period decrease.

SS32

(1) With parameters for unconditional stability, $\delta = O(\Delta t^2)$; $\nu < \sqrt{2}/2$:

$\delta < 0$ for smaller Δt, $\delta > 0$ for larger Δt;

(a) $0 < \nu < 0.15$ period increase

(b) $0.15 \leq \nu \leq 0.5$ period decrease

$\nu > \sqrt{2}/2$: $\delta > 0$ for smaller Δt, $\delta < 0$ for larger Δt (Figure 12).

(2) Explicit, conditionally stable; with $b = -\frac{1}{3}$, $\delta = O(\Delta t^3)$. With $0 < \nu \leq 0.5$, $a > 0$; $\delta > 0$; period increase for smaller Δt, decrease for larger Δt.

Example: $\theta_1 = 0.7887$, $\theta_2 = 0.6220$, $\theta_3 = 0$ (3E3).

Because of the nature of the solution of the differential equation (1) changes when approaches unity we have restricted the discussion of this backward error analysis to values of ν less than or equal to 0.75.

Conclusions

The results presented here of the backward error analysis of the numerical solution of the dynamic vibration equation (1) by the single-step methods proposed in references [1,2,3] give some general rules which should be useful in choosing parameters for a particular purpose. For example, we know that most structures have modes with damping < 10%, sometimes as low as 4 or 5% [9], hence results obtained here for $\nu < 0.1$ will be relevant. We can see that it is not possible to have unconditional stability with $\delta \geq 0$ guaranteed for all sizes of time step. If we want $\delta \geq 0$ in order to give no overshoot then we must accept some limitation on the time step and 3E3 with its extra accuracy looks well worth considering.

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TABLE 1

ν	θ_1	θ_2	A	B	$\lvert A/B \rvert$	Stability Limit
0.1	0.5	0.5	0	-2.5×10^{-2}	-	-
0.1	0.6	0.605	4.8×10^{-2}	-2.9×10^{-2}	1.6	-
0.1	0.5	0	0	3.3×10^{-4}	-	2
0.1	0.6	0	4.8×10^{-2}	1.5×10^{-3}	32	1.8
0.4	0.5	0.5	0	-7.9×10^{-2}	-	-
0.4	0.6	0.605	1.8×10^{-2}	-8.5×10^{-2}	2.1×10^{-1}	-
0.4	0.5	0	0	2.1×10^{-2}	-	2
0.6	0.6	0.605	-2.2×10^{-2}	-6.9×10^{-2}		

SS22 $\delta = A\Delta t + B\Delta t^2$

TABLE 2

ν	θ_1	θ_2	$-C/\overline{w}$	$-D/\overline{w}$	$\lvert C/D \rvert$
0.1	0.5	0.5	0	8×10^{-2}	
0.1	0.6	0.605	1.5×10^{-2}	8.8×10^{-2}	
0.1	0.5	0	0	-4.4×10^{-2}	
0.1	0.6	0	1.5×10^{-2}	-6.3×10^{-2}	0.24

SS22 $\varepsilon = C\Delta t + D\Delta t^2$

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LIST OF FIGURES

1) 2U1 : SS22 with $\theta_1 = \theta_2 = 0.5$
   2E1 : S222 with $\theta_1 = 0.5$, $\theta_2 = 0$ } $\nu = 0$, $\Delta t = 0.5$, $f = 0$

2) 3W : SS32 Wilson-θ; $\theta_1 = 1.4$, $\theta_2 = 1.96$, $\theta_3 = 2.744$
   3H : SS32 Houbolt; $\theta_1 = 2$, $\theta_2 = 3.6667$, $\theta_3 = 6$
   3U1 : SS32 with $\theta_1 = 0.5$, $\theta_2 = 0.6667$, $\theta_3 = 0.75$ } $\nu = 0$, $\Delta t = 0.25$, $f = 0$

3) 3E1 : SS32 with $\theta_1 = 0.5$, $\theta_2 = 0.16667$, $\theta_3 = 0$
   3E2 : SS32 with $\theta_1 = 2.16607$, $\theta_2 = 2$, $\theta_3 = 0$ } $\nu = 0$, $\Delta t = 0.5$, $f = 0$

4) Percentage period increases and decreases for the algorithms represented in Figures, 1,2 and 3.

5) SS22 : $\theta_1 = 0.6$, $\theta_2 = 0.605$, $\nu = 0.1$, $\Delta t = 0.25$, $f = 0$ shows $\delta > 0$ and period increase.

6) SS22 : $\theta_1 = 0.5$, $\theta_2 = 0.5$, $\nu = 0.1$, $\Delta t = 0.75$, f = step function shows $\delta < 0$ and period increase.

7) SS22 : $\theta_1 = 0.6$, $\theta_2 = 1.0$, $\nu = 0.4$, $\Delta t = 0.75$, f = step function shows $\delta < 0$.

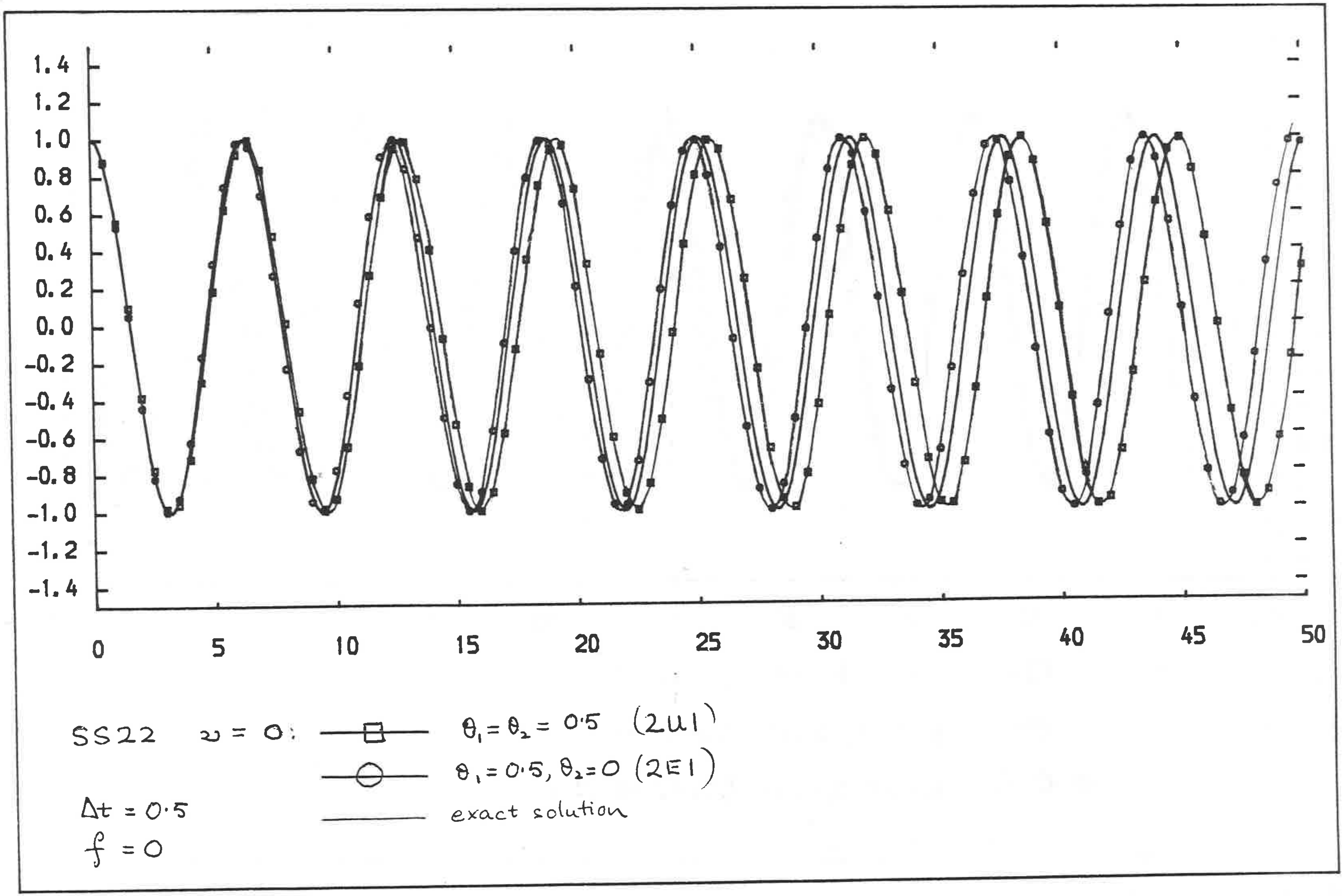
8) SS32 Houbolt $\nu = 0.3$, $\Delta t = 0.5$, $f = 0$ shows $\delta < 0$, period increase.

9) As (8) but with $\nu = 0.5$ shows $\delta < 0$, period decrease.

10) 3E1 : SS32 with $\theta_1 = 0.5$, $\theta_2 = 0.16667$, $\theta_3 = 0$, $\nu = 0.1$ unstable.

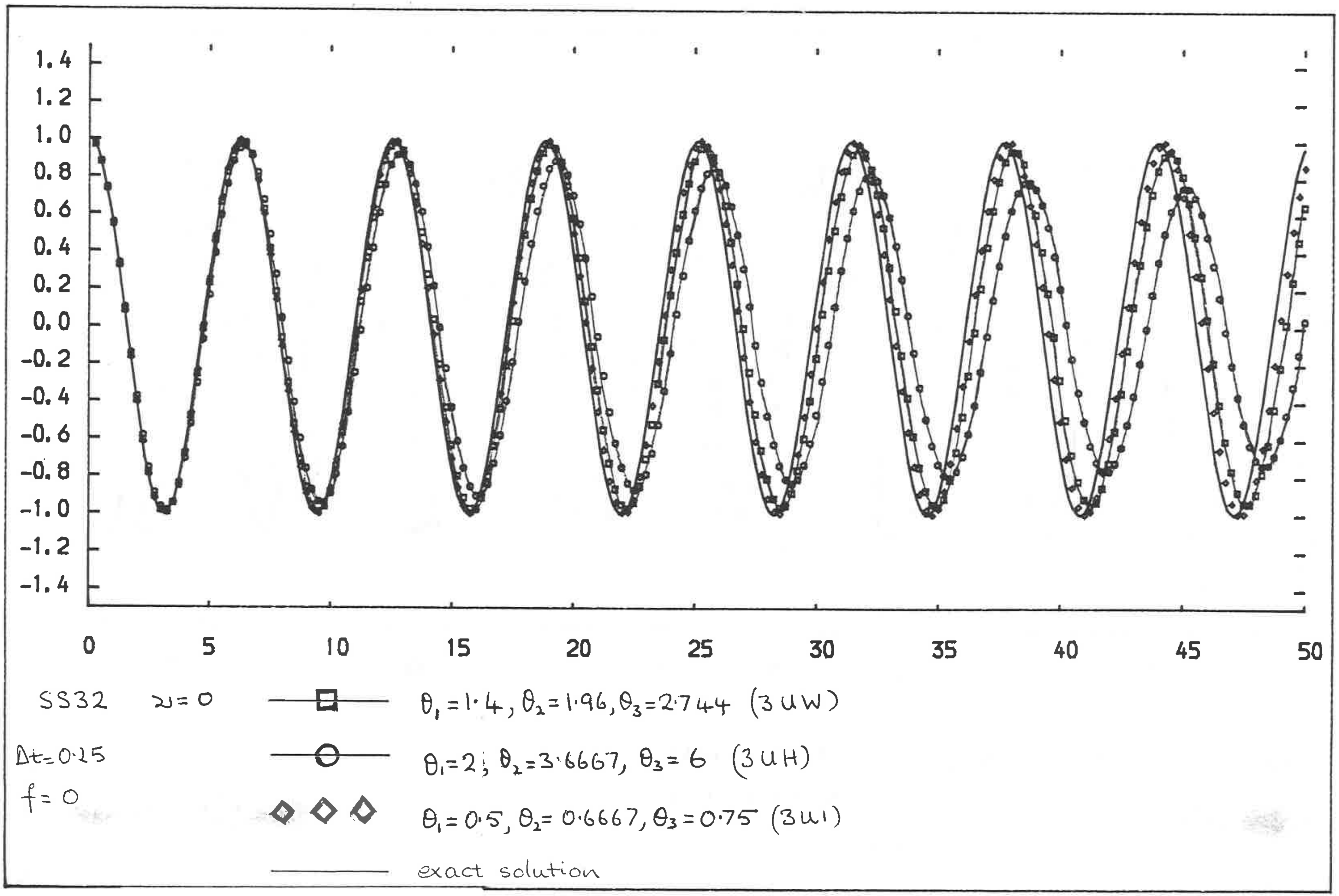
11) 3E3 : SS32 with $\theta_1 = 0.7887$, $\theta_2 = 0.6220$, $\theta_3 = 0$, $\nu = 0.1$, $\Delta t = 1.0$, $f = 0$.

12) SS32 Houbolt with $\nu = 0.75$, $\Delta t = 0.5$, shows $\delta < 0$.



1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
0
5
10
15
20
25
30
35
40
45
50
SS22
ν = 0:
θ1 = θ2 = 0·5 (2U1)
θ1 = 0·5, θ2 = 0 (2E1)
exact solution
Δt = 0·5
f = 0

Figure 1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1.0
0.8
0.6
0.4
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-0.4
-0.6
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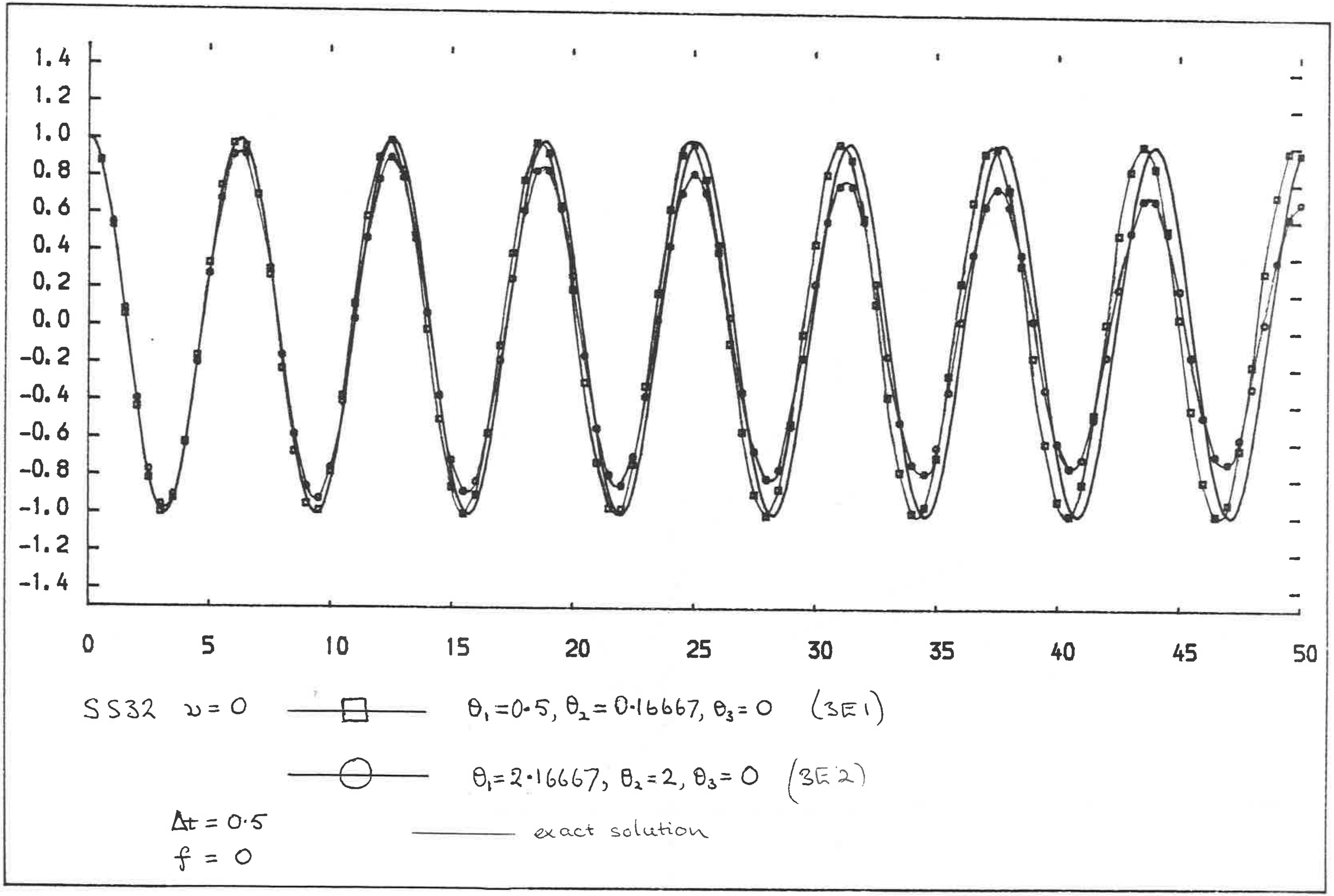
-1.0
-1.2
-1.4
0
5
10
15
20
25
30
35
40
45
50
SS32
ν = 0
Δt = 0.25
f = 0
θ1 = 1.4, θ2 = 1.96, θ3 = 2.744 (3UW)
θ1 = 2, θ2 = 3.6667, θ3 = 6 (3UH)
θ1 = 0.5, θ2 = 0.6667, θ3 = 0.75 (3UI)
exact solution

Figure 2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1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
0
5
10
15
20
25
30
35
40
45
50
SS32 ν = 0
θ1 = 0·5, θ2 = 0·16667, θ3 = 0 (3E1)
θ1 = 2·16667, θ2 = 2, θ3 = 0 (3E2)
exact solution
Δt = 0·5
f = 0

Figure 3

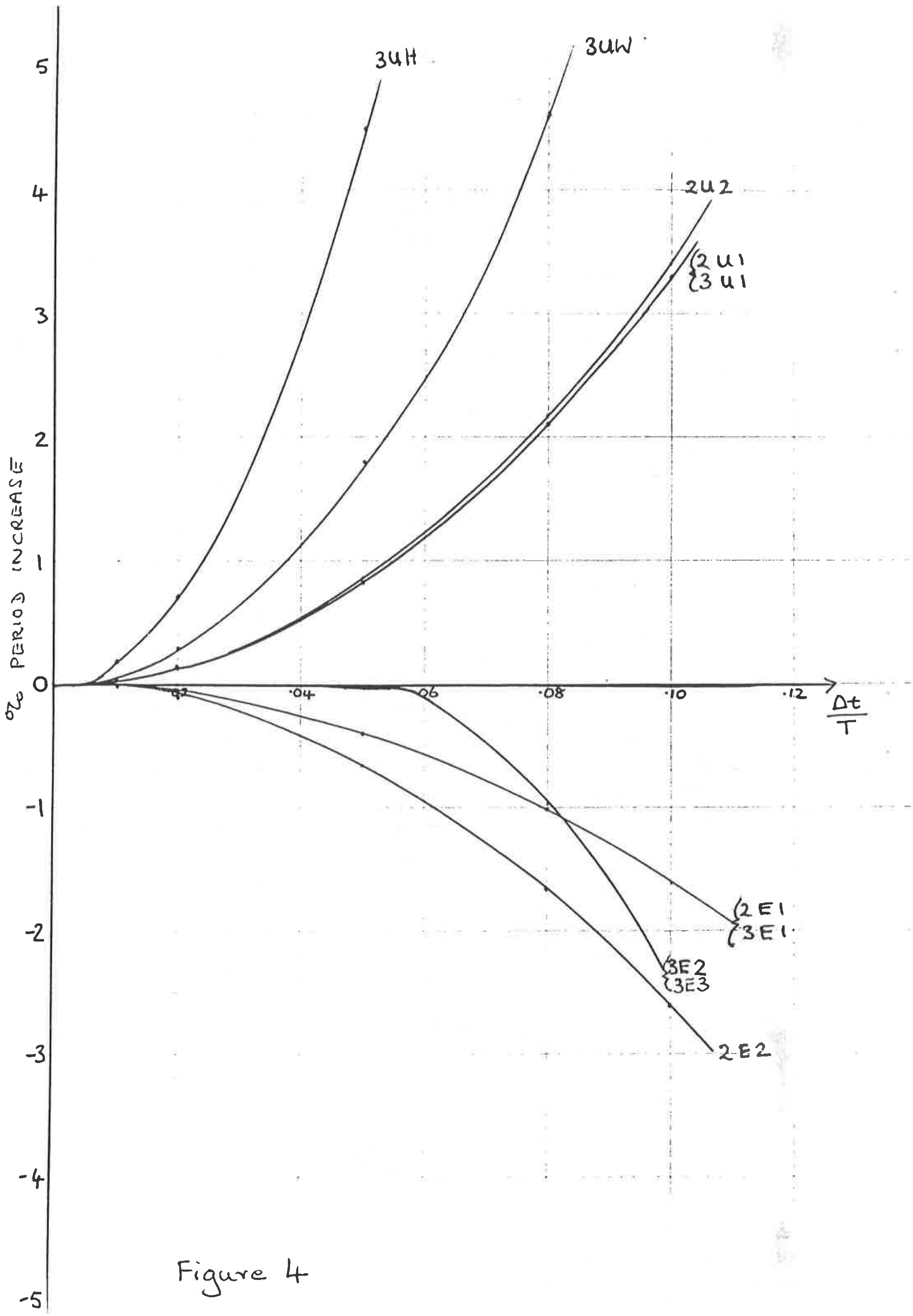
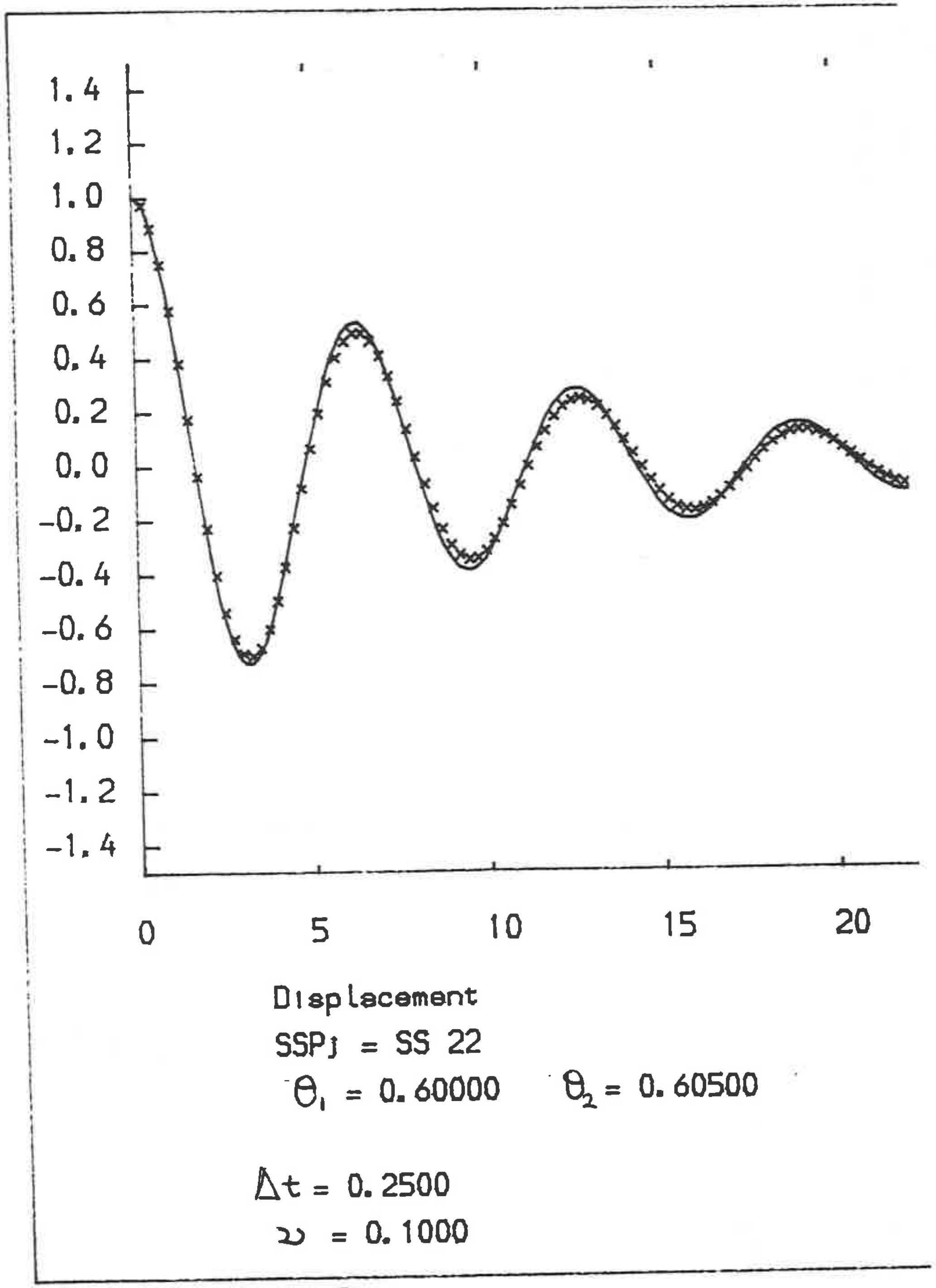

5
4
3
2
1
0
-1
-2
-3
-4
-5
% PERIOD INCREASE
·04
·06
·08
·10
·12
Δt
T
3UH
3UW
2U2
2U1
3U1
2E1
3E1
3E2
3E3
2E2

Figure 4



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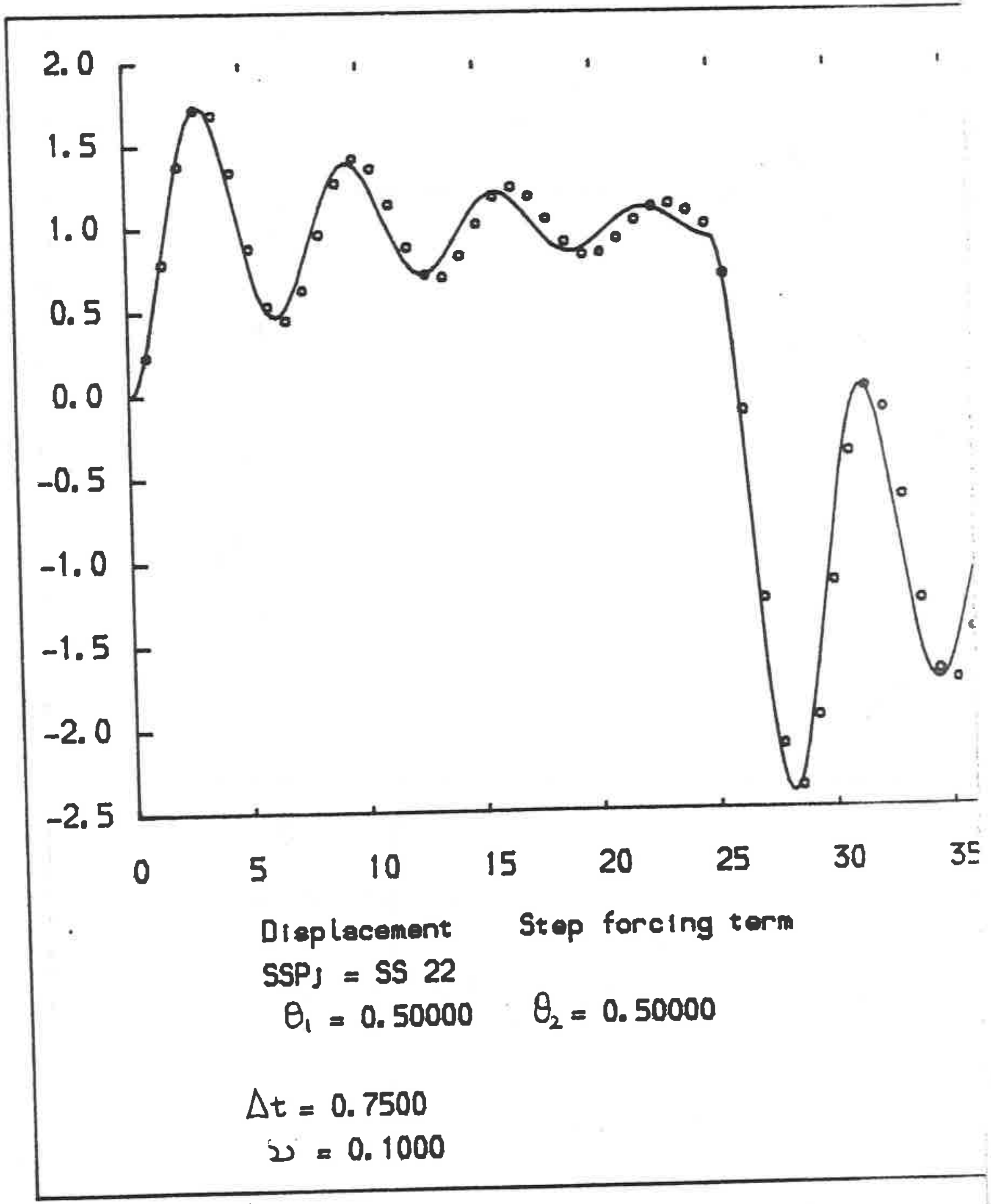

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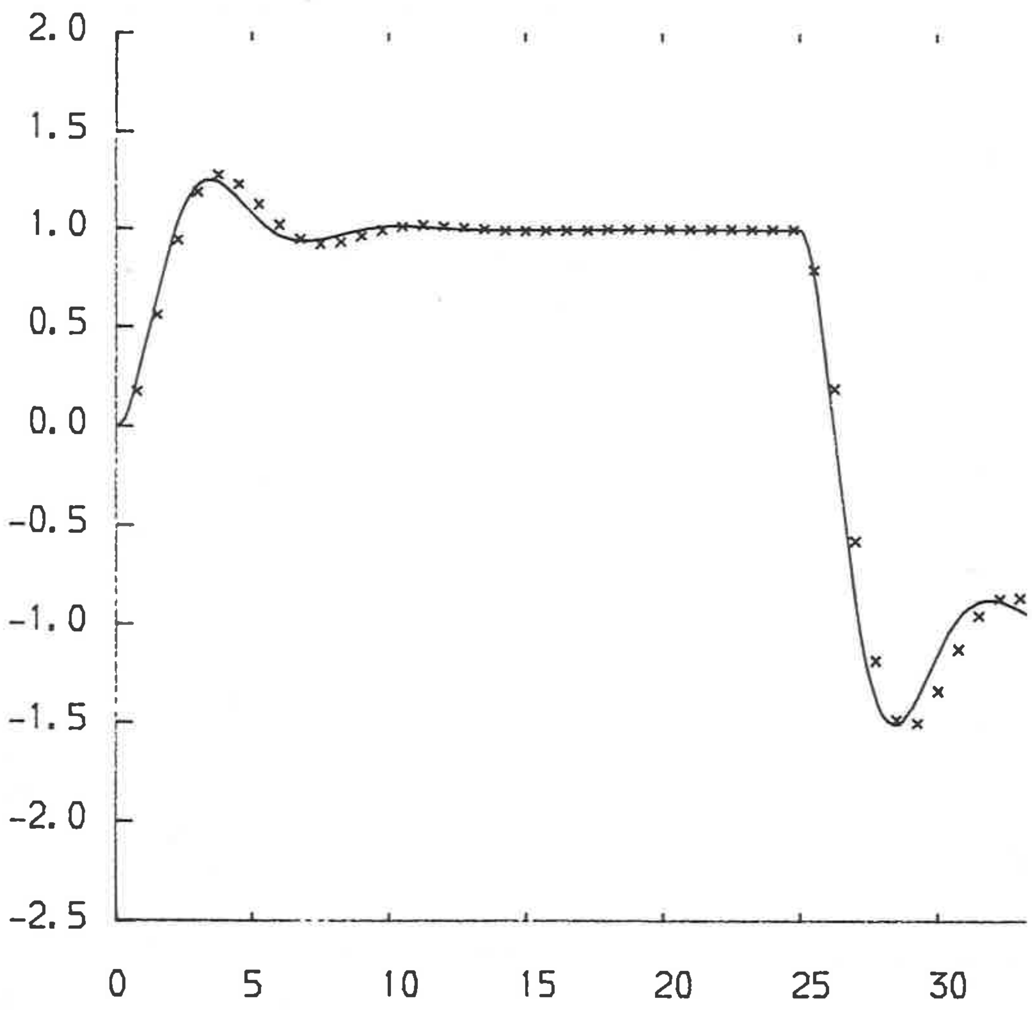
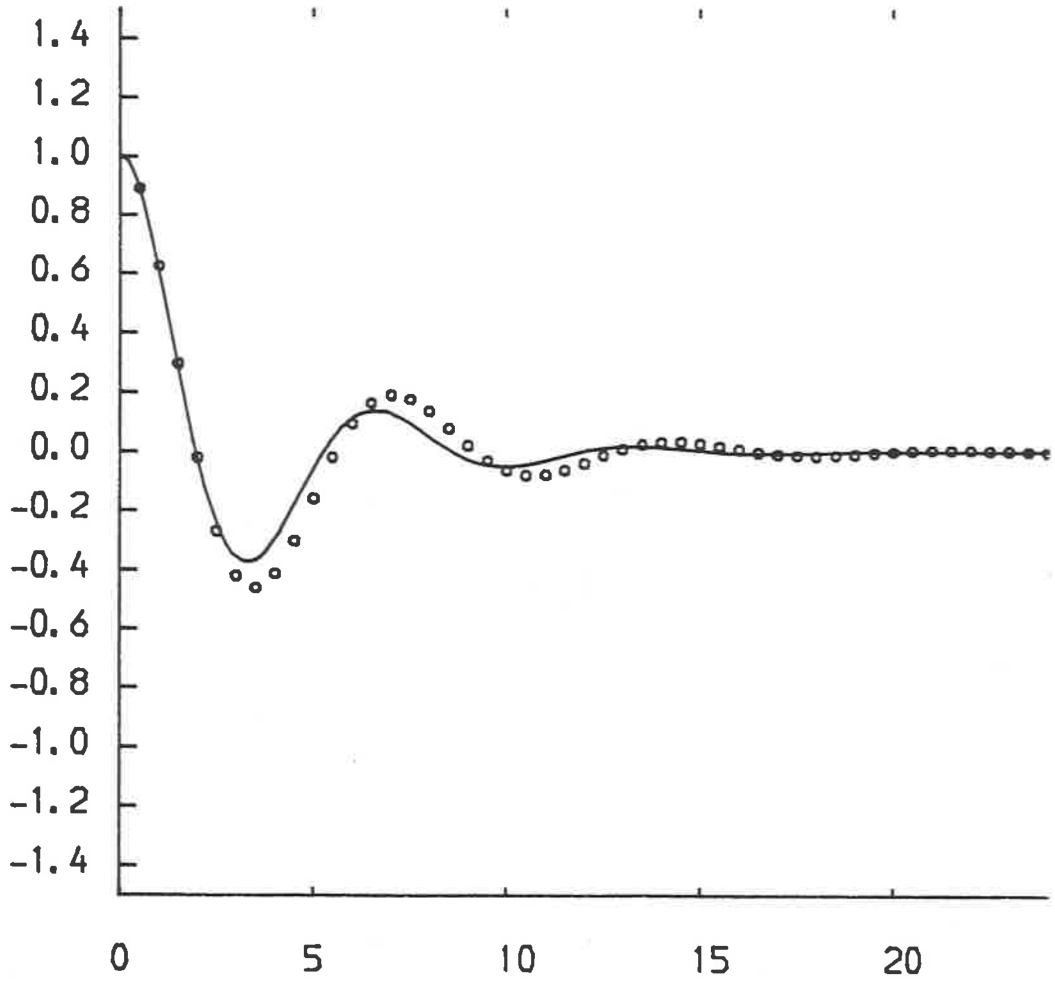


Displacement Step forcing term
SSPJ = SS 22
θ_1 = 0.60000 θ_2 = 1.00000

Δt = 0.7500
ν = 0.4000

Figure 7



Displacement
SSPJ = SS 32
θ_1 = 2.00000 θ_2 = 3.66667 θ_3 = 6.000(

Δt = 0.5000
ν = 0.3000

Figure 8

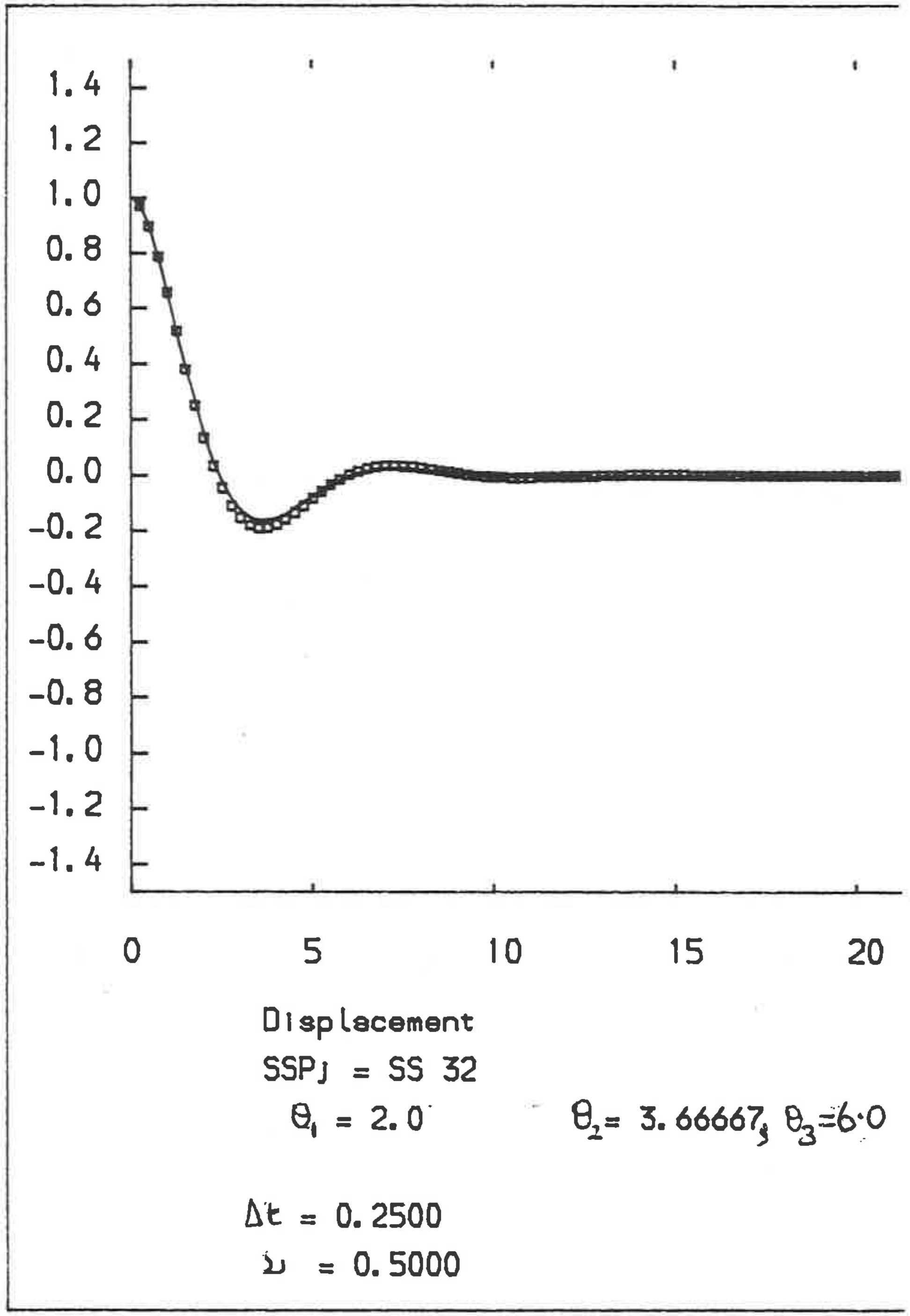


Figure 9

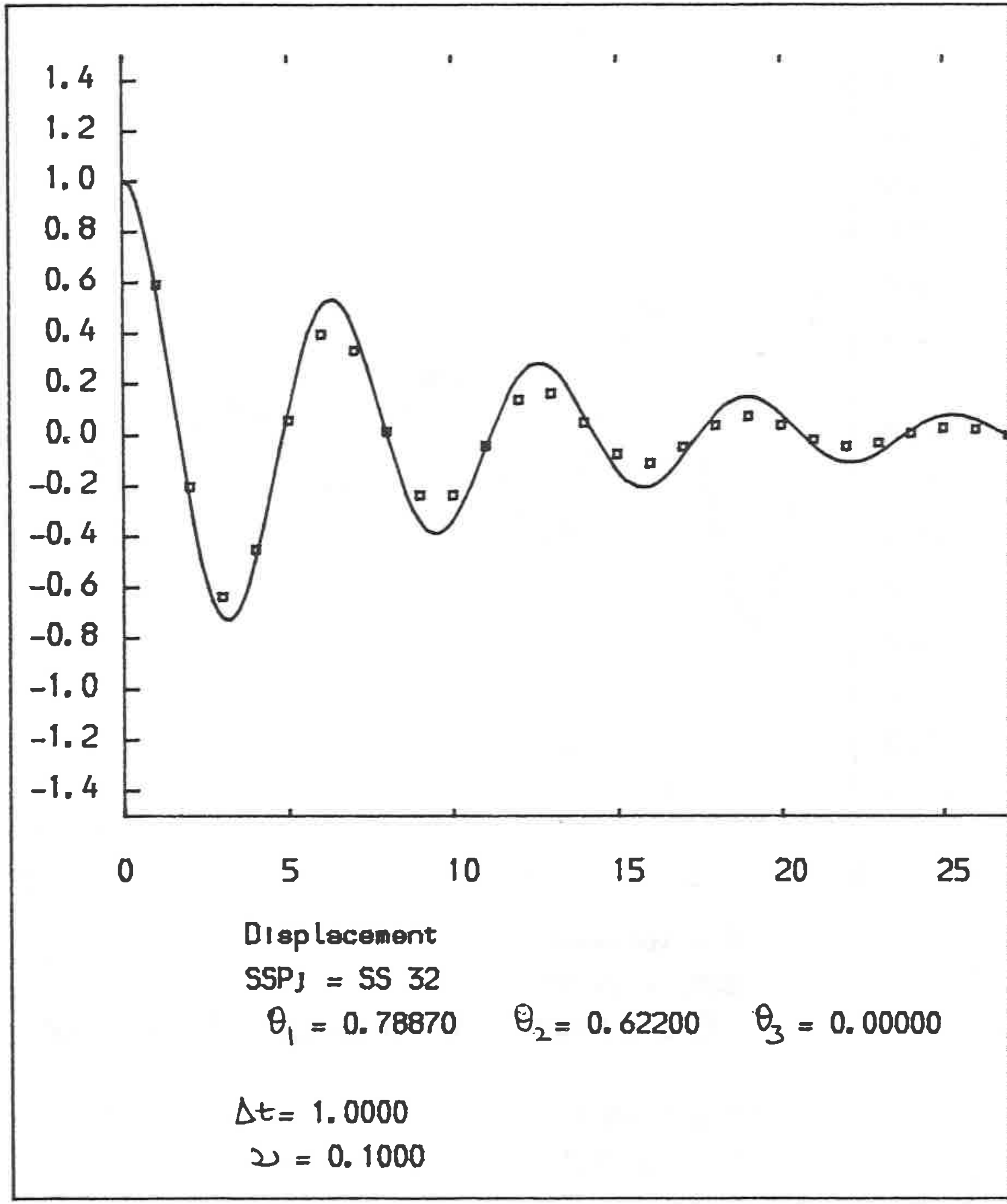
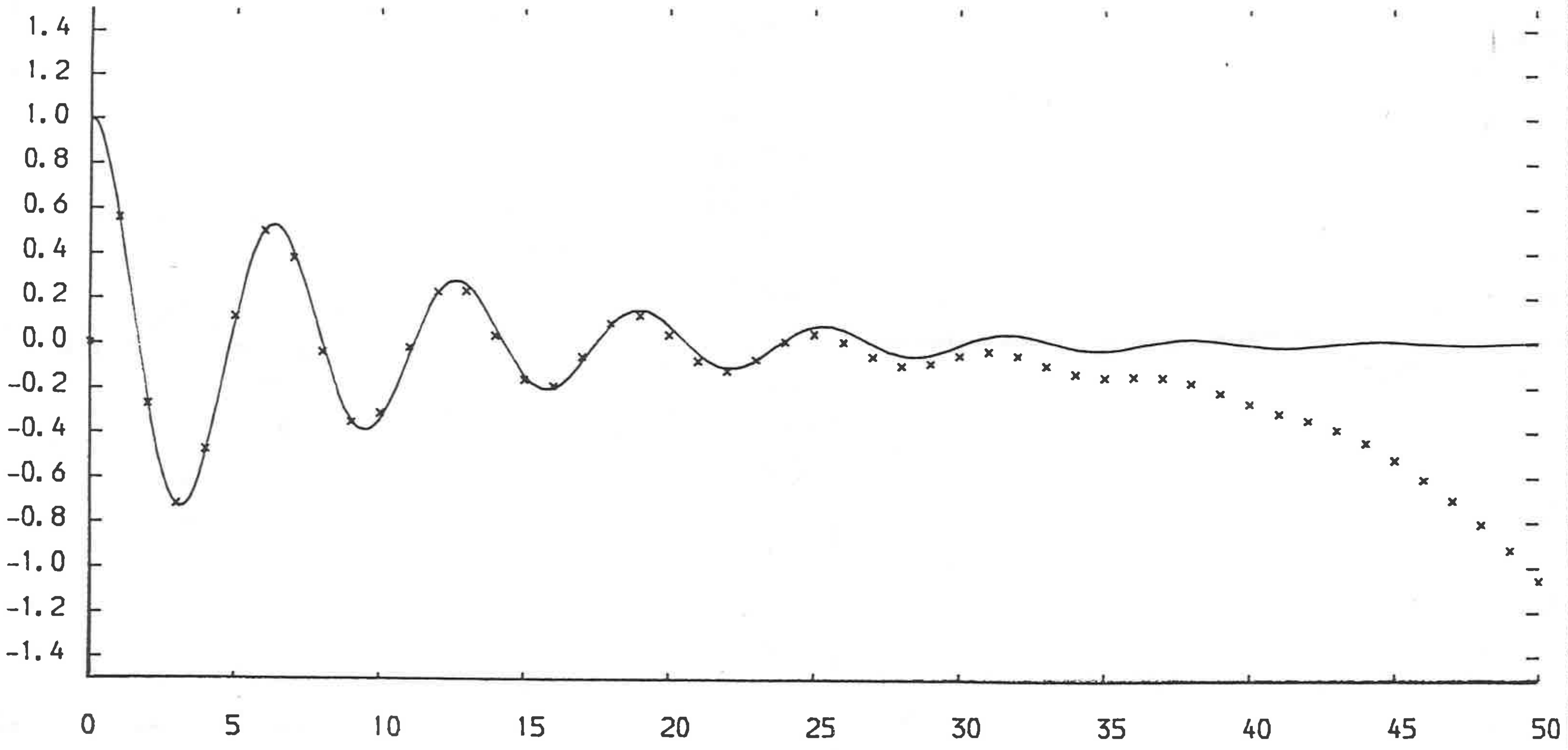


Figure 11



Displacement

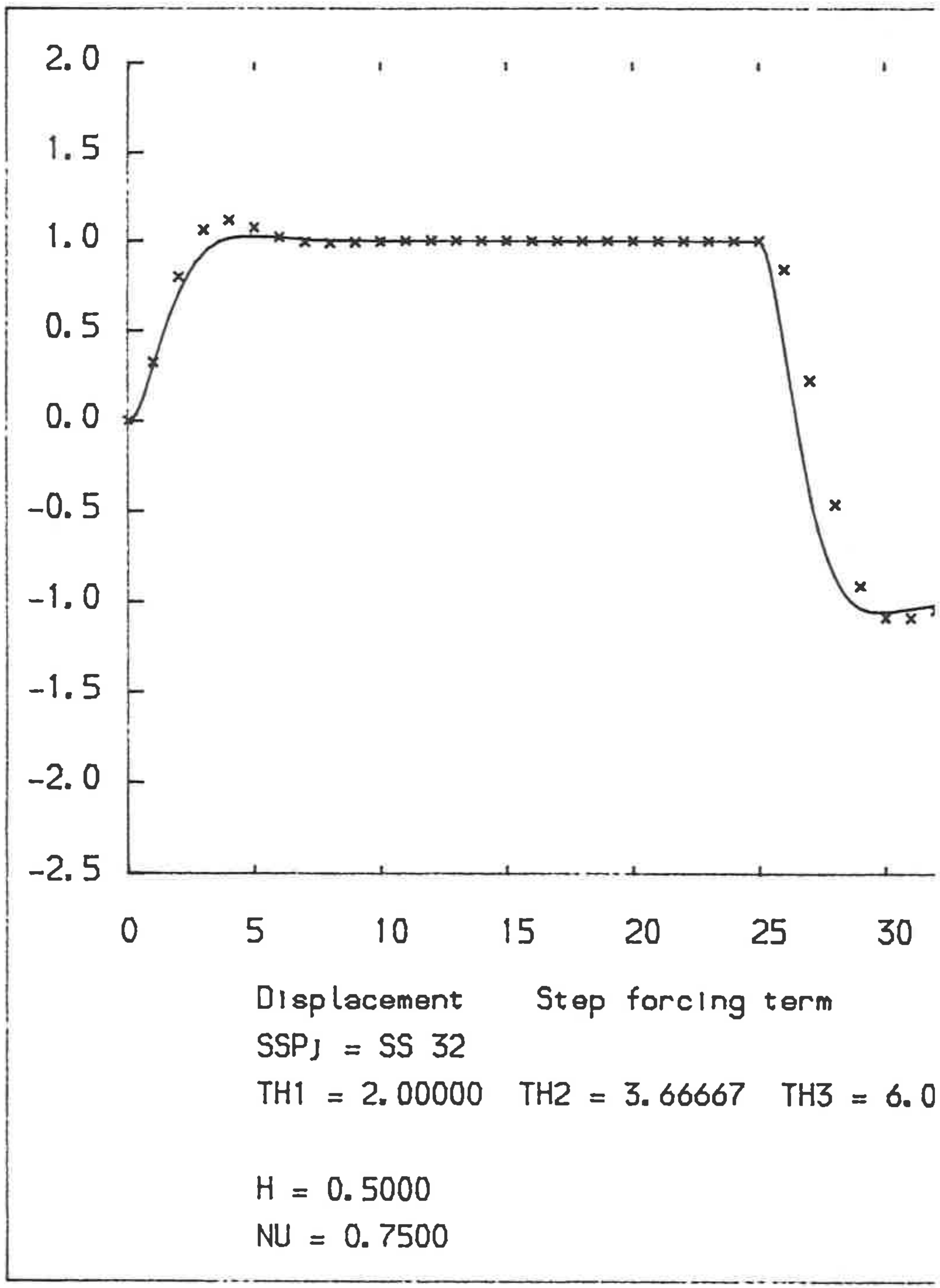
SSPJ = SS 32

θ_1 = 0.50000 θ_2 = 0.16667 θ_3 = 0.00000

Δt = 0.5000

ν = 0.1000

Figure 10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1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
0
5
10
15
20
25
30
Displacement Step forcing term
SSPJ = SS 32
TH1 = 2.00000 TH2 = 3.66667 TH3 = 6.0
H = 0.5000
NU = 0.7500

Figure 12