AN INVESTIGATION OF THE FORM OF AN OPTIMAL CONTROLLER IN A TIDAL POWER GENERATION SCHEME WITH TWO CONTROLS

Z.G.Xu and N.K.Nichols

Numerical Analysis Report 5/91

The work reported here forms part of the research programme of the Reading/Oxford Institute for Computational Fluid Dynamics.

Abstract

The optimal control problem in a tidal power generation scheme with two controls (sluices and turbines operating independently) is formulated and the optimal controllers are derived analytically. It is shown that the optimal controller for sluice operation is bang-bang in all circumstances, whereas the optimal controller for turbine operation is dependent on the form of the prescribed power function. For a linear power function a bang-bang solution results. For a non-linear power function, however, the solution is no longer bang-bang but takes interior values. To obtain the precise form of the solution, an extensive numerical investigation is carried out. Several gradient based optimisation algorithms are employed in the investigation and the behaviour of the algorithms is examined in detail. The results obtained confirm the analytical work and are in support of the findings made in a previous investigation where only one control was used.

Contents

List of Figures

List of Tables

1 Introduction

In a previous paper [1], we reported the results of an investigation into the form of an optimal controller in a tidal power generation scheme. There were two aspects to the investigation: analytical and numerical. The main emphasis was on the numerical solution procedure, especially on how various optimisation algorithms function under different circumstances. Three gradient based optimisation algorithms were used. They are the conditional gradient algorithm (CGA), the projected gradient algorithm (PGA) and the revised conditional gradient algorithm (NCGA) respectively. It was established basically by the investigation that the behaviour of the algorithms varies with the form of the power function. For linear models, the conditional gradient algorithm is the most effective both in terms of rate of convergence and smoothness of the optimal control function obtained. In contrast, for the non-linear model, the projected gradient algorithm is the most effective. As regards the revised conditional gradient algorithm, a quadratic step length rule is used which is supposed to maximise the functional in the step direction at each new iteration. Unfortunately this was not supported by the investigation.

The investigation carried out in [1] was extensive and systematic, with different choices of the coefficients in the power function and different forms of constraint on the turbine flux. However, the scope of the investigation was limited in the sense that only one control was employed, i.e. the control may be used for sluice operation as well as for turbine operation. As shown in [1], some numerical difficulty was encountered in the ebb generation only scheme, when the sluices are required to open the moment the turbines are shut down. Whether this switch point can be predicted was found to be very much affected by the initial guess of the optimal control function. In reality, it is quite unlikely that only one control is employed in a practical tidal power project, as the range of the operation would be too limited.

With this view in mind, we seek to extend our investigation to the tidal power generation scheme with two independent controls for turbines and sluices. As in [1], the estuary is treated as a flat basin. Firstly the optimal control problem for a tidal power generation scheme with two controls is formulated and the form of the optimal controllers is derived. Then the numerical solution of the problems is sought, using two optimisation algorithms, viz. the conditional and the projected gradient algorithm. In the process the behaviour of the algorithms is compared.

2 The Mathematical Model

2.1 The Equation of Flow

As in [1], the estuary is treated as a flat basin and hence the flow across the tidal barrage can be described by an ordinary differential equation (ODE). The exact form of the equation is

$$\dot{\eta} = -\frac{1}{A}[\alpha_S X_S(h) + \alpha_T X_T(h)] \tag{1}$$

where

$\eta(t)$ is the water surface elevation above the datum in the estuary,
$h = \eta - f(t)$ is the head difference across the tidal barrage,
f(t) is the tidal elevation above the datum,
$A = A(\eta)$ is the flat basin surface area at elevation η,
α_S and α_T are the sluice control and the turbine control respectively. They satisfy the inequalities

$$0 \leq \alpha_S \leq 1 \tag{2}$$

$$0 \leq \alpha_T \leq 1 \tag{3}$$

and X_S, X_T are used to denote the fluxes through sluices and turbines respectively. They are further defined as

$$X_S(h) = \begin{cases} -Q_{S1}(h) & \text{if } h \leq 0 \\ Q_{S2}(h) & \text{otherwise} \end{cases}$$

$$X_T(h) = \begin{cases} -Q_{T1}(h) & \text{if } h \leq 0 \\ Q_{T2}(h) & \text{otherwise} \end{cases}$$

where

$Q_{S1}(h)$ is the maximum sluice flow into the estuary for head h,
$Q_{S2}(h)$ is the maximum sluice flow out of the estuary for head h,
$Q_{T1}(h)$ is the maximum turbine flow into the estuary for head h,
$Q_{T2}(h)$ is the maximum turbine flow out of the estuary for head h.

Over short intervals of time the tides are approximately periodic, hence $f(t + T) = f(t)$, where T is the tidal period. This requires that η is also periodic satisfying

$$\eta(0) = \eta(T) \tag{4}$$

2.2 The Optimal Control Problem

The optimal control problem for a tidal power generation scheme with two controls is to determine a sluice control function α_S and a turbine control function α_T which maximise the power functional

$$E = \int_o^T e(X_T(h)\alpha_T, h)dt \tag{5}$$

subject to the constraints expressed by Eqns.(1)-(4).

The integrand in Eqn.(5) is the instantaneous power function, depending upon turbine flow and head difference across the barrage. As will be shown below, the form of the optimal controller is to a great extent determined by whether it contains non-linear terms or not.

3 The Form of the Optimal Controllers

3.1 Theorem

In order to solve the optimal control problem (1)-(5), we apply Pontryagin's Maximum Principle [2]. It states that a necessary condition for an admissible control vector (α_S, α_T) and its response $\eta(t)$ to be optimal is the existence of an adjoint $\lambda(t)$ satisfying

$$\dot{\lambda} = -\frac{\partial H}{\partial \eta} \tag{6}$$

$$\lambda(0) = \lambda(T) \tag{7}$$

and such that

$$H = e(X_T(h)\alpha_T, h) - \frac{\lambda}{A}[\alpha_S X_S(h) + \alpha_T X_T(h)] \tag{8}$$

is maximised with respect to the controls (α_S, α_T). H is known as the Hamiltonian.

In what follows, the form of the optimal controllers is determined analytically for various choices of the power function e(.,.). It is shown that the optimal control for sluices is bang-bang under any circumstances, while the optimal control for turbines is dependent on whether the power function contains non-linear terms or not. For the sake of clarity the ebb generation only scheme is dealt with. However it is not difficult to incorporate two-way generation schemes in a similar fashion.

3.2 A Linear Power Function

Here the power function is assumed to take the following simple form:

$$e = \begin{cases} 0 & \text{if } h \leq 0 \\ Q_{T2}(h)\alpha_T h & \text{otherwise.} \end{cases} \tag{9}$$

The Hamiltonian associated with this problem can then be written as

$$H = \begin{cases} \frac{\lambda}{A}(\alpha_S Q_{S1} + \alpha_T Q_{T1}) & \text{if } h \leq 0 \\ Q_{T2}\alpha_T h - \frac{\lambda}{A}(\alpha_S Q_{S2} + \alpha_T Q_{T2}) & \text{otherwise} \end{cases} \tag{10}$$

and hence the adjoint equation is

$$\dot{\lambda} = \begin{cases} -\frac{\lambda}{A}[\alpha_S Q'_{S1} + \alpha_T Q'_{T1} - \frac{A'}{A}(\alpha_S Q_{S1} + \alpha_T Q_{T1})] & \text{if } h \leq 0 \\ -\alpha_T(Q_{T2} + Q'_{T2}h) + \frac{\lambda}{A}[\alpha_S Q'_{S1} + \alpha_T Q'_{T1} - \frac{A'}{A}(\alpha_S Q_{S1} + \alpha_T Q_{T1})] & \text{otherwise} \end{cases} \tag{11}$$

and

$$\lambda(0) = \lambda(T). \tag{12}$$

This equation can be simplied by introducing another adjoint function $\mu = \frac{\lambda}{A}$. In terms of this new variable the adjoint equation becomes

$$A(\eta)\dot{\mu} = \begin{cases} -\mu(\alpha_S Q'_{S1} + \alpha_T Q'_{T1}) & \text{if } h \leq 0 \\ -Q_{T2}\alpha_T + \mu\alpha_S Q'_{S2} - \alpha_T(h-\mu)Q'_{T2} & \text{otherwise} \end{cases} \tag{13}$$

and

$$\mu(0) = \mu(T). \tag{14}$$

By the Maximum Principle, the Hamiltonian is maximised with respect to α_S and α_T by the optimal controller (α_S^*, α_T^*) where

$$\alpha_S^* = \begin{cases} 1 & \text{if } \nabla E_S > 0 \\ 0 & \text{if } \nabla E_S < 0 \end{cases} \tag{15}$$

$$\alpha_T^* = \begin{cases} 1 & \text{if } \nabla E_T > 0 \\ 0 & \text{if } \nabla E_T < 0. \end{cases} \tag{16}$$

Here ∇E_S and ∇E_T are the functional gradients with respect to α_S and α_T, and can be calculated as follows

$$\nabla E_S = \frac{\partial H}{\partial \alpha_S} = \begin{cases} \mu Q_{S1} & \text{if } h \leq 0 \\ \mu Q_{S2} & \text{otherwise} \end{cases} \tag{17}$$

$$\nabla E_T = \frac{\partial H}{\partial \alpha_T} = \begin{cases} \mu Q_{T1} & \text{if } h \leq 0 \\ Q_{T2}(h-\mu) & \text{otherwise.} \end{cases} \tag{18}$$

3.3 A Non-linear Power Function

Here the power function is assumed to take the following form:

$$e(X_T(h)\alpha_T, h) = \begin{cases} 0 & \text{if } h \leq 0 \\ Q_{T2}\alpha_T(h - cQ_{T2}^2\alpha_T^2) & \text{otherwise} \end{cases} \tag{19}$$

where c is some constant.

The Hamiltonian associated with this problem can then be written as

$$H = \begin{cases} \frac{\lambda}{A}(\alpha_S Q_{S1} + \alpha_T Q_{T1}) & \text{if } h \leq 0 \\ Q_{T2}\alpha_T(h - cQ_{T2}^2\alpha_T^2) - \frac{\lambda}{A}(\alpha_S Q_{S2} + \alpha_T Q_{T2}) & \text{otherwise} \end{cases} \tag{20}$$

and hence the adjoint equation, in terms of $\mu = \lambda/A$ is

$$A(\eta)\dot{\mu} = \begin{cases} -\mu(\alpha_S Q_{S1}^{'} + \alpha_T Q_{T1}^{'}) & \text{if } h \leq 0 \\ -Q_{T2}\alpha_T + \mu\alpha_S Q_{S2}^{'} - \alpha_T Q_{T2}^{'}(h - \mu - 3c\alpha_T^2 Q_{T2}^2) & \text{otherwise} \end{cases} \tag{21}$$

and

$$\mu(0) = \mu(T). \tag{22}$$

By the Maximum Principle, the Hamiltonian is maximised with respect to α_S and α_T by the optimal controller (α_S^*, α_T^*) where

$$\alpha_S^* = \begin{cases} 1 & \text{if } \nabla E_S > 0 \\ 0 & \text{if } \nabla E_S < 0 \end{cases} \tag{23}$$

$$\alpha_T^* = \begin{cases} 1 & \text{if } h < 0, \mu(t) > 0 \\ 0 & \text{if } h < 0, \mu(t) < 0 \\ 0 & \text{if } 0 \leq h < \mu(t) \\ \left[\frac{h-\mu(t)}{3cQ_{T2}^2}\right]^{\frac{1}{2}} & \text{if } h - \mu(t) \leq 3cQ_{T2}^2 \\ 1 & \text{if } h - \mu(t) > 3cQ_{T2}^2. \end{cases} \tag{24}$$

Clearly the optimal solution for the turbine control is no longer bang-bang. Instead it contains an internal non-extreme optimal control as part of the solution.

The functional gradients ∇E_S and ∇E_T can be calculated as:

$$\nabla E_S = \begin{cases} \mu Q_{S1} & \text{if } h \leq 0 \\ \mu Q_{S2} & \text{otherwise} \end{cases} \tag{25}$$

$$\nabla E_T = \begin{cases} \mu Q_{T1} & \text{if } h \leq 0 \\ Q_{T2}(h - \mu) - 3c\alpha_T^2 Q_{T2}^3 & \text{otherwise.} \end{cases} \tag{26}$$

4 Numerical Solution Procedures

The problems as formulated in the preceding sections are state constrained optimal control problems. To solve these optimal control problems, a numerical solution procedure is used. It determines the optimal admissible controls $\alpha_S(t)$ and $\alpha_T(t)$ with corresponding response $\eta(t)$ and $\mu(t)$ satisfying the state and adjoint equations. The procedure consists of a constrained optimisation algorithm for iteratively determining the optimal control functions, together with a finite difference scheme for solving the state and adjoint equations.

Two main optimisation algorithms are used in the current work and are known as the conditional gradient algorithm (CGA) and the projected gradient algorithm (PGA). The structure of the two algorithms is illustrated in the flow charts given in Fig.1 and Fig.2 respectively. These two algorithms have been described in detail in [1] for the optimal control problem with only one control. In a two control situation, some modifications of the basic form of the algorithms are necessary. This is particularly true as far as the projected gradient algorithm is concerned. The main modification carried out for this algorithm is the normalisation of the functional gradients with regard to the search directions. As shown in the flow chart for this algorithm, the normalisation takes the form of dividing each functional gradient by a so-called normalisation factor, which is calculated as the square root of the sum of the squares of the functional gradients. It is worth pointing out that this normalisation factor at each time step depends directly on the individual gradients and hence is also a function of time. This differs from an other normalisation scheme which employs only a single fixed value for each iteration. It is believed the former scheme has the advantage over the latter in that, firstly local conditions can be taken into account and secondly the search direction is not critically affected by the accuracy of the functional gradients. Both algorithms employ a simple but efficient step length rule which halves the current step length if the current control functions fail to improve the functional value over the previous iteration. The iteration is terminated when the measure $M(\underline{\alpha}^K)$ is less than a given tolerance, where $M(\underline{\alpha})$ is given by

$$M(\underline{\alpha}) = \max_{\underline{\tilde{\alpha}} \in \underline{U}} < \nabla E(\underline{\alpha}), \underline{\tilde{\alpha}} - \underline{\alpha} > \tag{27}$$

where $\underline{U}$ is the set of admissible controls. In the conditional gradient algorithm $\underline{\tilde{\alpha}}$ is used as a new control along the search direction. In the projected gradient algorithm, however, it is used solely for calculating the first variation for the convergence criterion.

The numerical solution to the state and adjoint equations is achieved by a finite difference scheme, with the state equation being integrated forward in time from the initial condition $\eta(0)$ and the adjoint equation then being integrated backward in time from the final condition $\mu(T)$. Details of the finite difference schemes are given in Appendix I.

5 Results and Discussion

5.1 Test Case

The data used in the current study, including the tidal period and the estuary geometry, correspond to the Severn estuary and are listed as follows:

$$T = 4.32 \times 10^4 \text{ s}$$

$$A = 3.\dot{3}\dot{3} \times 10^3 \ m^2$$

It is further assumed that the forcing function imposed on the seaward side of the barrage is given by $f(t) = cos2\pi t$ over a complete tidal period $[0, T]$, and that the sluice flow functions Q_{S1}, Q_{S2} and the turbine flow functions Q_{T1}, Q_{T2} take the following simple form

$$-Q_{S1} = Q_{S2} = h$$

$$-Q_{T1} = Q_{T2} = h.$$

In the non-linear power function the constant c is taken as unity. The purpose of this choice is that a high degree of non-linearity can be introduced into the problems so that the algorithms can be vigorously tested. It should be noted that throughout the whole computation 500 time steps are used.

Both ebb and two-way generation schemes are computed, with particular emphasis on ebb generation with a non-linear power function. This is because numerical difficulties were experienced in a similar situation with one control (see [1]). In order to test the integrity of the numerical algorithms, the computation is carried out for a wide range of initial choices of sluice and turbine controls. Table.1 illustrates the computational results for the ebb scheme with the non-linear power function. The results for the ebb scheme with the linear power function and those for the two-way generation scheme are given in Tables 2-4. Some typical results are also shown graphically in Figs.3-12. All these results are obtained with a 1.0% tolerance.

A thorough examination of the results shows the following:

As far as the linear power function is concerned, the two algorithms perform well and a fast rate of convergence is achieved. By comparison, however, CGA yet again proves to be the most efficient both in terms of convergence rate and quality of the optimal control functions computed. This is an expected result from the work in [1]. With PGA the optimal control functions are not very satisfactory. Therefore it is obvious that CGA is recommended for this type of optimal control problem.

In the case where the power function is non-linear, it can be seen that PGA has a very fast rate of convergence, and typically only 14 iterations are needed to achieve convergence for 1.0% tolerance. The convergence is achieved for all initial values of turbine and sluice controls. With this tolerance the optimal control functions obtained are in general quite satisfactory with very smooth curves, although some oscillations are observed in the turbine control function for certain initial values of turbine and sluice control. Nevertheless it must be pointed out that the oscillation can easily be dealt with, as it is caused by too large a convergence tolerance. If a smaller tolerance, say 0.1% is employed, the oscillation disappears, as shown in Fig.21. More importantly, there is no substantial increase in the number of iterations when this smaller tolerance is used. By comparison CGA has a poor performance, with an extremely slow rate of convergence and low quality optimal control functions. For this reason the results for CGA are obtained only for two initial values 1.0 and 0.1. Consequently, for a non-linear power function, PGA is highly recommended.

It was mentioned briefly in the last section that various normalisation schemes are possible with the projected gradient algorithm. Apart from the one adopted above, it was proposed in a previous investigation by Birkett that normalisation factors should be taken as the maximum functional gradients for sluice and turbine controls respectively at each optimisation iteration. To be more specific, the two controls at the new iteration are computed from the following:

$$\alpha_S^{K+1} = \alpha_S^K + S\nabla_S E^K / \max_n |\nabla_S E^K(n)|$$

$$\alpha_T^{K+1} = \alpha_T^K + S\nabla_T E^K / \max_n |\nabla_T E^K(n)|$$

where n is time step number and

$$n = 0, 1, ..., N-1$$

It is clear that the normalisation factors as defined above are single values at each optimisation iteration and have the advantage of being simple to implement. However this scheme fails to take into account the local conditions at each time step and hence slow convergence is expected. This has been confirmed by the results obtained by this scheme, which are given in Tables 1-4 under the heading NPGA and in Figs.20-21. It can be seen that apart from the slow convergence associated with this algorithm, the predicted optimal control control functions are not satisfactory either. Consequently this scheme is not pursued any further.

In the process of testing the PGA algorithm for the ebb-generation scheme with a non-linear power function, a new algorithm is developed. In view of the extensive work carried out here and in [1], it can be said with confidence that CGA is best suited for bang-bang type control problems while PGA is best suited for those containing interior values. Therefore it is reasonable to believe that an ideal algorithm for the type of problem under consideration is one which combines CGA for sluice control and PGA for turbine control. This algorithm is named CPGA and some tests are carried out. Good results are obtained for a wide range of initial controls, as shown in Tables 1-4 and Figs. 30-40. In comparison with PGA, this algorithm does not show strong dependence on the initial controls and the optimal controls obtained are generally better.

Even though only limited tests have been conducted on CPGA so far, the initial results have shown that it has a great potential in solving optimal control problems where one control is bang-bang and the other contains interior points. Further numerical tests using more practical data are currently under way.

Perhaps one of the most important discoveries from the present work is the overlapping of two controls, when sluices and turbines operate simultaneously. As shown in the various figures, this phenomenon is observed in both ebb and two-way schemes, but has to be interpreted differently. For an ebb scheme it happens in the sluicing stage of the cycle and indicates that the turbines are actually used for sluice operation. Thus the overall sluicing capacity is increased, contributing to more power output. At this point it should be noted that in the figures the turbine controls are assigned negative values when sluicing. The aim is simply to differentiate sluicing and generating operations for turbines. However it should be borne in mind that the turbine control can only vary in [0,1]. In the case of a two-way scheme, the overlapping occurs as soon as the head difference drops to a certain level. At such moment the sluices are switched on while the turbines are still generating. It is believed that this helps to let the water out of basin quickly. In the ebb scheme typically 8% more power can be generated by using turbines for sluicing operation when they are not generating. However this figure should be treated with caution, as it is obtained on the assumption that turbines have same flow characteristics whether they are used for generating or sluicing.

5.2 A Practical Example

It is assumed in the test case that both sluice and turbine fluxes are directly proportional to the head difference across the tidal barrage. However, in practical situations they often take more complex non-linear forms. For this reason the algorithms need to be tested with more realistic data.

Numerical results are obtained for a problem which approximates the Severn estuary, where the non-linearities model both the effects of drying sand banks and the variation of turbine efficiency with head difference. The relevant data and functions for this problem are taken from [4] and can be described as follows:

$$A(\eta) = 4.6 \times 10^8 + 2.6 \times 10^7 \eta$$

$$Q_{T1} = \begin{cases} 290 \times 140(1 - tanh(10(h + 1.7)) & \text{for two-way scheme} \\ 216 \times 140\sqrt{-2gh} & \text{for ebb scheme} \end{cases}$$

$$Q_{T2} = 290 \times 140(1 + tanh(10(h - 1.7))$$

$$Q_{S1} = 216 \times 160\sqrt{-2gh}$$

$$Q_{S2} = 216 \times 160\sqrt{2gh}$$

$$T = 4.46 \times 10^4$$

$$f(t) = F_o cos(2\pi t/T)$$

$$e = \frac{\rho g \times 140}{T}\alpha_T(t) \cdot X_T(h) \cdot h \cdot ef(h)$$

where ef(h) is turbine efficiency and

$$ef(h) = 0.14 + 0.68 tanh(0.7(|h| - 1.7))$$

As the power function is in linear form, bang-bang type optimal controls for sluices and turbines are expected. With this in mind, we only present those results obtained using CGA. Table 5 shows the power outputs obtained at springs with tidal amplitude $F_o = 5.2$ metres, and at neaps with tidal amplitude $F_o = 2.65$ metres, using firstly 100% efficient turbines and then the variable efficiency machines. These results are also illustrated in Figs.20-28. In the computation 800 time steps are used, with the initial controls taken as $\alpha_S^o = \alpha_T^o = 0.1$.

Generally speaking, the results tabulated above confirm what was reported in [4]. However one obvious discrepancy between the two is observed concerning variable efficiency turbines. In [4] the results predicted that for the variable efficiency machines the ebb scheme produces more power than the two-way scheme, an unexpected conclusion. Our computation contradicts this result and it can be seen from the table that the two-way scheme is superior to the ebb scheme in all the cases, with the exception of the variable efficiency machines at springs. This may have been caused by our currently adopting the sluice discharge coefficient for turbines, which tends to overestimate the turbine sluicing capacity.

6 Conclusions

In this report we examine the optimal control problem for a tidal power generation scheme with two controls. By Pontryagin's Maximum Principle, the analytical forms of the optimal controllers are evaluated for linear and non-linear power functions, and for ebb and two-way generation schemes. It is shown that with a linear power function both optimal sluice and turbine controls are bang-bang in nature. On the other hand, with a non-linear power function the optimal turbine control takes some interior values. These are confirmed by numerical work.

Two iterative optimisation algorithms are used to solve the optimal control problems numerically. The results generally confirm those conclusions made in [1]; that is to say, if the power function is in linear form, the conditional gradient algorithm is the most efficient in terms of rate of convergence and smoothness of the control functions obtained; if the power function is in non-linear form, the projected gradient algorithm is the most efficient.

Acknowledgements

The research reported here has been conducted with financial support from National Power (NPTEC) and this support is gratefully acknowledged.

Appendix I

The finite difference approximation to the state equation Eqn.(1) is given by

$$\eta^{n+1} - \eta^n = \frac{\Delta t}{2}\left\{\frac{\alpha_S^n X_S(h^n) + \alpha_T^n X_T(h^n)}{A(\eta^n)} + \frac{\alpha_S^{n+1} X_S(h^{n+1}) + \alpha_T^{n+1} X_T(h^{n+1})}{A(\eta^{n+1})}\right\} \tag{28}$$

$$n = 0, 1, ..., N-1$$

where N is the total number of time steps, $\Delta t = T/N$ is the time step and n the point number on the finite difference grid. All the quantities with superscript n are evaluated at $t = n\Delta t$ and those with superscript n+1 are evaluated at $t = (n+1)\Delta t$. At each time step the non-linear algebraic equation Eq.(28) is solved for η^{n+1} using a simple iteration procedure. It starts with an initial guess for η^{n+1}. Based upon this initial guess, all the quantities on the r.h.s. of Eq.(28) can be evaluated. Solving Eq.(28), a new η^{n+1} can then be obtained. If it is equal to the guessed value within a specified tolerance, it is accepted as required η^{n+1} and the iteration is terminated. If not, then this new η^{n+1} is taken as a new initial guess and the iteration is repeated.

The adjoint equation is similarly approximated by the finite difference scheme

$$\mu^n - \mu^{n-1} = \frac{\Delta t}{2}\left\{\frac{\alpha_S^{n-1} X_S'(h^{n-1}) + \alpha_T^{n-1} X_T'(h^{n-1})}{A(\eta^{n-1})}\mu^{n-1} + \frac{\alpha_S^n X_S'(h^n) + \alpha_T^n X_T'(h^n)}{A(\eta^n)}\mu^n\right\}$$

$$+ \frac{\Delta t}{2}\left\{\frac{1}{A(\eta^{n-1})}\frac{\partial e}{\partial \eta}^{n-1} + \frac{1}{A(\eta^n)}\frac{\partial e}{\partial \eta}^{n}\right\} \tag{29}$$

$$n = N, N-1, ..., 1$$

References

[1] Andrews, T.P., Nichols, N.K. and Xu, Z. The Form of Optimal Controllers for Tidal Power Generation Schemes, Numerical Analysis Tech. Rpt. NA 8/90, Department of Mathematics, University of Reading, 1990.

[2] Pontryagin, L.S., Boltranskii, V.G. and Gamkrelidze, R.V., The Mathematical Theory of Optimal Processes, Interscience, 1962.

[3] Birkett, N.R.C. and Nichols, N.K., Optimal Control Problems in Tidal Power Generation, Numerical Analysis Tech. Rpt. NA 8/83, Department of Mathematics, University of Reading, 1983.

[4] Birkett, N.R.C., Nonlinear Optimal Control Problems in Tidal Power Calculations, Numerical Analysis Tech. Rpt. NA 2/85, Department of Mathematics, University of Reading, 1985.

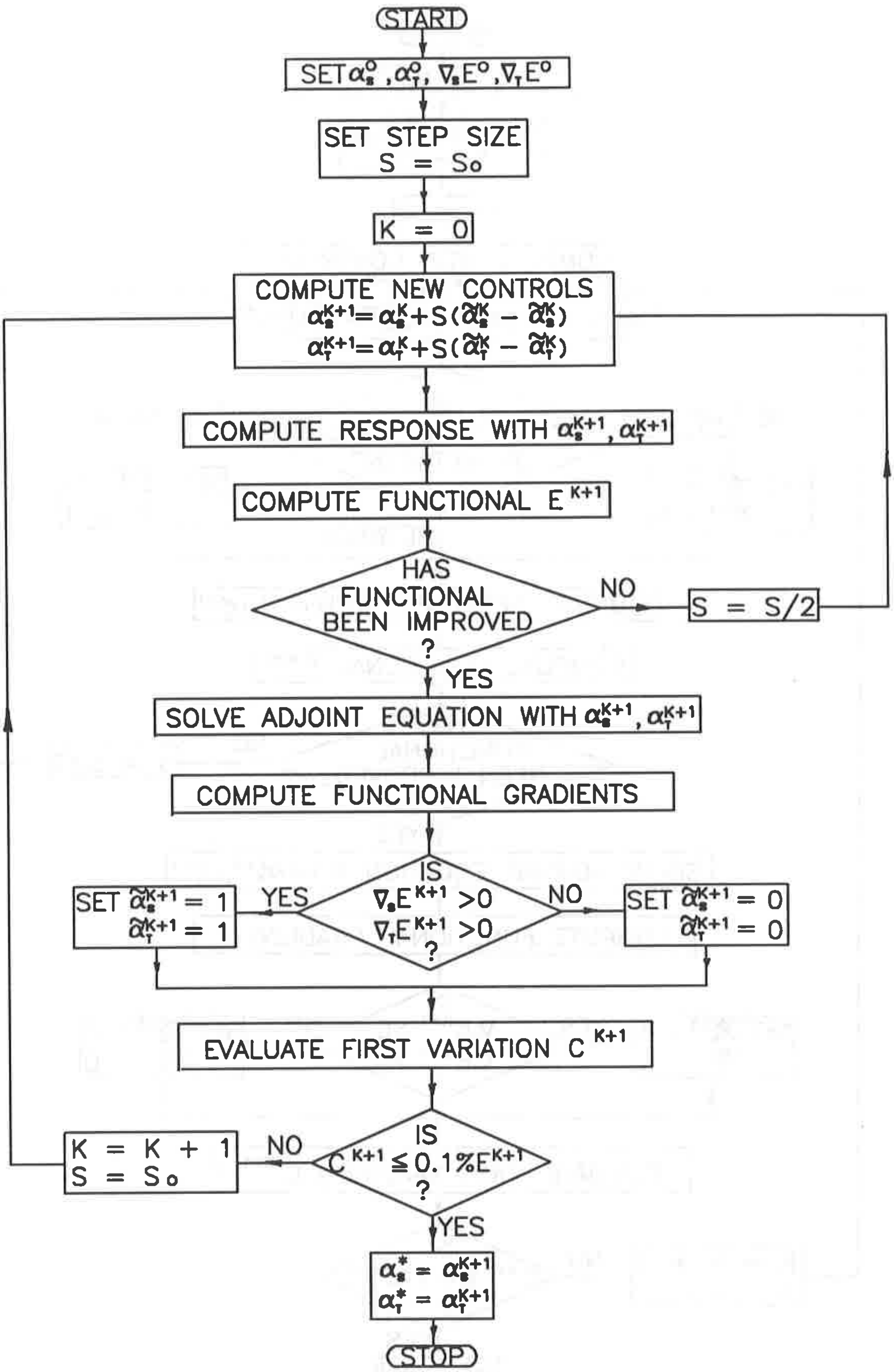

FIG.1 FLOW CHART — CONDITIONAL GRADIENT ALGORITHM (CGA)

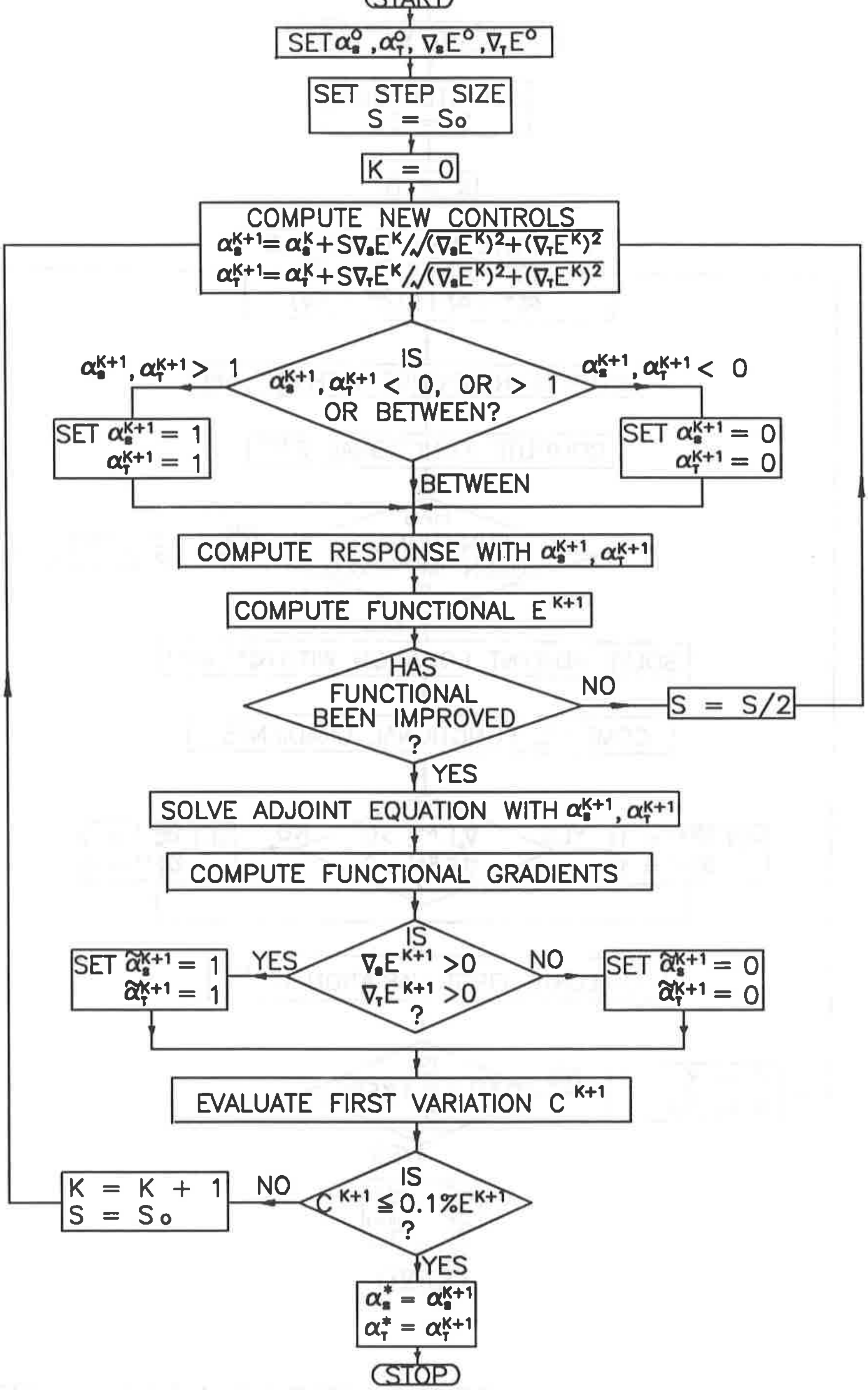

FIG.2 FLOW CHART — PROJECTED GRADIENT ALGORITHM (PGA)

Table 1. EBB SCHEME WITH NON-LINEAR POWER FUNCTION

(A) TURBINES CANNOT BE USED FOR SLUICING

α_S^o	α_T^o	Power Output				No. of Iterations			
		CGA	PGA	CPGA	NPGA	CGA	PGA	CPGA	NPGA
1.0	1.0	0.10163	0.10173	0.10177	0.10043	68(351)	15(28)	16(28)	237(731)
0.5	1.0		0.10178	0.10175			21(39)	14(24)	
0.1	1.0		0.10178	0.10178			18(33)	14(25)	
1.0	0.5		0.10177	0.10179			20(37)	14(25)	
1.0	0.1		0.10179	0.10179			21(39)	14(25)	
0.5	0.5		0.10176	0.10173			14(26)	13(24)	
0.1	0.1	0.10164	0.10178	0.10178	0.10097	79(445)	15(28)	19(35)	98(298)

(B) TURBINES CAN BE USED FOR SLUICING

α_S^o	α_T^o	Power Output				No. of Iterations			
		CGA	PGA	CPGA	NPGA	CGA	PGA	CPGA	NPGA
1.0	1.0	0.10937	0.10945	0.10946	0.10834	58(272)	14(26)	16(29)	232(716)
0.5	1.0		0.10947	0.10946			14(26)	15(27)	
0.1	1.0		0.10948	0.10950			17(31)	14(26)	
1.0	0.5		0.10946	0.10951			14(26)	13(23)	
1.0	0.1		0.10950	0.10951			18(34)	13(23)	
0.5	0.5		0.10942	0.10951			12(23)	13(25)	
0.1	0.1	0.10935	0.10949	0.10951	0.10885	72(393)	14(26)	18(34)	100(293)

Table 2. TWO-WAY SCHEME WITH NON-LINEAR POWER FUNCTION

α_S^o	α_T^o	Power Output				No. of Iterations			
		CGA	PGA	CPGA	NPGA	CGA	PGA	CPGA	NPGA
1.0	1.0	0.16533	0.16490	0.16508	0.16411	146(885)	36(123)	36(130)	79(257)
0.1	0.1	0.16531	0.16547	0.16508	0.16479	80(420)	24(72)	33(120)	55(178)

Table 3. EBB SCHEME WITH LINEAR POWER FUNCTION

α_S^o	α_T^o	Power Output				No. of Iterations			
		CGA	PGA	CPGA	NPGA	CGA	PGA	CPGA	NPGA
1.0	1.0	0.14134	0.14090	0.14101	0.13981	4(4)	15(15)	13(13)	64(64)
0.1	0.1	0.14214	0.14088	0.14089	0.14122	5(5)	13(13)	13(13)	36(36)

Table 4. TWO-WAY SCHEME WITH LINEAR POWER FUNCTION

α_S^o	α_T^o	Power Output				No. of Iterations			
		CGA	PGA	CPGA	NPGA	CGA	PGA	CPGA	NPGA
1.0	1.0	0.24091	0.23858	0.23931	0.23774	6(6)	11(11)	10(10)	26(26)
0.1	0.1	0.24068	0.23987	0.23961	0.23971	7(7)	11(11)	11(11)	18(18)

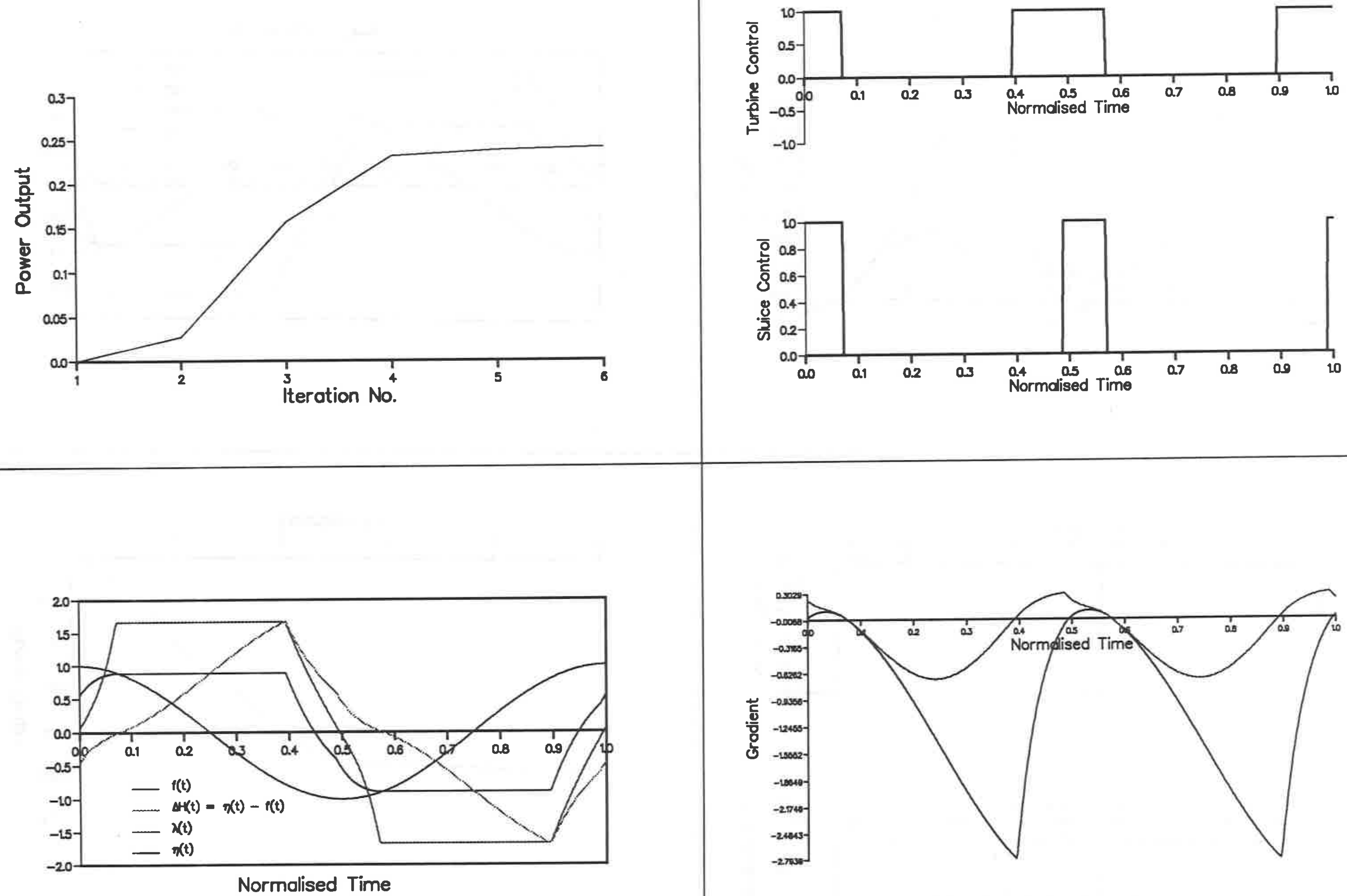

Fig. 3 Linear - CGA
Tol = 1.0%, 2-Way

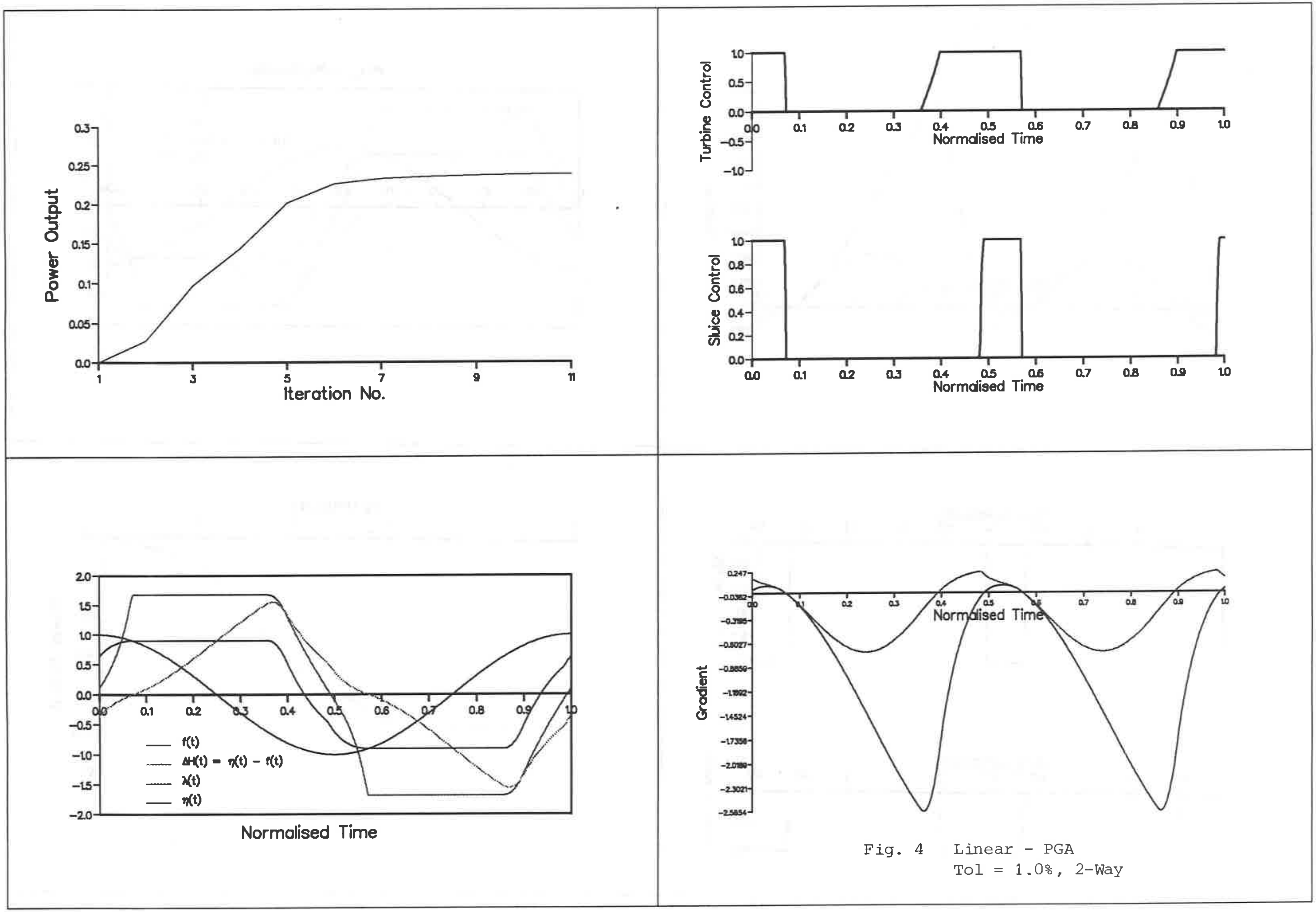

Fig. 4 Linear - PGA
Tol = 1.0%, 2-Way

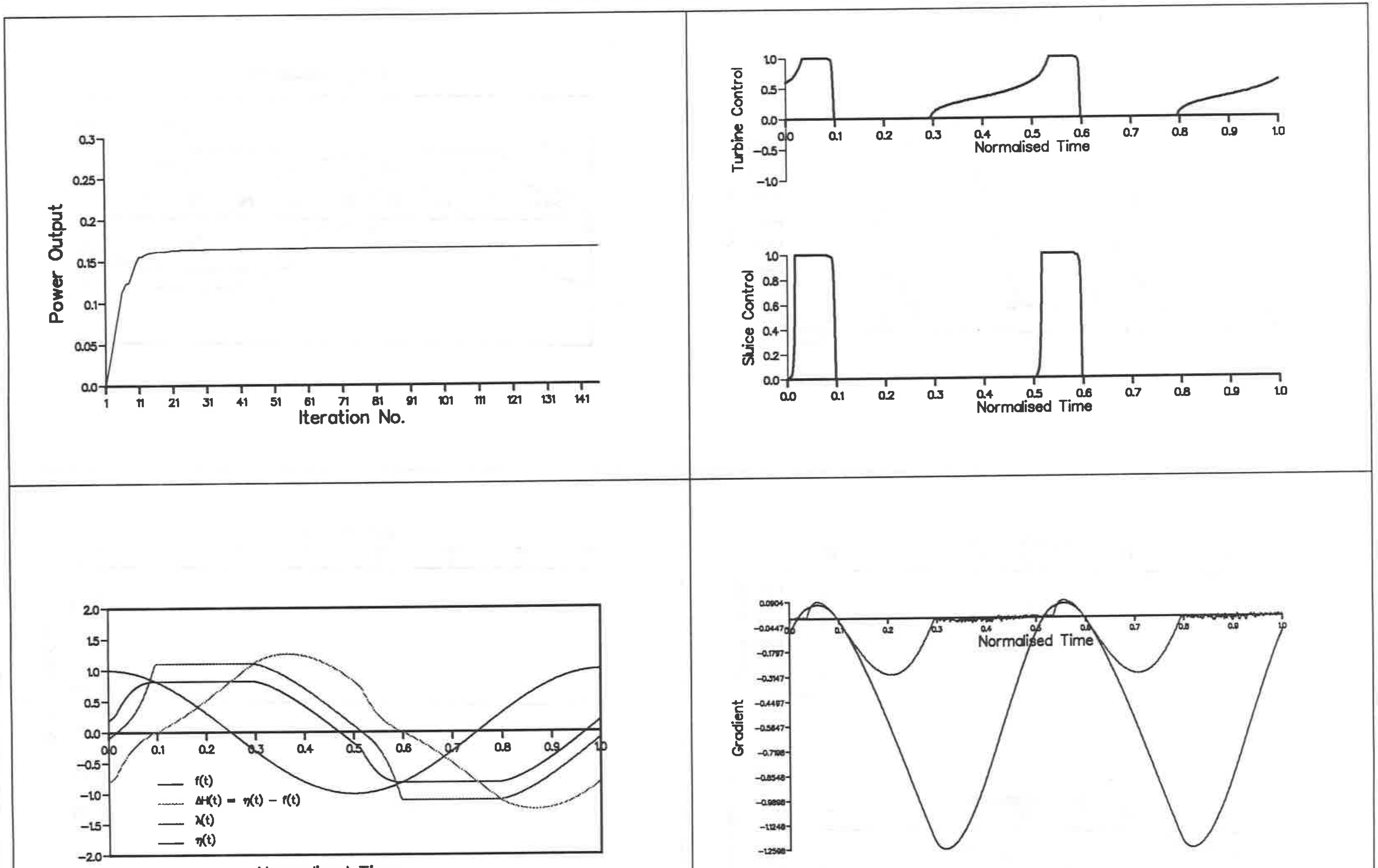

Fig. 5 Non-linear - CGA
Tol = 1.0%, 2-Way

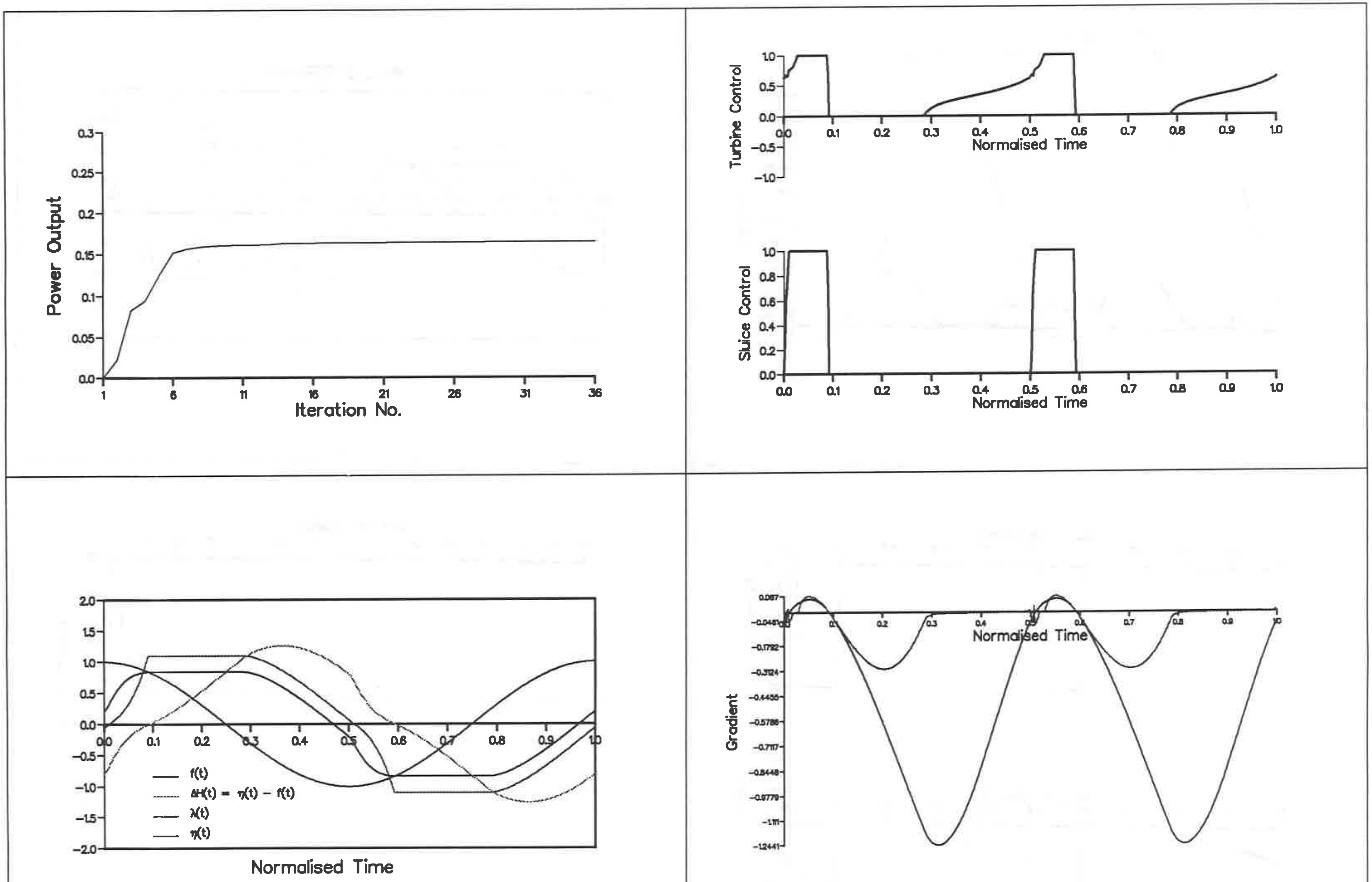

Fig. 6 Non-linear - PGA
Tol = 1.0%, 2-Way

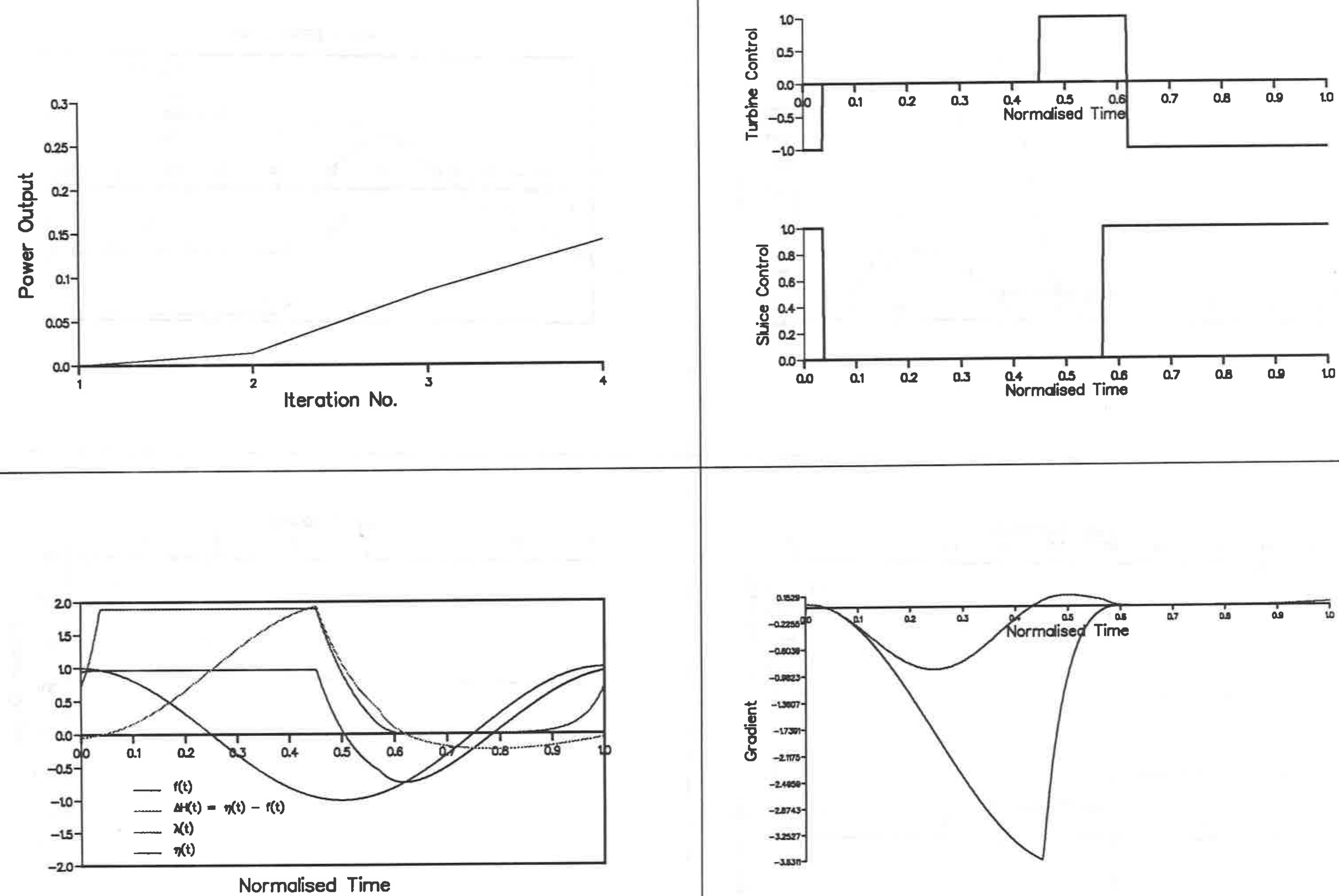

Fig. 7 Linear - CGA
Tol = 1.0%, Ebb

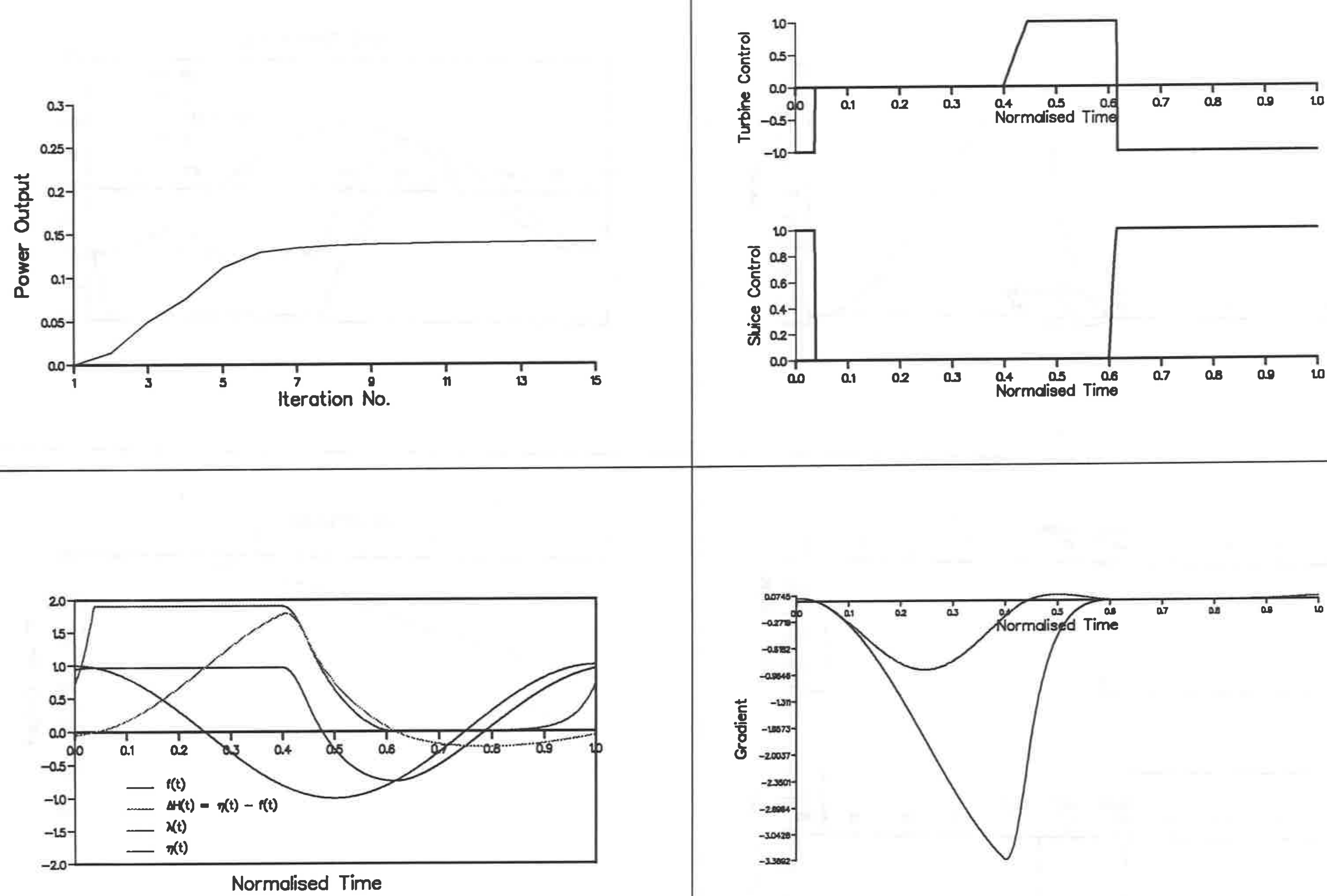

Fig. 8 Linear - PGA
Tol = 1.0%, Ebb

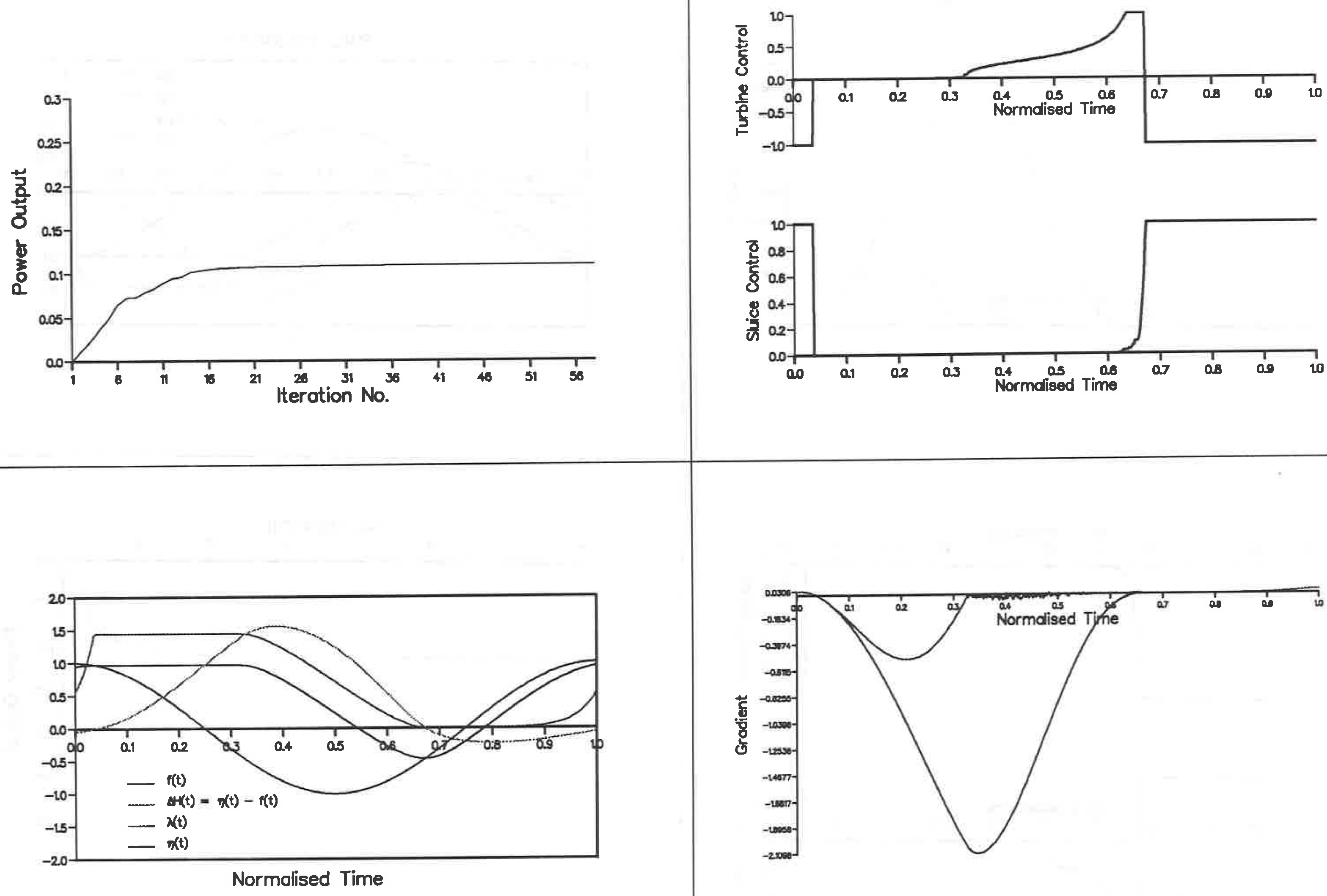

Fig. 9 Non-linear - CGA
Tol = 1.0%, Ebb

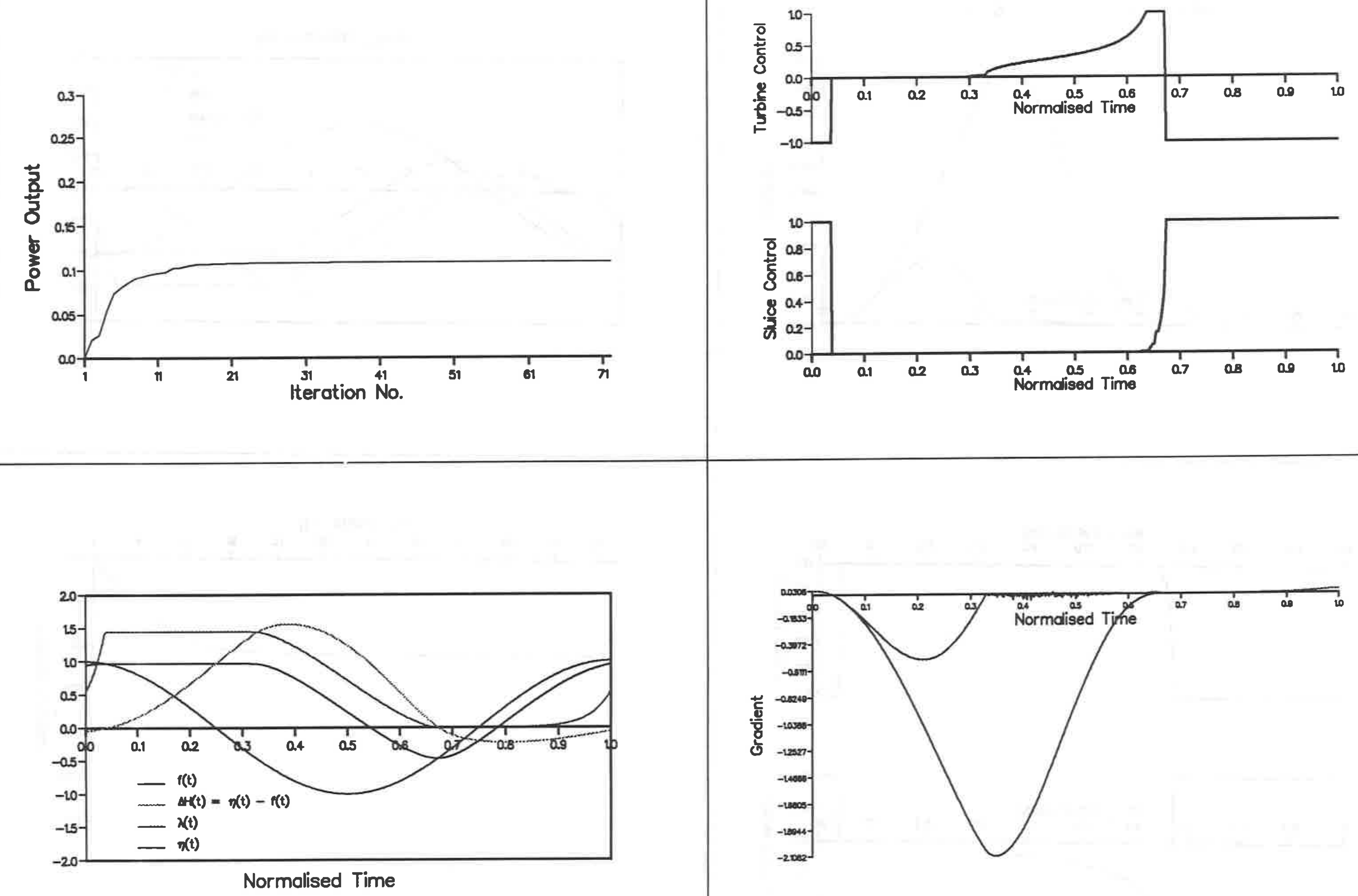

Fig. 10 Non-linear - CGA
Tol = 1.0%, Ebb

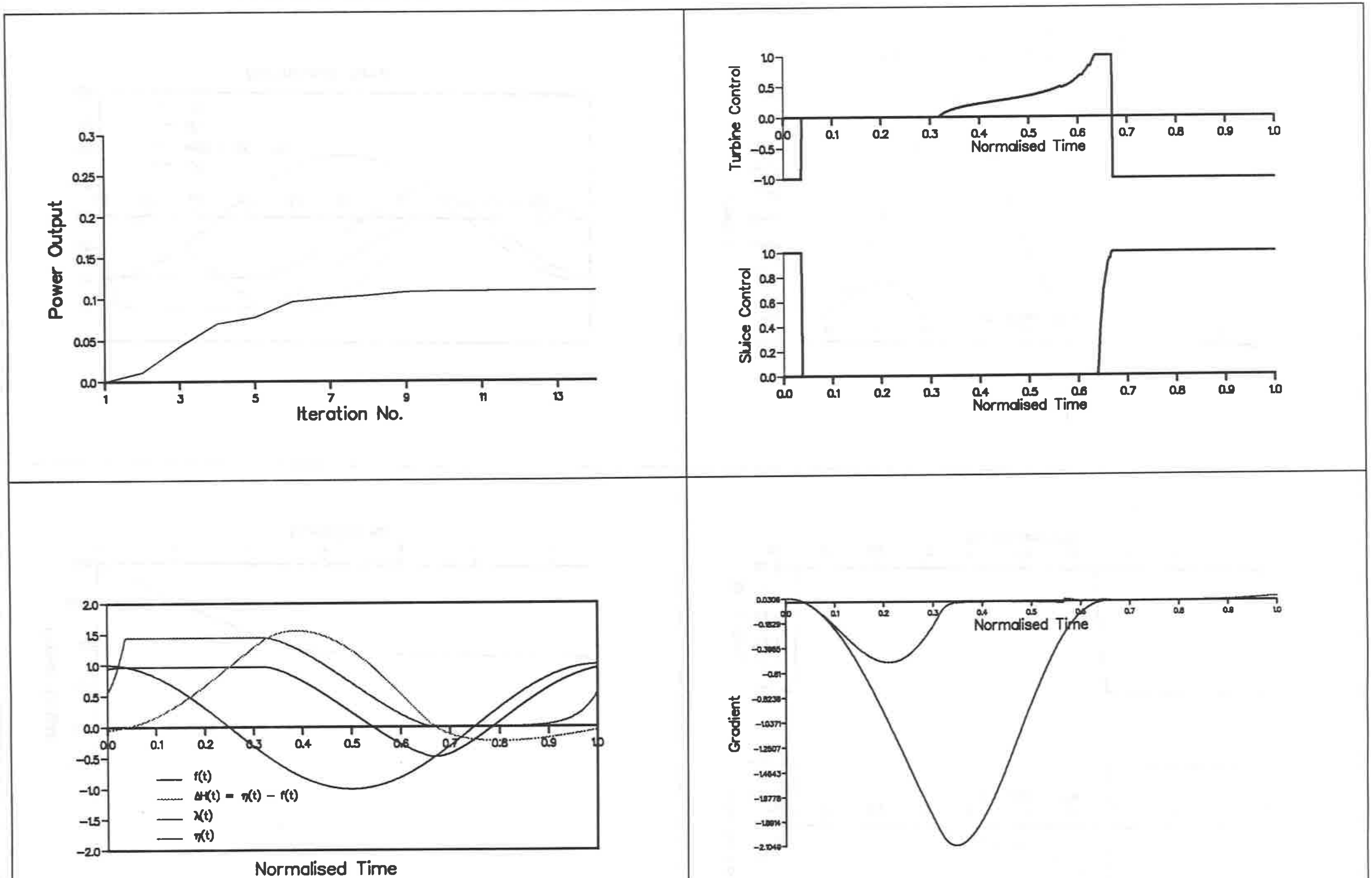

Fig. 11 Non-linear - PGA
Tol = 1.0%, Ebb

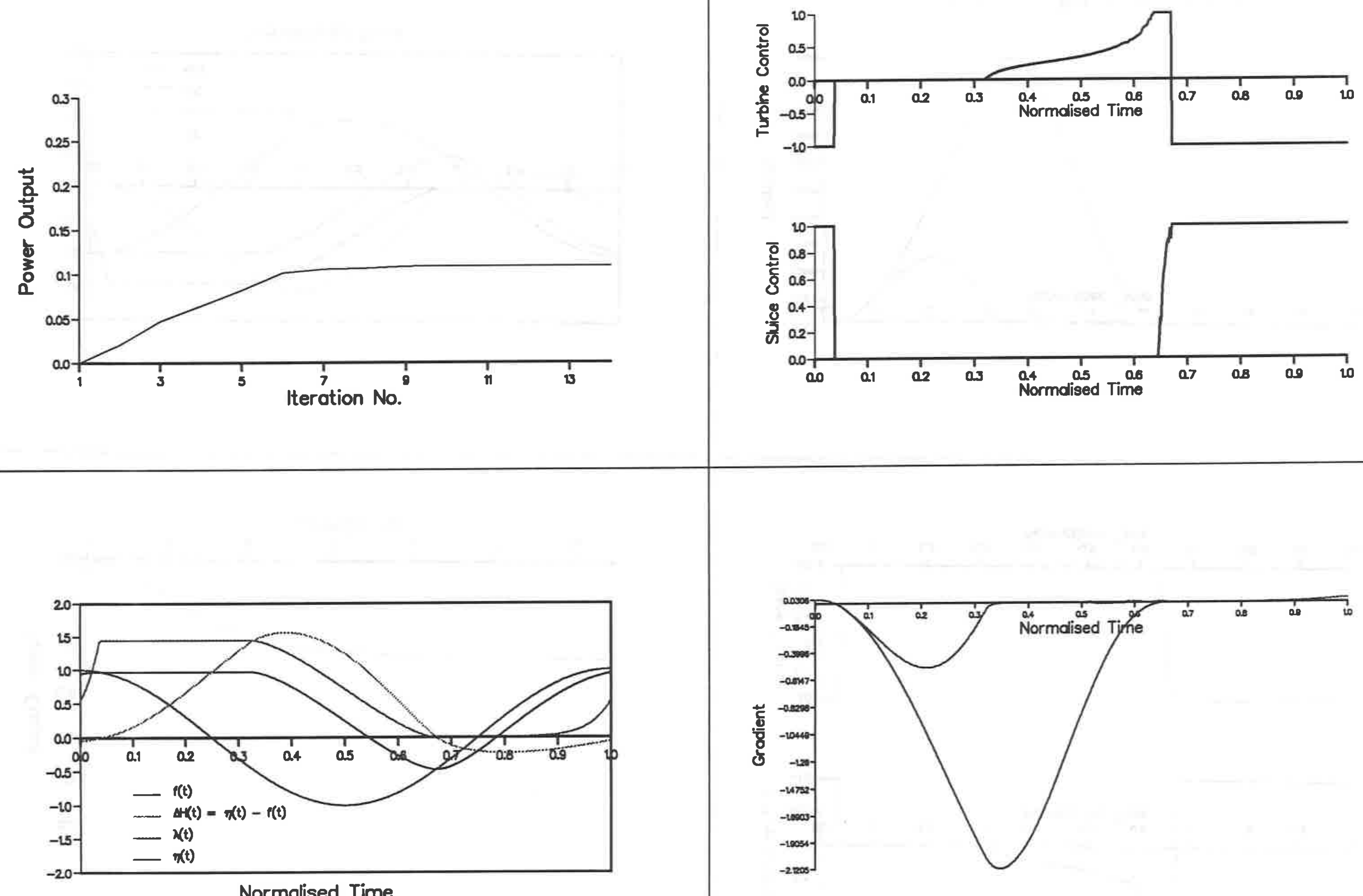

Fig. 12 Non-linear - PGA
Tol = 1.0%, Ebb

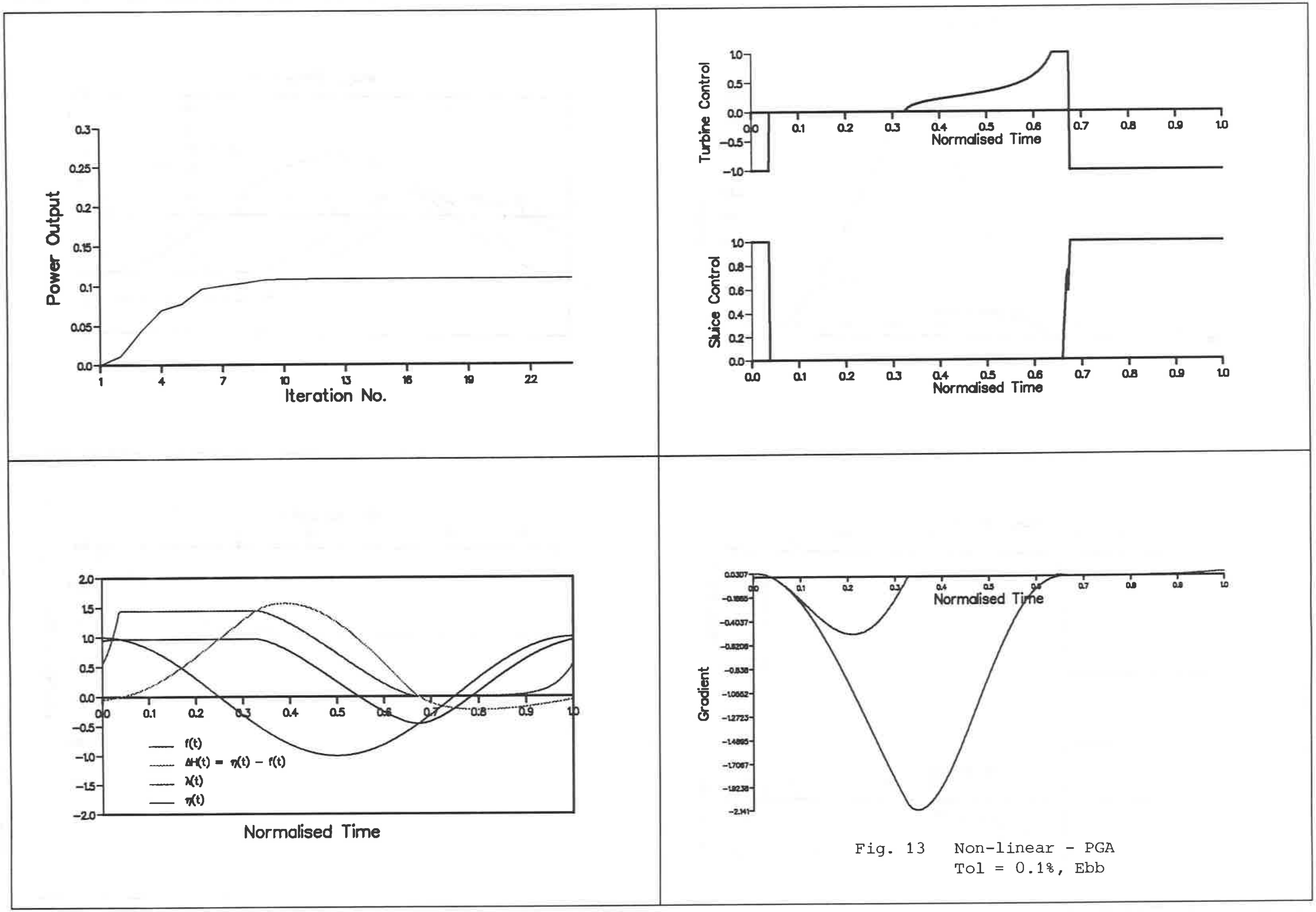

Fig. 13 Non-linear - PGA
Tol = 0.1%, Ebb

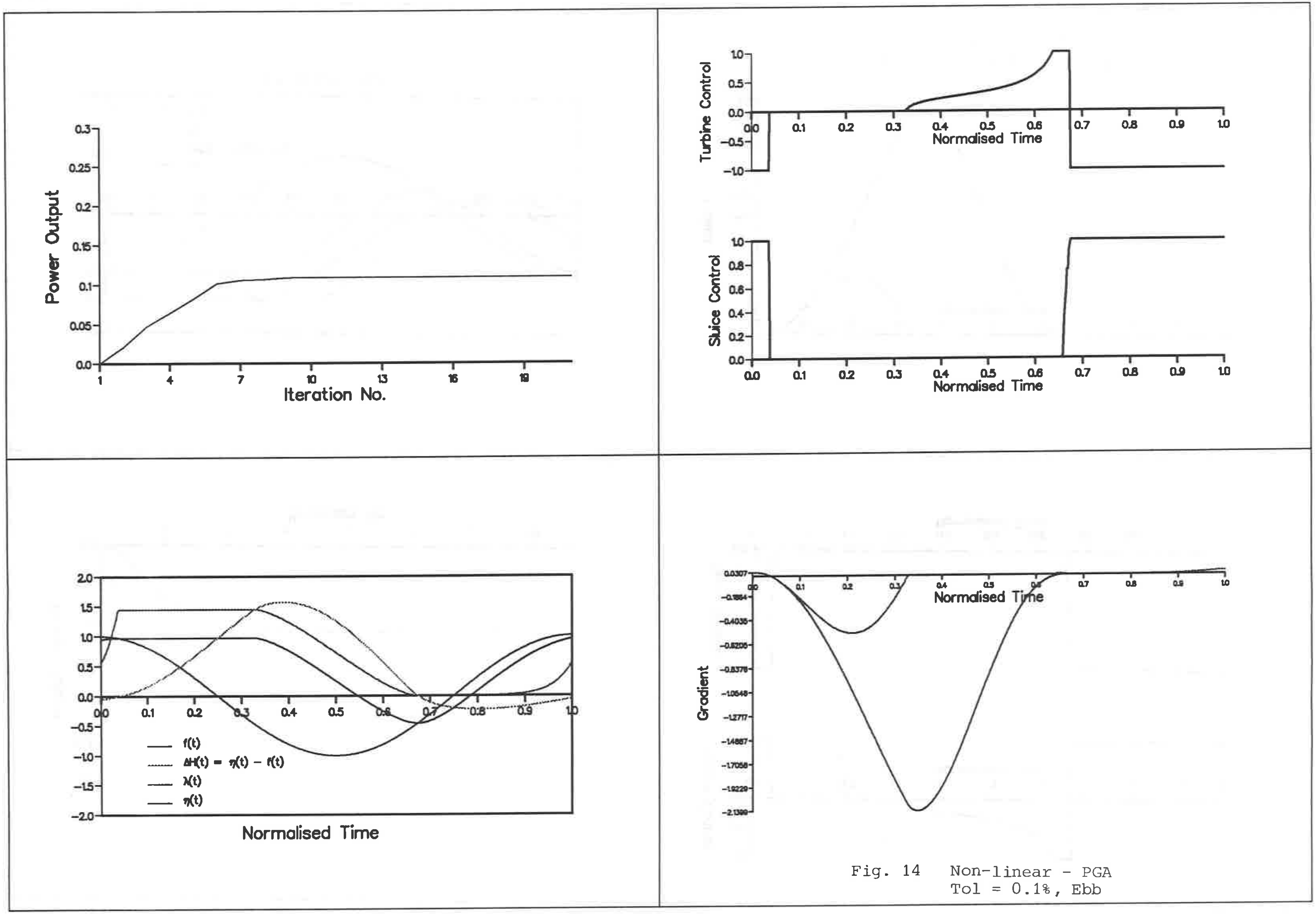

Fig. 14 Non-linear - PGA
Tol = 0.1%, Ebb

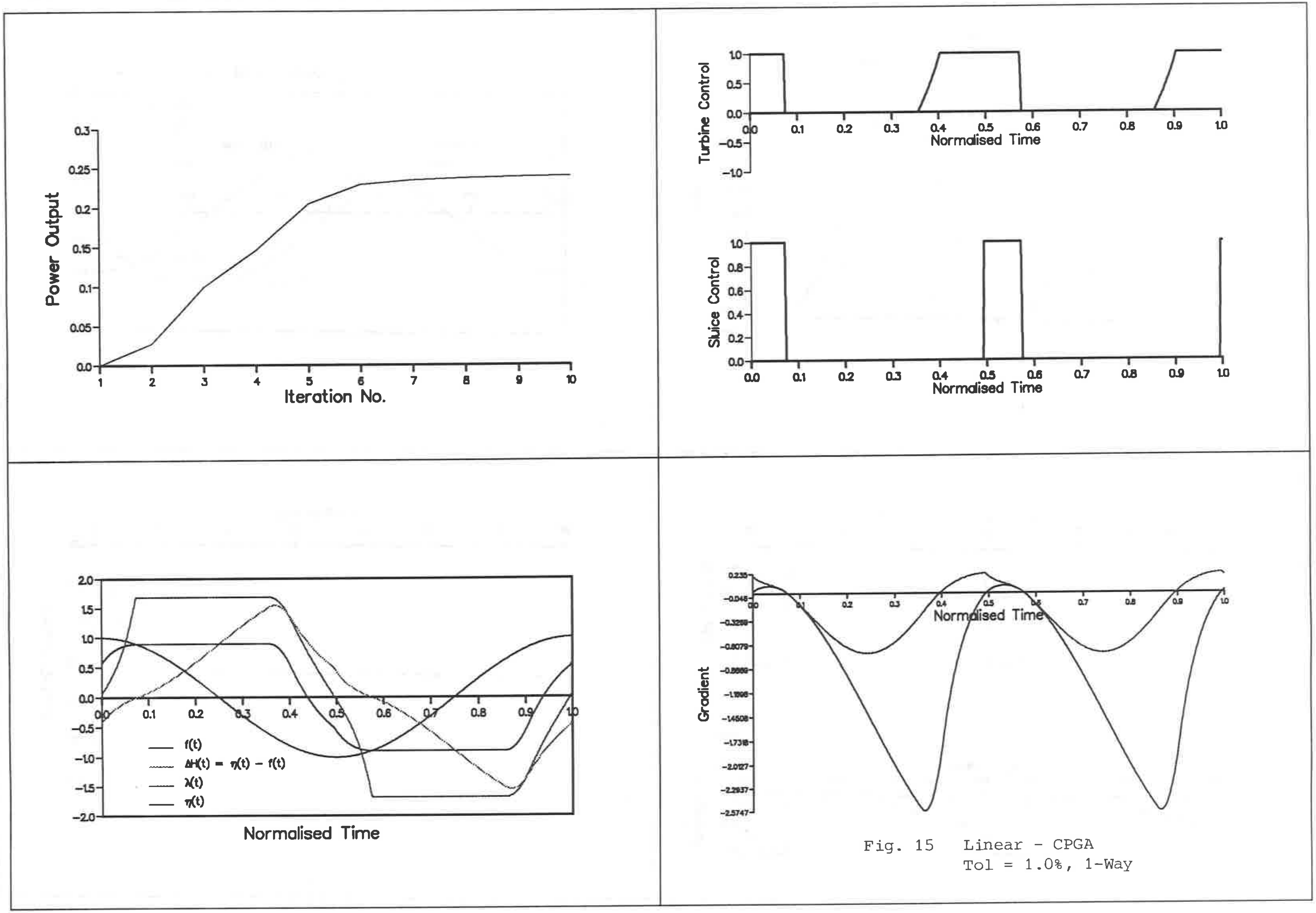

Fig. 15 Linear - CPGA
Tol = 1.0%, 1-Way

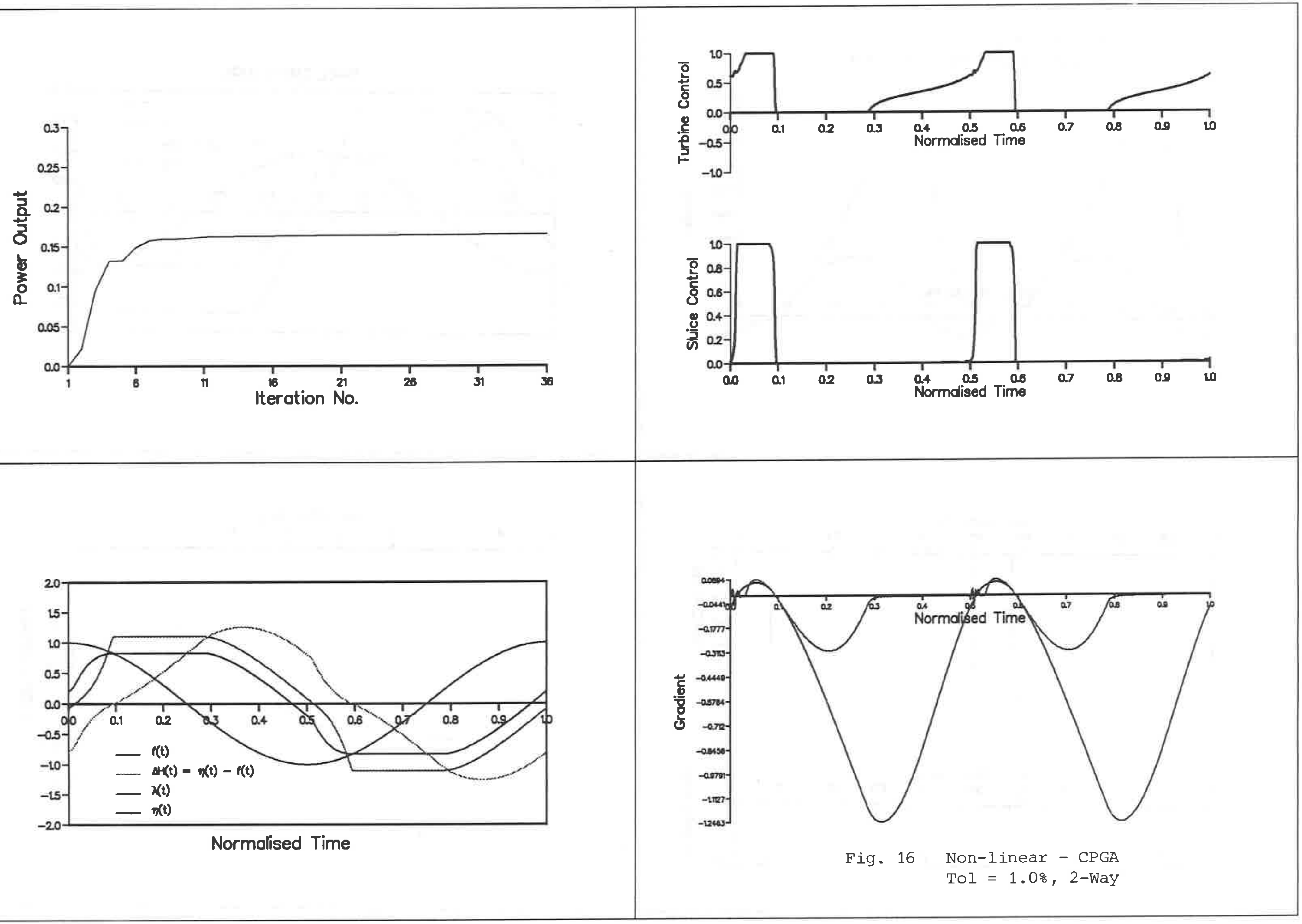

Fig. 16 Non-linear - CPGA
Tol = 1.0%, 2-Way

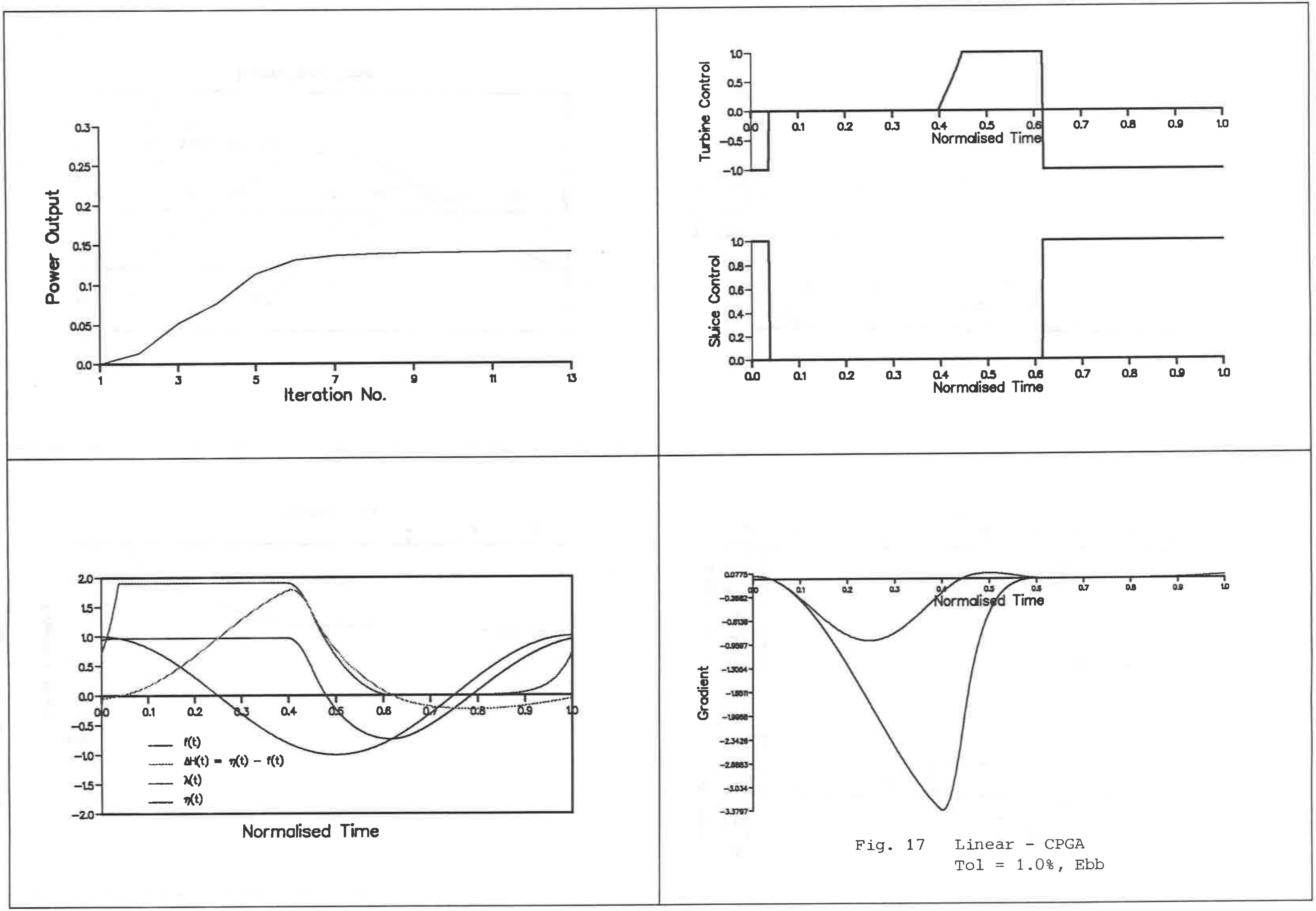

Fig. 17 Linear - CPGA
Tol = 1.0%, Ebb

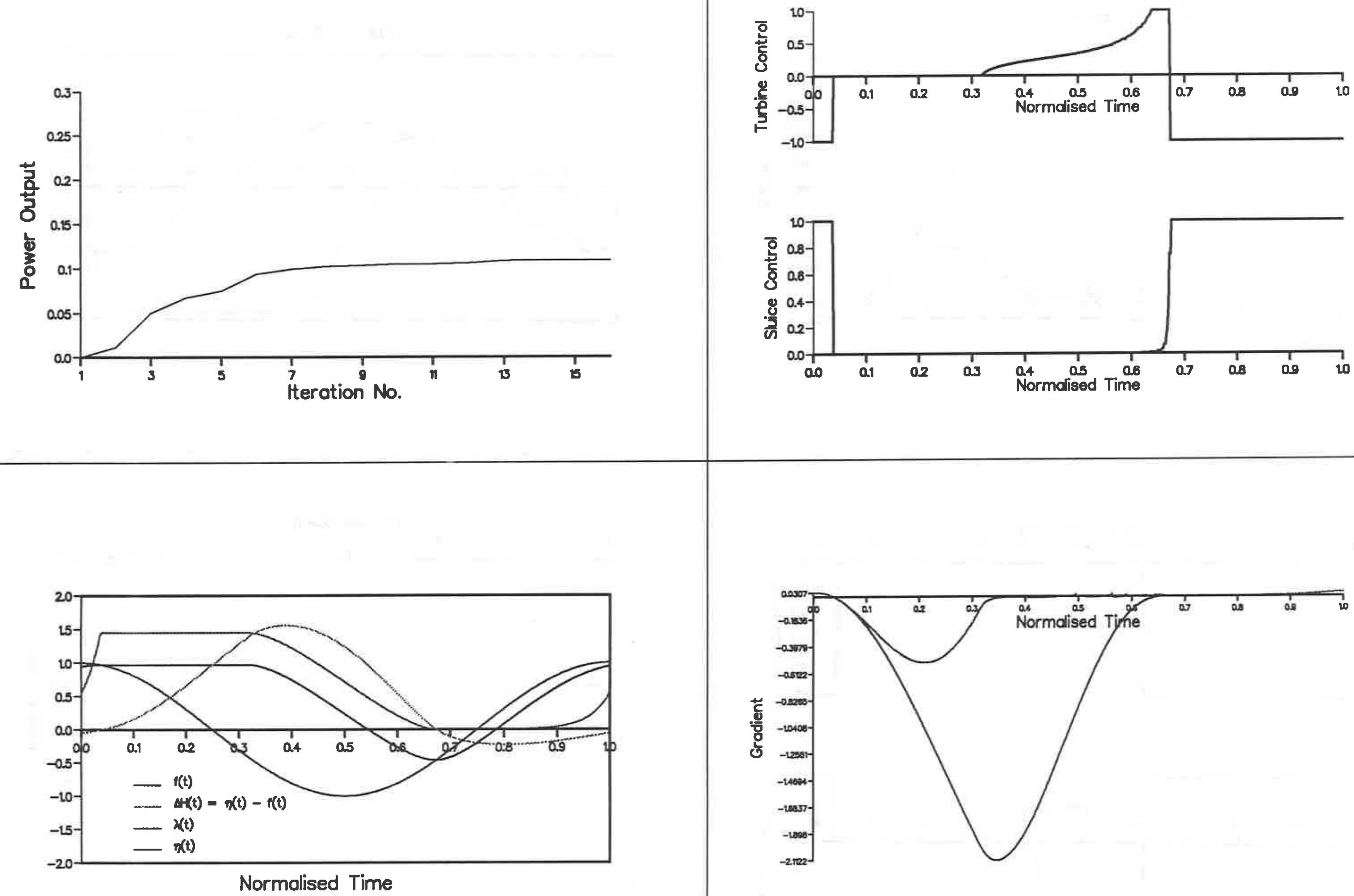

Fig. 18 Non-linear - CPGA
Tol = 1.0%, Ebb

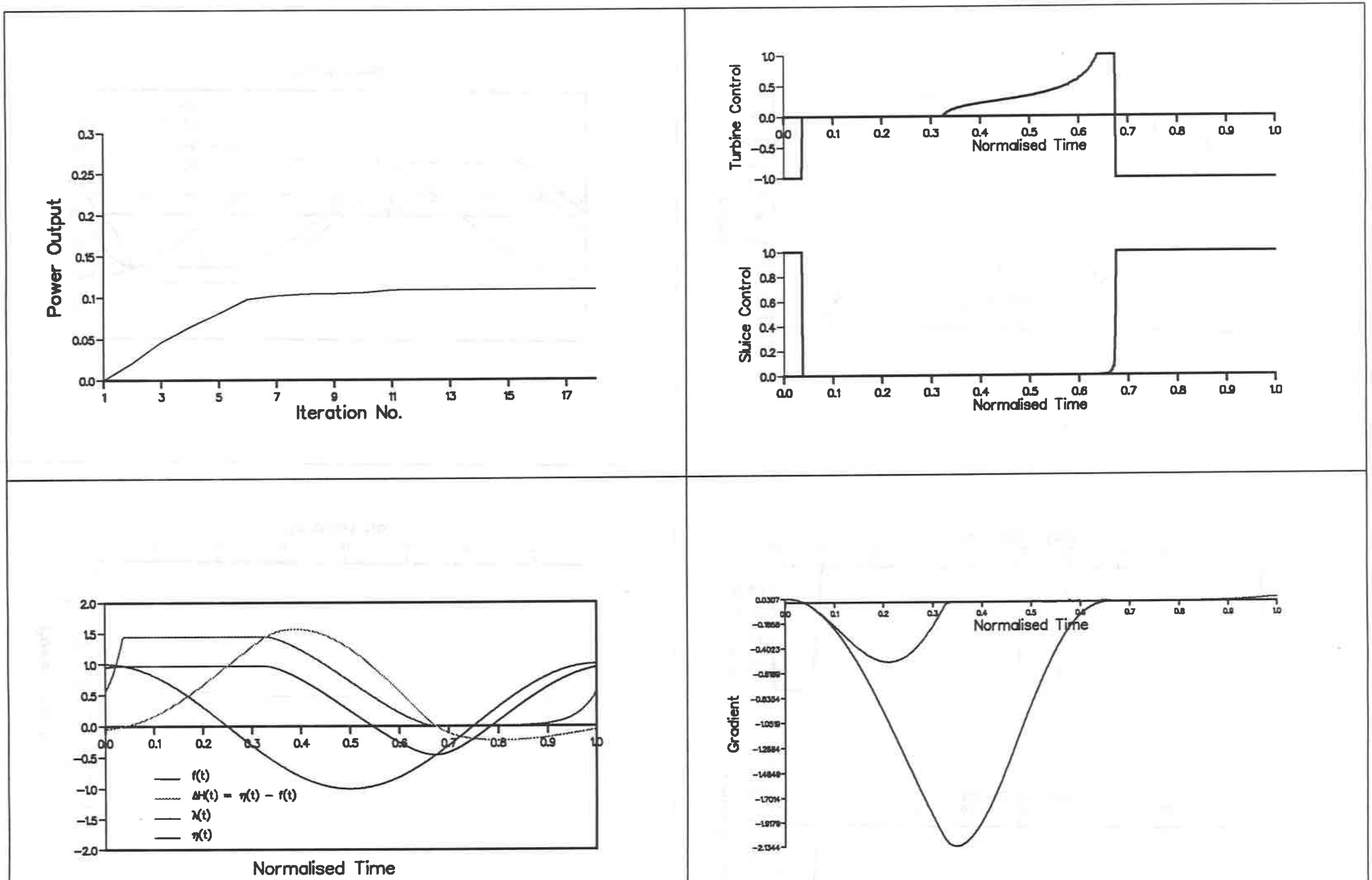

Fig. 19 Non-linear - CPGA
Tol = 1.0%, Ebb

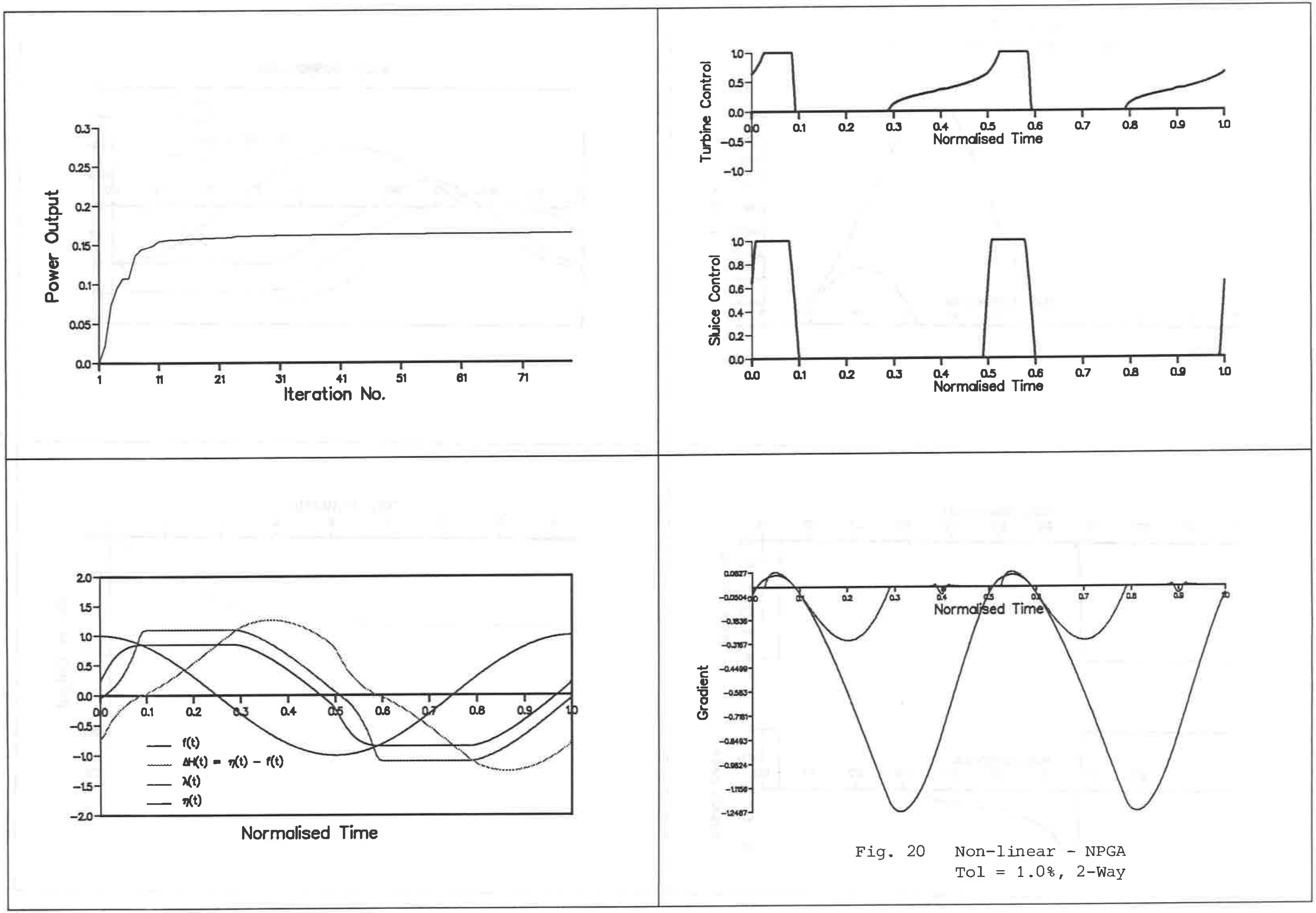

Fig. 20 Non-linear - NPGA
Tol = 1.0%, 2-Way

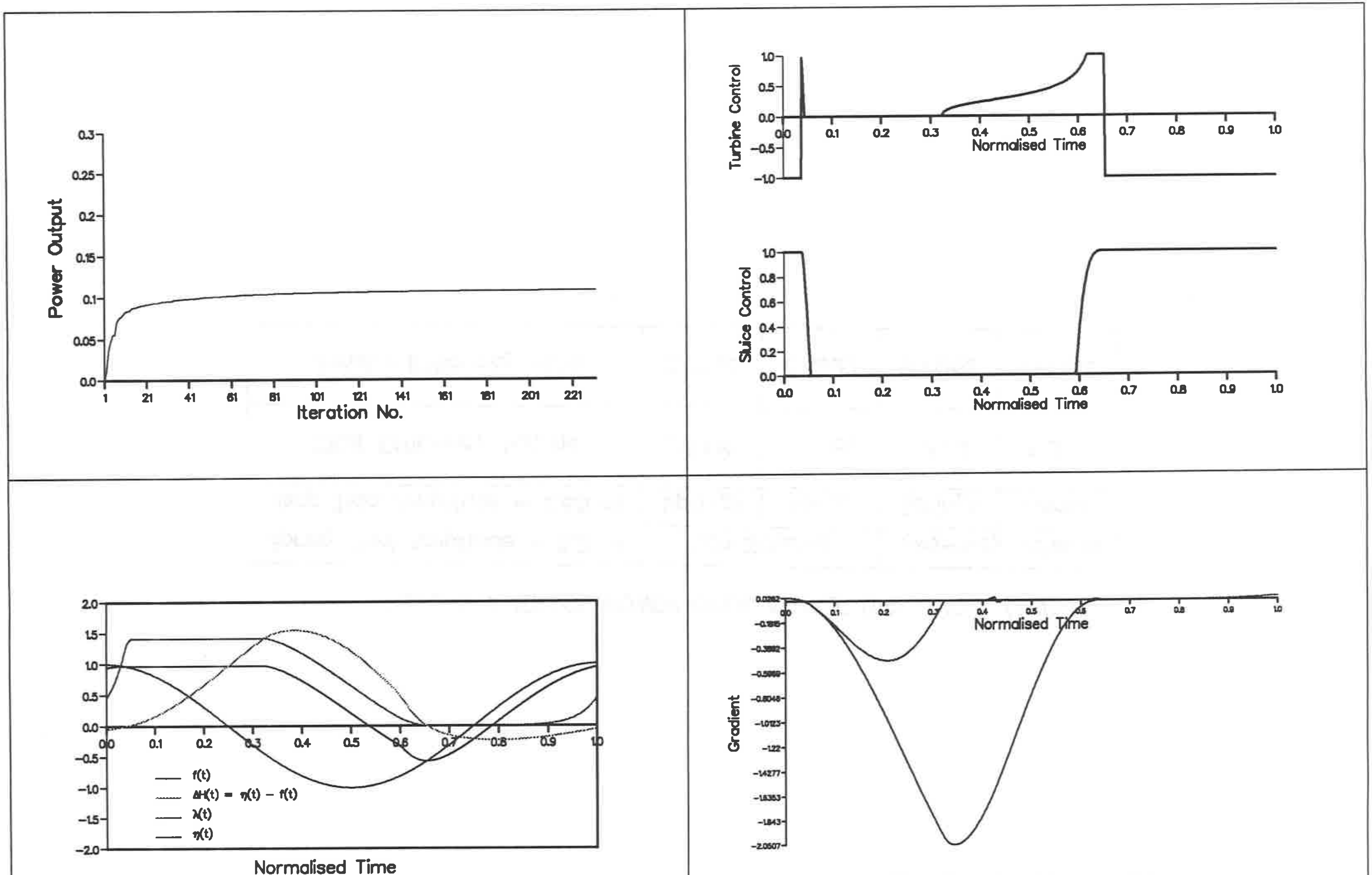

Fig. 21 Non-linear - NPGA
Tol = 1.0%, Ebb

Table 5. AVERAGE POWER OVER A HALF DAY CYCLE (GW)

Spring Tidal Amplitude = 5.2 m Neap Tidal Amplitude = 2.65 m	Ebb Scheme		Two-Way Scheme	
	Springs	Neaps	Springs	Neaps
100% Efficiency Turbine	2.7410	0.9561	2.8114	1.1334
Variable Efficiency m/c	2.2085	0.6922	2.2052	0.7133

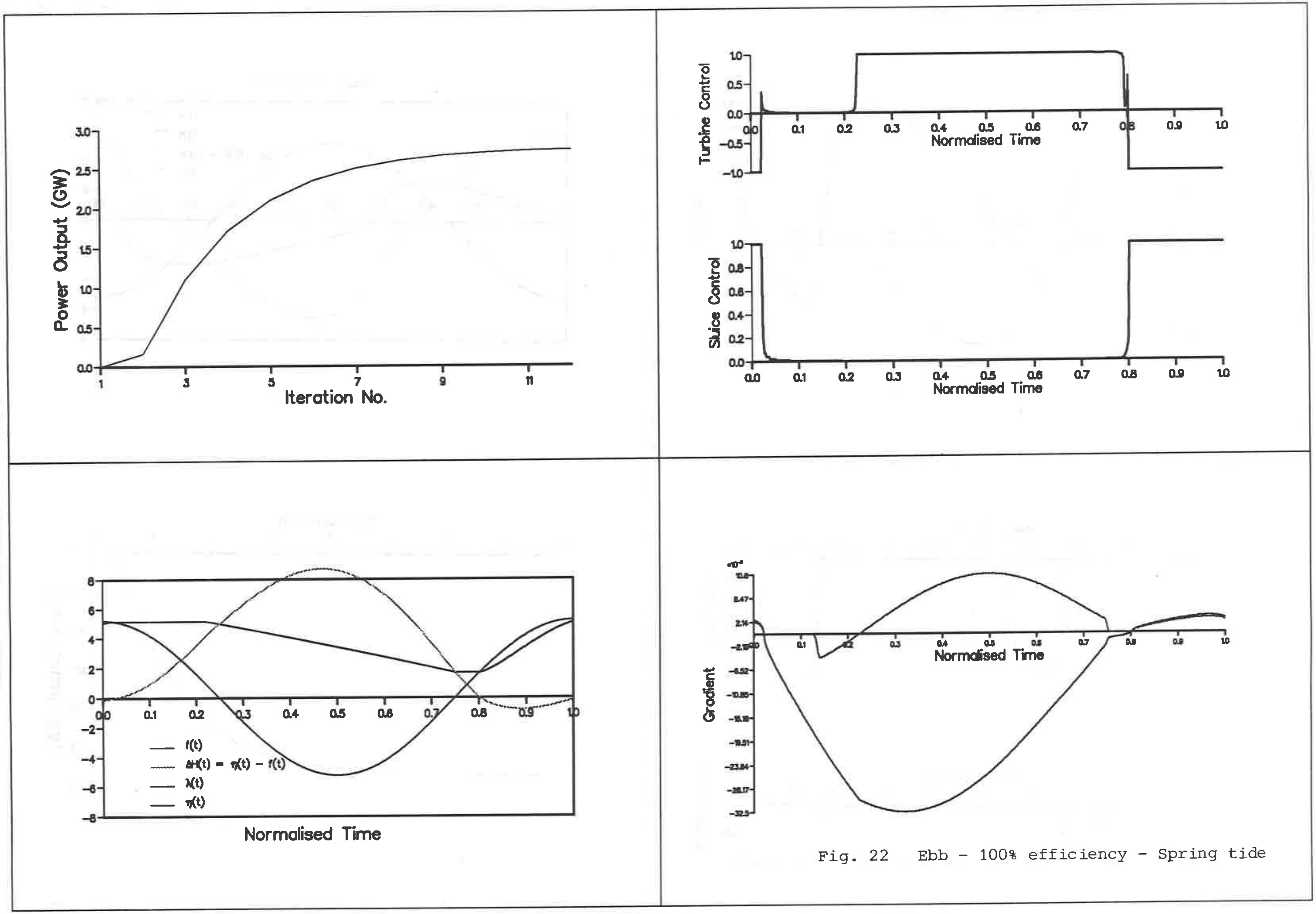

Fig. 22 Ebb - 100% efficiency - Spring tide

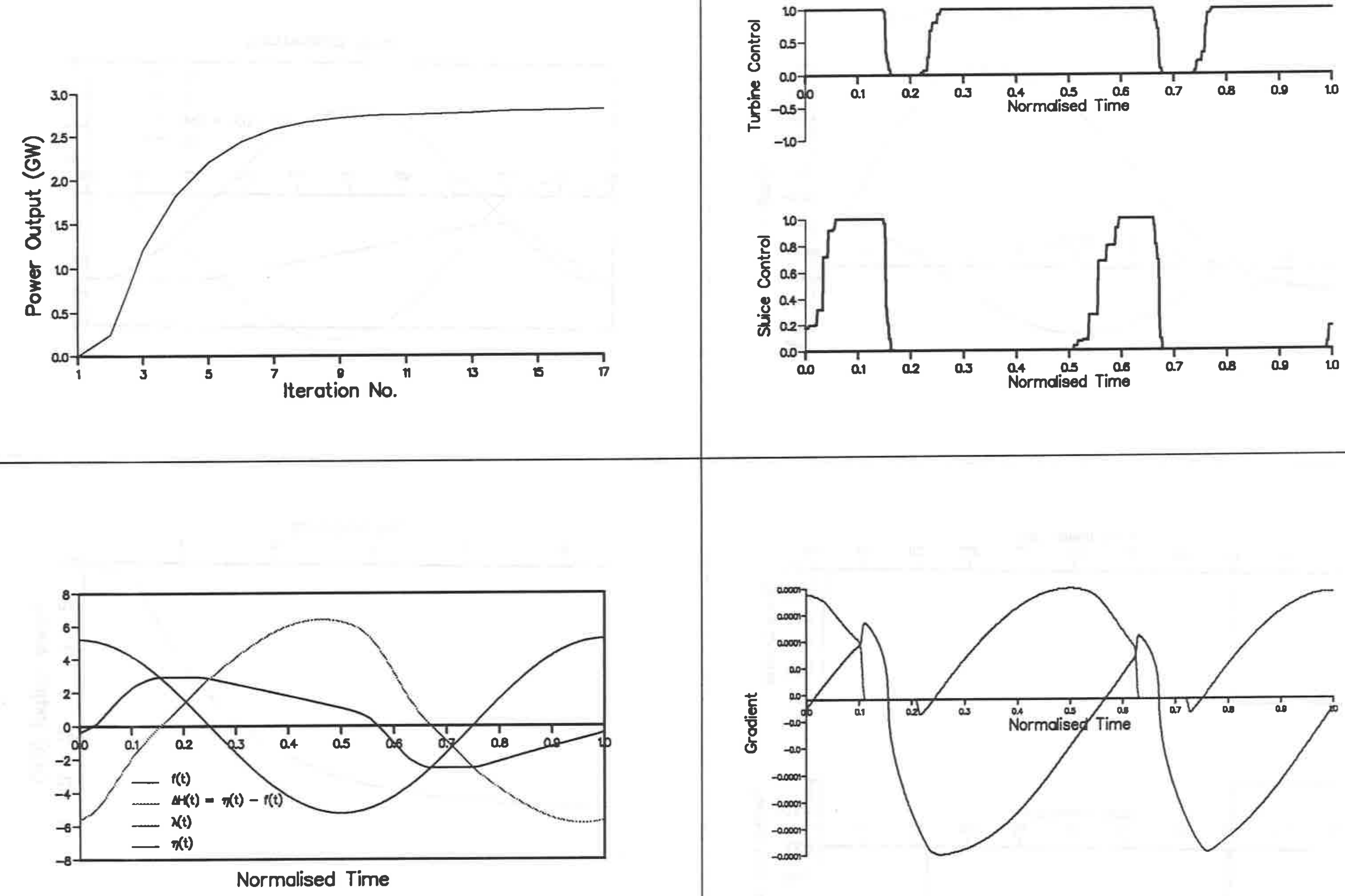

Fig. 23 2-Way - 100% efficiency - Spring tide

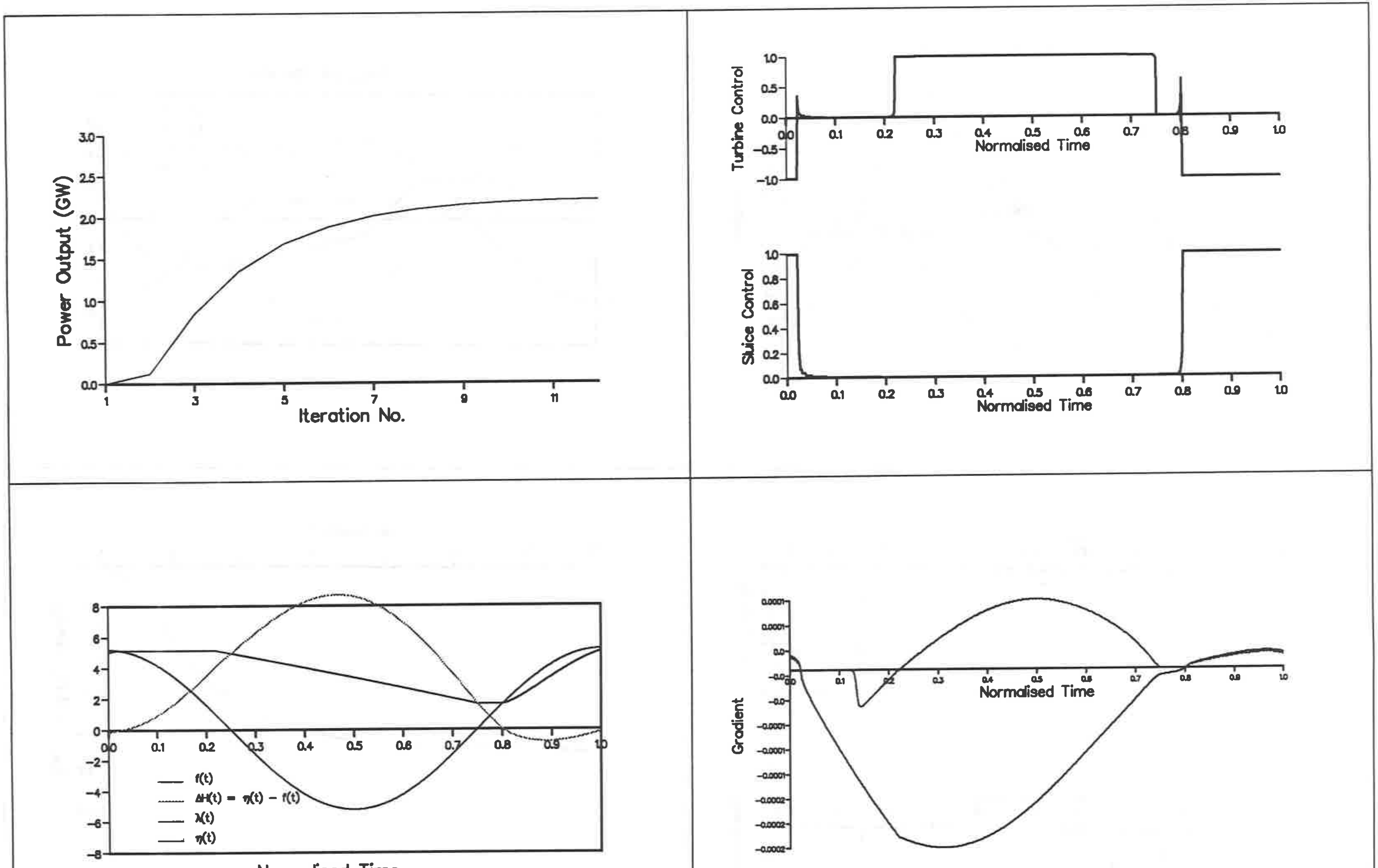

Fig. 24 Ebb - Variable efficiency - Spring tide

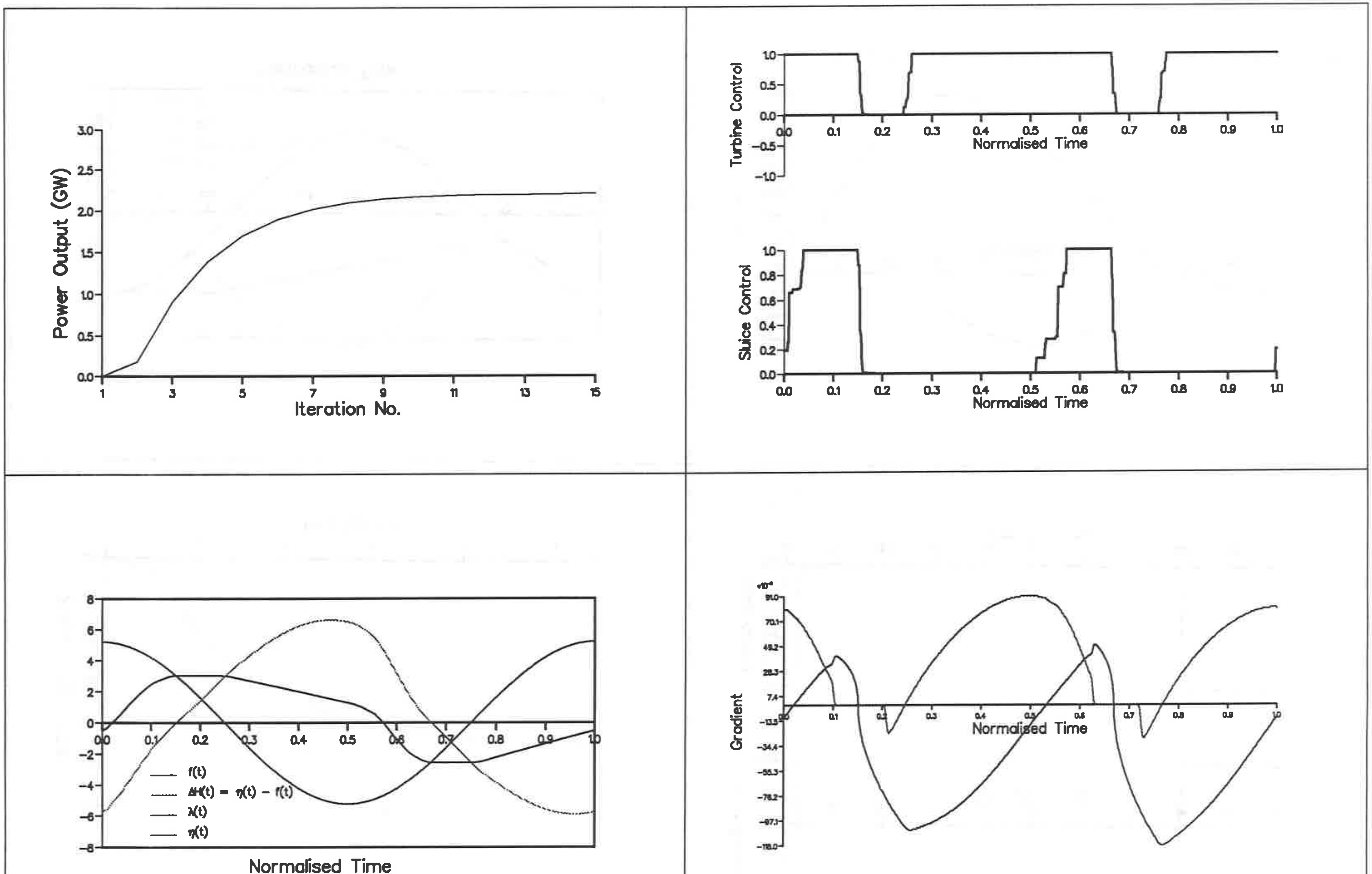

Fig. 25 2-Way - Variable efficiency - Spring tide

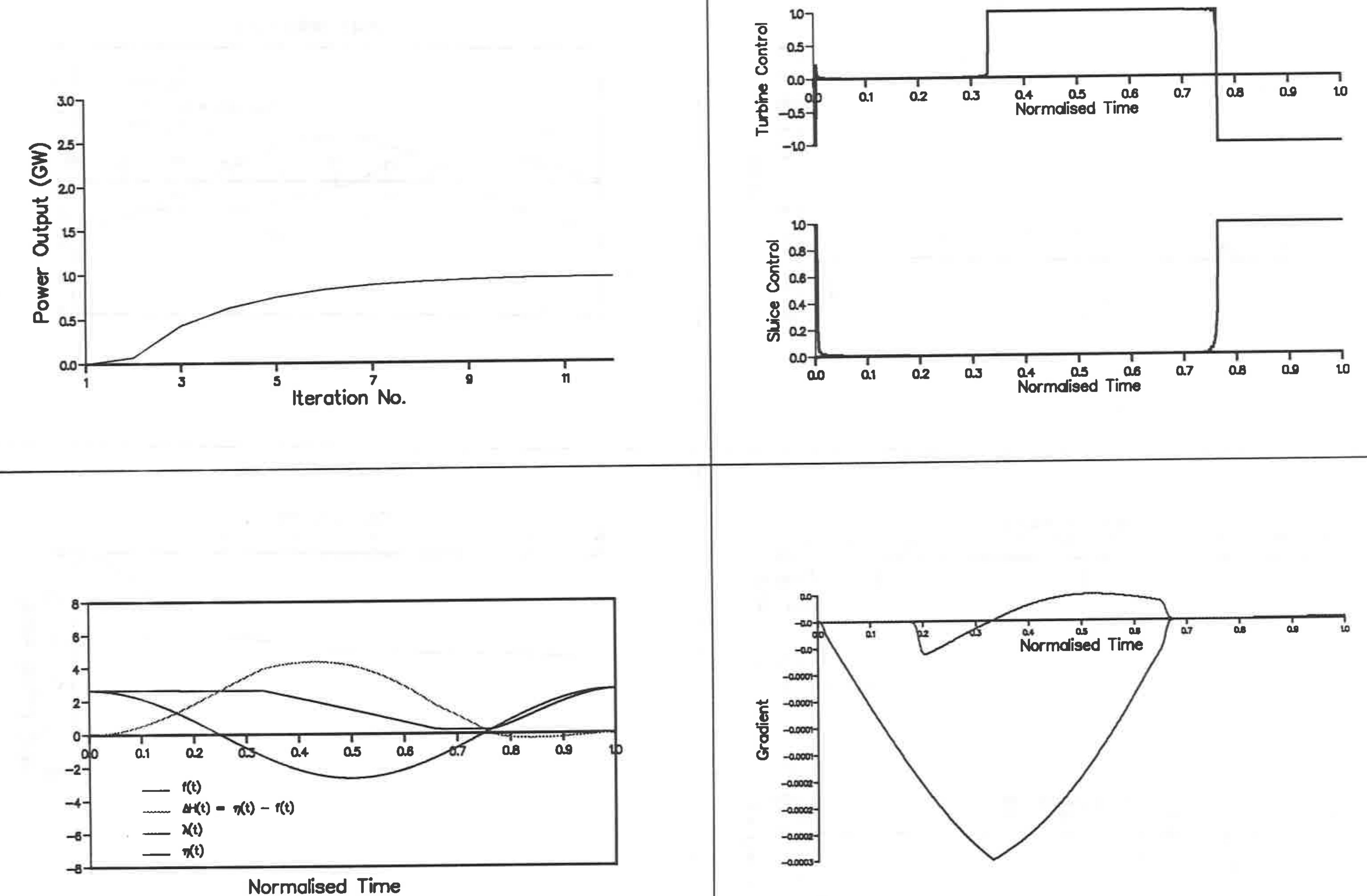

Fig. 26 Ebb - 100% efficiency - Neap tide

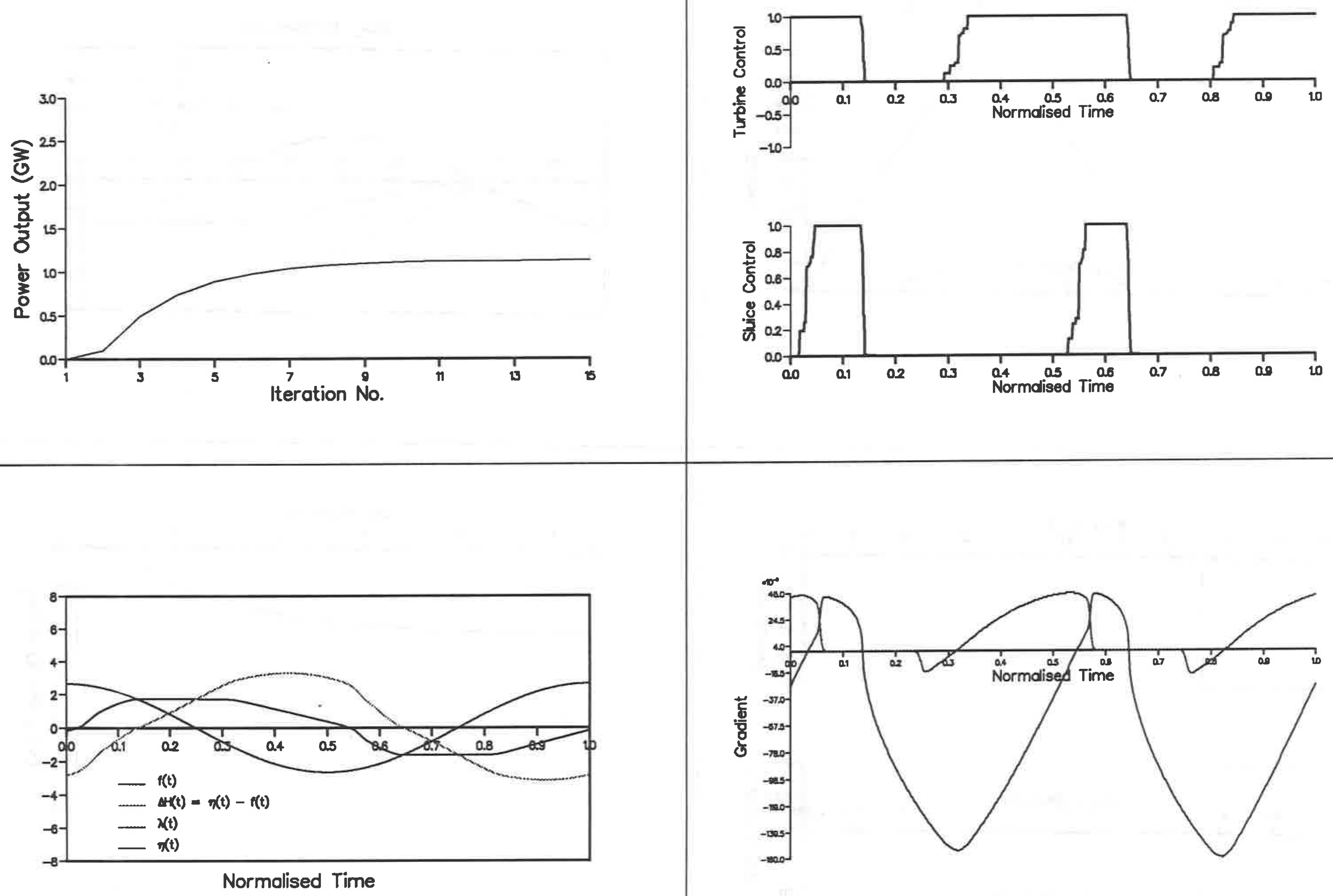

Fig. 27 2-Way - 100% efficiency - Neap tide

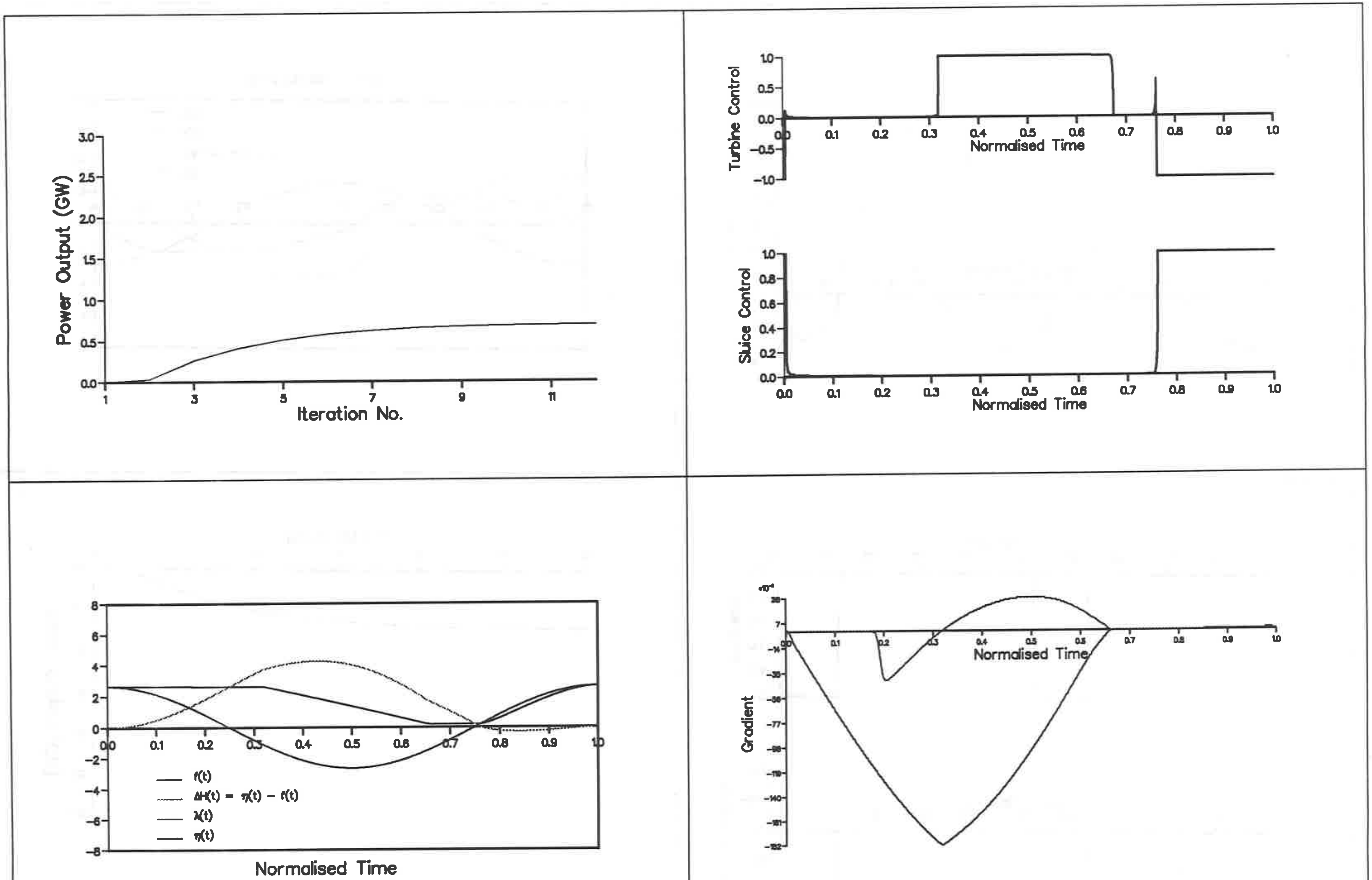

Fig. 28 Ebb - Variable efficiency - Neap tide

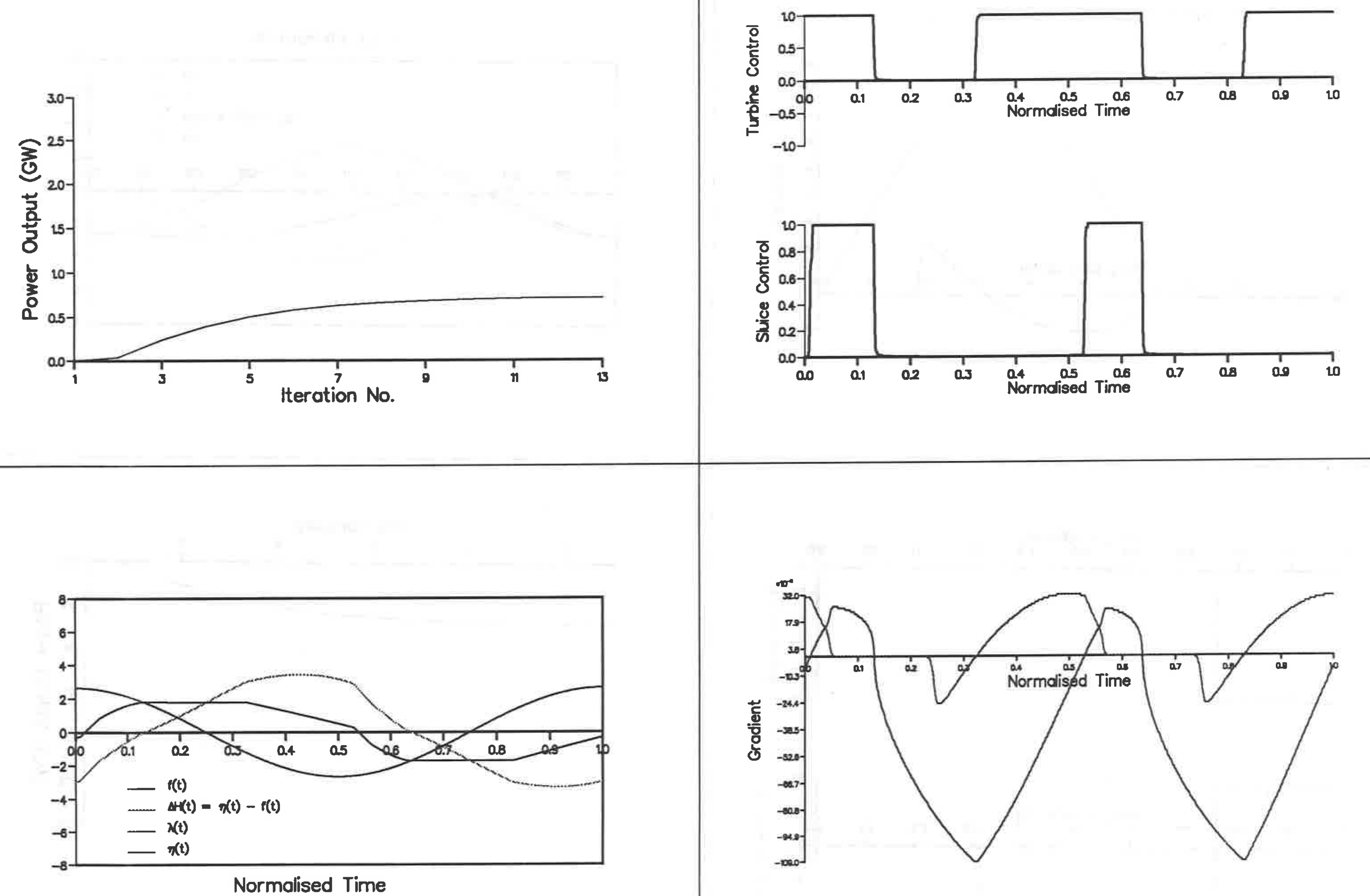

Fig. 29 2-Way - Variable efficiency - Neap tide