QUASI-VARIATIONAL INEQUALITIES RELATED TO FLOW THROUGH AN ANISOTROPIC POROUS MEDIUM

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Introduction

The theory of fluid flow through porous media gives rise to free boundary problems governed by elliptic partial differential equations. Traditionally these problems have been solved using the trial free boundary method (see e.g. Birkhoff [5]). In 1971 Baiocchi [2], [3] introduced a technique which allowed many porous flow free boundary problems to be reformulated as variational inequalities. This approach has two advantages, firstly we can now prove existence and uniqueness theorems for the problem and secondly the reformulated problem can be solved using simple numerical techniques which compare very favourably, both in ease of programming and in speed of execution, with the older heuristic algorithms.

In this report we shall consider an extension of this technique to a problem where the flow region consists of anisotropic material, in particular where the principal directions of flow are not parallel to the horizontal and vertical axes, as would occur, for example, in a natural dam composed of tilted strata. We show that, under some restrictions on the geometry of the dam, this problem can be reformulated as a quasi-variational inequality.

The Physical Problem.

Two water reservoirs are separated by a porous dam and water seeps through this dam from the higher reservoir to the lower one. We wish to find the quantities associated with the motion, e.g. flow region, equipotentials, streamlines, pressure, discharge etc.. We limit ourselves to a homogeneous dam on a horizontal impermeable base for steady, irrotational, incompressible, two dimensional flow. We also neglect capilliarity and evaporation effects. However in this problem the principal directions are not horizontal and vertical as in previous cases which have been studied.

Equations of flow.

The governing law of flow through a porous medium is Darcy's law which relates the potential u to the velocity vector $\underline{v}$.

If we define u as

$$u = y + \frac{p(x,y)}{\gamma} \tag{1}$$

where y is the height above the dam base

p is the fluid pressure

γ is the specific weight of the fluid

then Darcy's law states that

$$\underline{v} = -K\underline{\nabla}u \tag{2}$$

(see e.g. Bear [4], Polubori nova-Kochina [8]).

In the case of an isotropic, homogeneous medium then $K \equiv k$ a positive constant, but in our case $K \equiv \{k_{i,j}\}$ a full two by two matrix.

<u>Remark</u> In a practical case this matrix K would not be given directly but rather the angle of tilt of the principal directions, θ, and the permeabilities along these directions m and n. The matrix would then be recovered from the fact that the vectors in the principal directions $(\cos\theta,\sin\theta)^T$ and $(-\sin\theta,\cos\theta)^T$ are eigenvectors of K with eigenvalues m and n respectively. This leads us to the equations:

$$k_{11} = m\cos^2\theta + n\sin^2\theta$$

$$k_{12} = k_{21} = (m-n)\sin\theta\cos\theta$$

$$k_{22} = m\sin^2\theta + n\cos^2\theta$$

from which we have that K is positive definite.

<u>Mathematical Statement of Problem.</u> [see Fig.1]

We define

$$D = \{(x,y) : a<x<b;\ 0<y<Y(x)\} \tag{3}$$

where $Y(x) \in C^2[(a,b)]$, such that $Y(a) = Y(b) = 0$ (4)
and $Y(x)$ is concave on $[a,b]$, $a<0$.
Further we suppose that $C \in (0,b)$, such that

$$Y(0) = y_1 \quad \text{(i)}$$
$$Y(C) = y_2 \quad \text{(ii)} \qquad (5)$$

Given $K = \{k_{ij}\}$, $i,j = 1,2$, where

$$k_{ii} > 0 \quad , i = 1,2 \quad \text{(i)}$$
$$k_{ij} = k_{ji} \quad , i,j = 1,2 \quad \text{(ii)} \qquad (6)$$
$$k_{11}k_{22} - k_{12}k_{21} > 0 \quad \text{(iii)}$$

then we seek to find a subset $\Omega \subset D$ such that there exists a function $u(x,y)$ satisfying the following conditions:

$$\underline{\nabla} \cdot K \underline{\nabla} u = 0 \quad \text{in} \quad \Omega \tag{7}$$

$$U|_{AF} = y_1; \quad U|_{BC} = y_2; \quad U|_{\Gamma_s} = y \tag{8}$$

$$U|_{\Gamma_f} = y \tag{9}$$

$$(K\underline{\nabla} u) \cdot \underline{n}|_{AB} = 0 \tag{10}$$

$$(K\nabla u) \cdot \underline{n}|_{\Gamma_f} = 0 \tag{11}$$

Remark Equation (7) comes from Darcy's law and the continuity equation $\underline{\nabla} \cdot \underline{v} = 0$; boundary conditions (8) and (9) are derived by assuming zero atmospheric pressure and from the fact the boundaries in contact with stationary bodies of water are equipotentials. Boundary conditions (10) and (11) are no flow conditions across streamlines.

In order to exclude some non-physical solutions we shall also impose the relation

$$(K\underline{\nabla} u) \cdot \underline{n}|_{\Gamma_s} \leq 0 \tag{12}$$

which simply states that any flow across the face must occur from the inside to the outside of the dam.

Given this statement of the problem we are now in a position to define a weak solution to the problem.

Weak Formulation.

A triplet $\{\phi, \Omega, u\}$ is said to be a weak solution of the previous problem if:

$$\begin{cases} \phi \in C^0([0,c]);\ \phi \text{ decreasing on } [0,c] \\ \phi(0) = y_1;\ \phi(c) = y_2 \end{cases} \tag{13}$$

If we set [see fig.2]

$$\begin{aligned} D_1 &= \{(x,y) \in D : a<x\leq 0\} \\ D_2 &= \{(x,y) \in D : c\leq x<b\} \\ D_3 &= \{(x,y) \in D : 0<x<c\} \end{aligned} \tag{14}$$

then

$$\Omega = D_1 \cup D_2 \cup \{(x,y) \in D_3 : y < \phi(x)\} \tag{15}$$

$$u \in H^1(\Omega) \cap C^0(\bar{\Omega}) \tag{16}$$

$$u|_{AF} = y_1 \ ;\ \ u|_{BC} = y_2 \ ;\ \ u|_{\Gamma_s} = y \tag{17}$$

$$u|_{\Gamma_f} = y \tag{18}$$

$$\int_\Omega K\underline{\nabla}u \cdot \underline{\nabla}\psi \, dxdy = 0 \tag{19}$$

$$\forall \ \psi \in C^1(\bar{D}) \tag{19'}$$

with $\psi = 0$ in a neighbourhood of $y = Y(x)$.

Remark Equation (19) contains in the weak sense equation ($\dot{7}$) and boundary conditions (10) & (11).

Lemma 1 $u > y$ almost everywhere in Ω.

Proof. We have that $\underline{\nabla}\cdot K\underline{\nabla}(y-u) = 0$ in Ω and therefore the function $y-u$ satisfies a maximum principle on $\bar{\Omega}$. We have from the boundary conditions that $y-u \leq 0$ on $\partial\Omega$ -]AB[. We can show by an extension of the Hapf maximum principle (Gilbarg and Trudinger [6]) that if the maximum were to occur on]AB[then the directional derivative corresponding to $K\nabla(y-u)$ on]AB[must be in a direction moving strictly out of the region Ω. But from the boundary conditions on]AB[, $K\underline{\nabla}(y-u) = [k_{12},k_{22}]^T$ and from (6) (i), $k_{22} > 0$ so that the maximum must occur on $\partial\Omega$ -]AB[.

Reduction of the problem to a quasi-variational inequality.

We now introduce an extension of u to $\bar{D}$ by setting

$$\tilde{u}(x,y) = \begin{cases} u(x,y) & (x,y) \in \bar{\Omega} \\ y & (x,y) \quad \bar{D} - \bar{\Omega} \end{cases} \tag{20}$$

Lemma 2 $\tilde{u} \in H^1(D) \cap C^o(\bar{D})$

Proof. See Baiocchi [3]

Lemma 3 $\underline{\nabla}\cdot \underline{K}\underline{\nabla}\tilde{u} = -\frac{\partial}{\partial x}(k_{12}\chi_\Omega) - \frac{\partial}{\partial y}(k_{22}\chi_\Omega)$ in the sense of distributions on D, where χ_Ω is the characteristic function of Ω, i.e.

$$\chi_\Omega = \begin{cases} 1 & (x,y) \in \Omega \\ 0 & \text{elsehwere} \end{cases} \tag{21}$$

<u>Proof</u>. $$\left\langle \underset{\sim}{\nabla}\cdot K\underset{\sim}{\nabla}\tilde{u},\ \psi \right\rangle = -\int_D K\underset{\sim}{\nabla}\tilde{u}\cdot\nabla\psi\,dxdy$$

$$= -\int_\Omega K\underset{\sim}{\nabla}u\cdot\underset{\sim}{\nabla}\psi\,dxdy - \int_{D-\Omega} K\underset{\sim}{\nabla}y\cdot\underset{\sim}{\nabla}\psi\,dxdy$$

$$= -\int_{D-\Omega} (k_{12}\psi_x + k_{22}\psi_y)\,dxdy \qquad \text{(from equation (19))}$$

$$= -\int_D \chi_{D-\Omega}(k_{12}\psi_x + k_{22}\psi_y)\,dxdy$$

$$= \left\langle \frac{\partial}{\partial x}(k_{12}\ \chi_{D-\Omega}) + \frac{\partial}{\partial y}(k_{22}\ \chi_{D\backslash\Omega}),\ \psi \right\rangle$$

$$= \left\langle -\ \frac{\partial}{\partial x}(k_{12}\ \chi_\Omega) + \frac{\partial}{\partial y}(k_{22}\ \chi_\Omega),\ \psi \right\rangle$$

□

Now we can introduce a change of unknown functions by setting

$$w(x,y) = \int_{-\ y/k_{22}}^{0} \{y+k_{22}t - \tilde{u}(x+k_{12}t,\ y+k_{22}t\}dt \tag{22}$$

We introduce the following region [see figs. 3,4]

$$D_4 = \{(x,y) \in D : 0<y<Y(x);\ \frac{k_{12}}{k_{22}}(y-y_1) < x < \frac{k_{12}}{k_{22}}(y-y_2) + C\} \tag{23}$$

and impose the following condition on $y = Y(x)$.

$$Y'(x) \neq \frac{k_{22}}{k_{12}} \qquad x \in (a,b) \tag{24}$$

[This has the effect of restricting the geometry of the dam for the use of this method with certain values of the parameters m, n, θ].

Note that by introducing a change of variable in equation (22) we can write w in the following form.

$$w(x,y) = \frac{1}{k_{22}} \int_0^y \tau - \tilde{u}(x + \frac{k_{12}}{k_{22}}(\tau-y), \tau)\, d\tau \qquad (25)$$

Because $\tilde{u} \in H^1(D)$ we can differentiate this to obtain the following relation

$$k_{12} \frac{\partial w}{\partial x} + k_{22} \frac{\partial w}{\partial y} = y - \tilde{u}(x,y) \qquad (26)$$

therefore from w we can recover $\tilde{u}$.

Theorem 1 $\{\emptyset, \Omega, u\}$ is a weak solution of the problem if and only if $w(x,y)$ satisfies.

$$w|_{AB} = 0;\ k_{12}w_x + k_{22}w_y|_{AF} = y-y_1;$$
$$k_{12}w_x + k_{22}w_y|_{BC} = y-y_2;\ k_{12}w_x + k_{22}w_y|_{FEC} = 0 \qquad (27)$$

$$w(x, Y(x)) \text{ is concave } \forall\ x \in [0,c] \qquad (28)$$

$$\underline{\nabla}\cdot K\underline{\nabla}w = \chi_{D\backslash D_4} + \chi_{D_4}\cdot H(w - B(w)) = \chi_\Omega \qquad (29)$$

where $H(\cdot)$ is the multivalued Heaviside distribution defined by

$$H(t) = \begin{cases} \{0\} & t<0 \\ \{h: 0\le h\le 1\} & t=0 \\ \{1\} & t>0 \end{cases} \qquad (30)$$

and $B(\cdot)$ is defined in the following remark.

Remark Because of relation (26) and definition (20) we can define

$$\Omega = (D-D_4) \cup \{(x,y) \in D_4 : k_{12}\frac{\partial w}{\partial x} + k_{22}\frac{\partial w}{\partial y} < 0\} \qquad (31)$$

so that on $D\backslash\Omega$ we have,

$$k_{12}\frac{\partial w}{\partial x} + k_{22}\frac{\partial w}{\partial y} = 0 \quad \forall (x,y) \in D\backslash\Omega \qquad (32)$$

Note that these relations imply that the function $w(x,y)$ at each point (x,y) is bounded below by the value of w on the boundary $y = Y(x)$ at the point determined by the intersection of the vector (k_{12}, k_{22}), passing through (x,y), with the curve $y = Y(x)$. We shall refer to this value as $B(w(x,y))$ for each point (x,y).

Proof of Theorem 1. (i) Assume that $\{\emptyset, \Omega, u\}$ is a weak solution of the problem, then (27) is self-evident from (20), (22) and (26). To show that (28) holds we must prove that $w''(x, Y(x)) \leq 0$ i.e. that $w'(x, Y(x))$ is decreasing in the x-direction. From (25) we have that

$$w'(x,Y(x)) = -\frac{1}{k_{22}}\int_0^{Y(x)} \tilde{u}_x\left(x + \frac{k_{12}}{k_{22}}(\tau - Y), \tau\right) d\tau \qquad (33)$$

Now if we look at the discharge $q(x)$ through any section of the dam in the direction (k_{12}, k_{22}) then we can define

$$q(x) = -\int_0^{Y(x)} \frac{(\underline{K}\nabla\tilde{u})\cdot(k_{22}, -k_{12})dy}{k_{22}}$$

$$= -\int_0^{Y(x)} \frac{(k_{11}k_{22} - k_{12}^2)}{k_{22}}\tilde{u}_x\, dy \qquad (34)$$

therefore $$w'(x,Y(x)) = \alpha\, q(x) \tag{35}$$

for a constant $\alpha > 0$

but condition (12) asserts that the function $q(x)$ is decreasing for $s<x<c$. In particular it is constant for $0<x<s$ and decreases along Γ_s (due to seepage). Therefore we have shown that condition (28) holds.

In order to show that equation (29) holds we apply the operator $k_{12}\frac{\partial}{\partial x} + k_{22}\frac{\partial}{\partial y}$ to the difference between the two extreme sides of equation (29) and from equation (26) and lemma 3 we have that

$$(k_{12}\frac{\partial}{\partial x} + k_{22}\frac{\partial}{\partial y})(\underline{\nabla}\cdot\underline{K}\nabla w - \chi_\Omega) = 0 \tag{36}$$

Therefore $$\underline{\nabla}\cdot\underline{K}\nabla w - \chi_\Omega = \text{constant} \tag{37}$$

along the vectors (k_{12},k_{22}).

In order to show that this constant is zero it is sufficient to note that $\tilde{u}$ is analytic in a neighbourhood of the base of the dam then we can explicitly calculate $\underline{\nabla}\cdot\underline{K}\nabla w - \chi_\Omega$ from relation (22) in order to obtain (29). Hence w satisfies the conditions of the theorem.

(ii) Let w satisfy the conditions of the theorem and define

$$\begin{aligned} \tilde{u} &= y - k_{12}\frac{\partial w}{\partial x} - k_{22}\frac{\partial w}{\partial y} \text{ in } D \\ u &= \tilde{u}|_\Omega . \end{aligned} \tag{38}$$

Boundary conditions (17) on u follow directly from relation (38), the definition of Ω(31) and conditions (27).

In order to prove (19) first note that $(x,y) \in D - \Omega$ (the 'dry' section of the dam) $k_{12}\frac{\partial w}{\partial x} + k_{22}\frac{\partial w}{\partial y} = 0$ from equation (32). We can also show that $k_{11}\frac{\partial w}{\partial x} + k_{12}\frac{\partial w}{\partial y} = g$, where g is a constant multiple of the discharge, in the following manner:

We apply the operator $k_{11}\frac{\partial}{\partial x} + k_{12}\frac{\partial}{\partial y}$ to the function w as defined in equation (25), we obtain

$$k_{11}\frac{\partial w}{\partial x} + k_{12}\frac{\partial w}{\partial y} = \frac{1}{k_{22}}\{-k_{11} + \frac{k_{12}^2}{k_{22}}\}\int_o^y \tilde{u}_x(x + \frac{k_{12}}{k_{22}}(\tau - y), \tau)d\tau$$

$(x,y) \in D-\Omega$. But using the fact that $\tilde{u} = y$ in $D-\Omega$ we have

$$k_{11}\frac{\partial w}{\partial x} + k_{12}\frac{\partial w}{\partial y} = \frac{1}{k_{22}}\{-k_{11} + \frac{k_{12}^2}{k_{22}}\}\int_o^{Y(x)} \tilde{u}_x(x + \frac{k_{12}}{k_{22}}(\tau - y), \tau)d\tau$$

= constant (by equation (34) and the remark following). From the above we have that $(x,y) \in D-\Omega$ $\frac{\partial w}{\partial x}$ = constant, $\frac{\partial w}{\partial y}$ = constant. Now

$$\int_\Omega K\underline{\nabla}u\cdot\underline{\nabla}\psi \, dxdy$$

$$= \int_\Omega (k_{11}u_x + k_{12}u_y)\psi_x + (k_{12}u_x + k_{22}u_y)\psi_y \, dxdy$$

$$= \int_\Omega [k_{11}(-k_{12}w_{xx} - k_{22}w_{xy}) + k_{12}(1-k_{12}w_{xy} - k_{22}w_{yy})]\psi_x$$

$$+ [k_{12}(-k_{12}w_{xx} - k_{22}w_{xy}) + k_{22}(1-k_{12}w_{xy} - k_{22}w_{yy})]\psi_y \, dxdy$$

(substituting for u from equation (38))

$$= \int_\Omega [k_{11}(-k_{12}w_{xx} - k_{22}w_{xy}) + k_{12}(k_{11}w_{xx} + k_{12}w_{xy})]\psi_x$$

$$+ [k_{12}(-k_{12}w_{xx} - k_{22}w_{xy}) + k_{22}(k_{11}w_{xx} + k_{12}w_{xy})]\psi_y \, dxdy$$

(using $\underline{\nabla}\cdot K\underline{\nabla}w = 1$ in Ω from equation (24))

$$= \int_{\Omega} (k_{12}^2 - k_{11}k_{22}) \{w_{xy}\psi_x - w_{xx}\psi_y\} \, dxdy$$

$$= (k_{11}k_{22} - k_{12}^2) \int_D w_{xx}\psi_x - w_{xy}\psi_x \, dxdy$$

(using w_x = constant, w_y = constant in $D-\Omega$)

$$= (k_{11}k_{22} - k_{12}^2) \int_D \frac{\partial}{\partial x}(w_x\psi_y) - \frac{\partial}{\partial y}(w_x\psi_x) \, dxdy$$

$$= (k_{11}k_{22} - k_{12}^2) \int_{\partial D} w_x(\psi_x dx + \psi_y dy)$$

= 0, because

$w = 0$ on $]AB[$ and $\psi = 0$ on $y = Y(x)$ □

Let us now introduce the following functionals which are an extension of those for the isotropic case.

$$a(u,v) = \int_D \{v_x[k_{11}u_x + (k_{12} - \frac{(k_{11}k_{22}-k_{12}^2)Y'}{(k_{22}-k_{12}Y')}) u_y] + v_y[(k_{12} + \frac{(k_{11}k_{22}-k_{12}^2)Y'}{(k_{22}-k_{12}Y')}) u_x + k_{22} u_y] - \frac{k_{22}(k_{11}k_{22}-k_{12}^2)}{(k_{22}-k_{12}Y')^2} Y''vu_y\} \, dxdy \quad u,v\in H^1(D) \tag{40}$$

$$L(v) = -\int_{D-D_4} v \, dxdy + \int_b^c \frac{k_{11}Y'^2-2k_{12}Y'+k_{22}}{k_{12}Y'-k_{22}} v(x,Y(x))(Y(x)-y_2) \, dx + \int_0^a \frac{k_{11}Y'^2-2k_{12}Y'+k_{22}}{k_{12}Y'-k_{22}} v(x,Y(x))(Y(x)-y_1) \, dx \tag{41}$$

$v\in H^1(D)$

$$j(z,v) = \int_{D_4} (v(x,y) - B(z(x,y)))^+ \; dxdy \qquad v,z\in H'(D) \tag{41}$$

where $B(z(x,y))$ is defined in the remark following the statement of theorem 1 and

$$f^+ = \begin{cases} f, & f>0 \\ 0, & f<0 \end{cases} .$$

<u>Lemma 4</u> $a(.,.)$ is coercive for all functions $v\in H^1(D)$, $v = 0$ on $]AB[$.

<u>Proof</u>. Let $v\in H^1(D)$, $v = 0$ on $]AB[$

$$a(v,v) = \int_D K\underline{\nabla}v \cdot \underline{\nabla}v dxdy$$

$$- \int_D \frac{k_{22}(k_{11}k_{22}-k_{12}{}^2)}{(k_{22}-k_{12}Y')^2} Y''vv_y \; dxdy .$$

The first part is coercive by the ellipticity of K, therefore, by conditions (6) on K we must show that

$$- \int_D Y''vv_y \; dxdy \geq 0$$

but
$$- \int_D Y''vv_y \; dxdy = -\tfrac{1}{2} \int_D Y'' \frac{\partial}{\partial y} (v^2) \; dxdy$$

$$= -\tfrac{1}{2} \int_a^b Y'' \int_0^{Y(x)} \frac{\partial(v^2)}{\partial y} \, dydx = -\tfrac{1}{2} \int_a^b Y''v^2(x,Y(x)) \; dx$$

and so by the concavity of Y the lemma is proved. □

Remark. $a(.,.)$ is one of the bilinear forms associated with the operator $-\underline{\nabla}\cdot K\underline{\nabla}(.)$ which contains $(k_{12}\frac{\partial}{\partial x} + k_{22}\frac{\partial}{\partial y})(.)$, the expression which occurs in the natural boundary conditions (27). It is also easily shown that $j(.,.)$ is a proper, lower semi-continuous convex functional.

Definition:

We have now from standard variational inequality theory (see Glowinski [7]).

Lemma 5 For any $z \in C^1(D)$, a unique $w = w(x,y;\, z) \in H^1(D)$ with $w|_{AB} = 0$ such that for any $v \in H^1(D)$ with $v|_{AB} = 0$, we have

$$a(w,\, v-w) + j(z,v) \geq j(z,w) + L(v-w) \tag{42}$$

Lemma 6 We can regard variational inequality (42) as a weak formulation of the problem:

Find $w \in H^1(D)$ such that

$$\underline{\nabla}\cdot K\underline{\nabla} w = \chi_{D-D_4} + \chi_{D_4} H(w-B(z)) \tag{43}$$

for a given z with w satisfying boundary conditions (27) and D_4 shown in Figures

Proof. See Baiocchi [1].

Because of lemma 5 it is now obvious that the solutions which we are seeking are the fixed points of the map, $z \to w_z$, i.e. we must solve:

Problem 3: Find $w \in H^1(D)$; $w|_{AB} = 0$ and

$$a(w,v-w) + j(w,v) \geq j(w,w) + L(v-w)$$
$$v \in H^1(D) \text{ with } v|_{AB} = 0 \tag{46}$$

This problem is a quasi-variational inequality and we can prove that there exists a maximum solution w_{max} and a minimum solution w_{min}. It can also be proved that $(w_{max}(x, Y(x)))'' \leq 0$ in $(0,c)$ while in general $(w_{min}(x, Y(x))) \nleq 0$ in $(0,c)$ (thus violating condition (28)) but however there does exist, in the family of solutions, a minimum member of the family satisfying (28). The proofs of these statements can be found in Baiocchi [1], but it is useful here to give a sketch of the proof of existence for a general quasi-variational inequality as the proof is of a constructive nature and the numerical techniques used in this paper are based on it; this will be found in the Appendix. We therefore have an existence theorem for weak solutions but uniqueness is, in general, an open problem.

Numerical Results

The numerical algorithm is based on the constructive nature of the existence proof (see Appendix), so that we have to solve a series of variational inequalities for each of the maximum and minimum solutions.

We first solve

$$\underline{\nabla} \cdot K \underline{\nabla}\, w^o = 0 \tag{47}$$

with the given boundary conditions (27). From this solution we calculate $D(w^o)$ as described in the Remark following the statement of

Theorem 1, we then solve the variational inequality with this value of $B(\cdot)$, to obtain w^1, calculate $B(w^1)$ and so on. This gives a numerical approximation to the maximum solution.

For the minimum solution we start by solving

$$\underline{\nabla}\cdot K\cdot\underline{\nabla}\, w_o = 1 \tag{48}$$

From this we obtain $B(w_o)$ which will in general not satisfy the concavity condition (28). Hence we adjust the values to the minimum piecewise linear concave envelope of $B(w_o)$ (see Figure 5) and then use this to obtain a new solution.

We can discretize the equations using either the finite difference or finite element methods combined with successive overrelaxation.

This means that in theory we have an outer iteration to approximate the sequence of solutions, and an inner iteration (SOR), however in practice [as in the simpler case of Baiocchi [1]] we can retain convergence by doing one SOR iteration and then recalculating $B(\cdot)$. This means that the cost is equivalent to solving one variational inequality for each of the maximum and minimum solutions.

It is more efficient to use finite differences than finite elements. Firstly, in terms of programming effort, because the forcing function changes discontinuously in the region D at a different place in each iteration. It is difficult to evaluate the load vector in the matrix equations, and this has to be re-evaluated at each iteration. Secondly, the algorithm takes on a simpler form using finite differences.

If $A = \{a_{ij}\}$ is the finite difference operator approximating the operator $-\underline{\nabla}\cdot K\underline{\nabla}(\cdot)$ then the SOR equations (with relaxation parameter ω) are:

$$a_{ii}w_i^{(m+1)} = \omega f_i + (1-\omega)\, a_{ii}w_i^{(m)} - \omega \sum_{j=1}^{i-1} a_{ij}w_j^{(m+1)}$$

$$- \omega \sum_{j=i+1}^{n} a_{ij}w_j^{(m)} - \omega H(w_i^{(m+1)} - p_i) \qquad (49)$$

Notice that this is a nonlinear equation for $w_i^{(m+1)}$. However it is equivalent to finding the value of t which gives the minimum of the following functional

$$J(t) = \tfrac{1}{2}a_{ii}t^2 + \omega(t-p_i)^+$$

$$- [\omega f_i + (1-\omega)\, a_{ii}w_i^{(m)} - \omega \sum_{j=1}^{i-1} a_{ij}w_j^{(m+1)} - \omega \sum_{j=i+1}^{n} a_{ij}w_j^{(m)}]t$$

$$(50)$$

which is easily calculable.

The functional J has the following form

$$J(t) = \tfrac{1}{2}at^2 + \lambda(t-p)^+ + bt \quad , \quad a, \lambda > 0 \qquad (51)$$

that is

$$J(t) = \begin{cases} \tfrac{1}{2}at^2 + (\lambda+b)t - p\lambda = J_+ & t \geq p \\ \tfrac{1}{2}at^2 + bt = J_- & t < p \end{cases} \qquad (52)$$

We shall show that this has a unique minimum. Obviously, as the two parts of the function are convex then there are only three possible places for local minima to occur, at the zeros of $J'(t)$ of which there are possibly two, corresponding to the zeros of J_+' and J_-', or at the point $t = p$.

Firstly we shall show that it is impossible, given values of a, λ, b, p. for J_+' and J_-' to be simultaneously zero.

$$\begin{aligned} J_+' &= at + (b+\lambda) \quad , \quad t > p \\ J_-' &= at + b \quad , \quad t < p \end{aligned} \tag{53}$$

Then, if there were two minima, they would occur at the points

$$\begin{aligned} t &= -\frac{(b+\lambda)}{a} \quad , \quad t > p \\ t &= -b/a \quad , \quad t < p \end{aligned} \tag{54}$$

but this implies

$$-\frac{b}{a} < -\frac{b}{a} - \frac{\lambda}{a} \tag{55}$$

i.e. $\frac{\lambda}{a} < 0$ which contradicts definition (51).

If the derivatives of J_+ and J_- do not have zeros within their range of definition then the minimum obviously occurs at $t = p$. Thus the minimum of J occurs at the point t_o given by

$$t_o = \begin{cases} -\frac{(b+\lambda)}{a} , & \text{if } -\frac{(b+\lambda)}{a} > p \\ -\frac{b}{a} , & \text{if } -\frac{b}{a} < p \\ p & \text{otherwise} \end{cases} .$$

Also note that it is only possible to use finite elements if the last term in equations (49) is $H(w_i^{(m)} - p_1)$. This has the effect of slowing the convergence of the sequence of solutions.

Appendix Sketch of existence theorem for solutions of quasi-variational inequalities.

Suppose that

V is a real Hilbert space defined over D (A1)

$a: V \times V \to R$ is a continuous bilinear, V-coercive form (A2)

We have two members v_1, $v_2 \epsilon V$ such that

$v_1 \leq v_2$ (pointwise) (A3)

We have a mapping $\emptyset: [v_1, v_2] \times V \to (-\infty, \infty]$

such that $u \epsilon V$ and such that $u \epsilon [v_1, v_2]$, $\emptyset(u, \cdot): V \to (-\infty, \infty]$ is convex, proper, and lower-semi-continuous, such that if $u_1 \leq u_2$ (pointwise) then $\emptyset(u_1, \cdot) \leq \emptyset(u_2, \cdot)$ (A4)

Let us look at the following problem:

Problem AI

Find $u \epsilon V$ such that

$$a(u, v-u) + \emptyset(u, v) \geq \emptyset(u, u) \quad v \epsilon V \tag{A5}$$

Find $u \epsilon V$ s.t. given $w \epsilon V$

$$a(u, v-u) + \emptyset(w, v) \geq \emptyset(w, u) \quad v \epsilon V \tag{A6}$$

for which we have existence and uniqueness Glowinski [7], and define a mapping $\Phi(w) = u_w$. We now have to look for fixed points of this mapping i.e.

find $u \epsilon V$ such that

$$\Phi(u) = u \tag{A7}$$

Denote by $\{U_m\}$ the sequence generated by

$$U_o : \underline{\nabla}\cdot K\underline{\nabla}U_o = 0 \quad \text{on the region D with the appropriate boundary condition} \tag{A8}$$

$$U_{m+1} = \Phi(U_m)$$

and similarly by $\{U^m\}$ the sequence

$$U^o : \underline{\nabla}\cdot K\underline{\nabla}U^o = 1 \quad \text{on the same region D with the same boundary conditions as for (A8)} \tag{A9}$$

$$U^{m+1} = \Phi(U^m)$$

It is then possible to prove, using a maximum principle argument, the following theorem [see Baiocchi [1]].

<u>Theorem</u>

$$U_m \le U_{m+1} \le U^o$$
$$U^m \ge U^{m+1} \ge U_o \tag{A10}$$

Therefore if we set

$$U_\infty = \lim_{m\to\infty} U_m \le U^o$$
$$U^\infty = \lim_{m\to\infty} U^m \ge U_o \tag{A11}$$

we can be assured that they both exist as we have bounded monotonic sequences. So we have a range of fixed points

$$U_\infty \le U \le U^\infty \tag{A12}$$

where U is a solution of Problem AI.

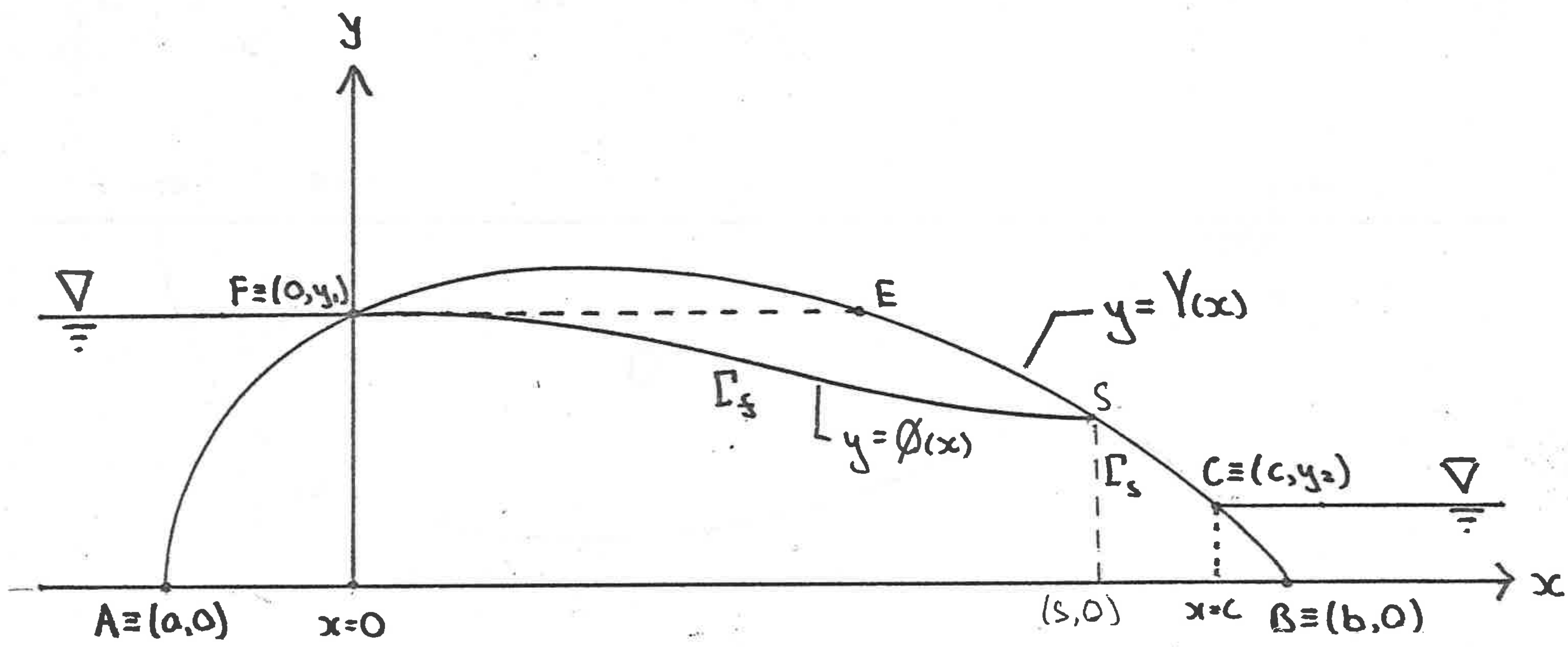


Fig 1 Geometrical Configuration of Dam

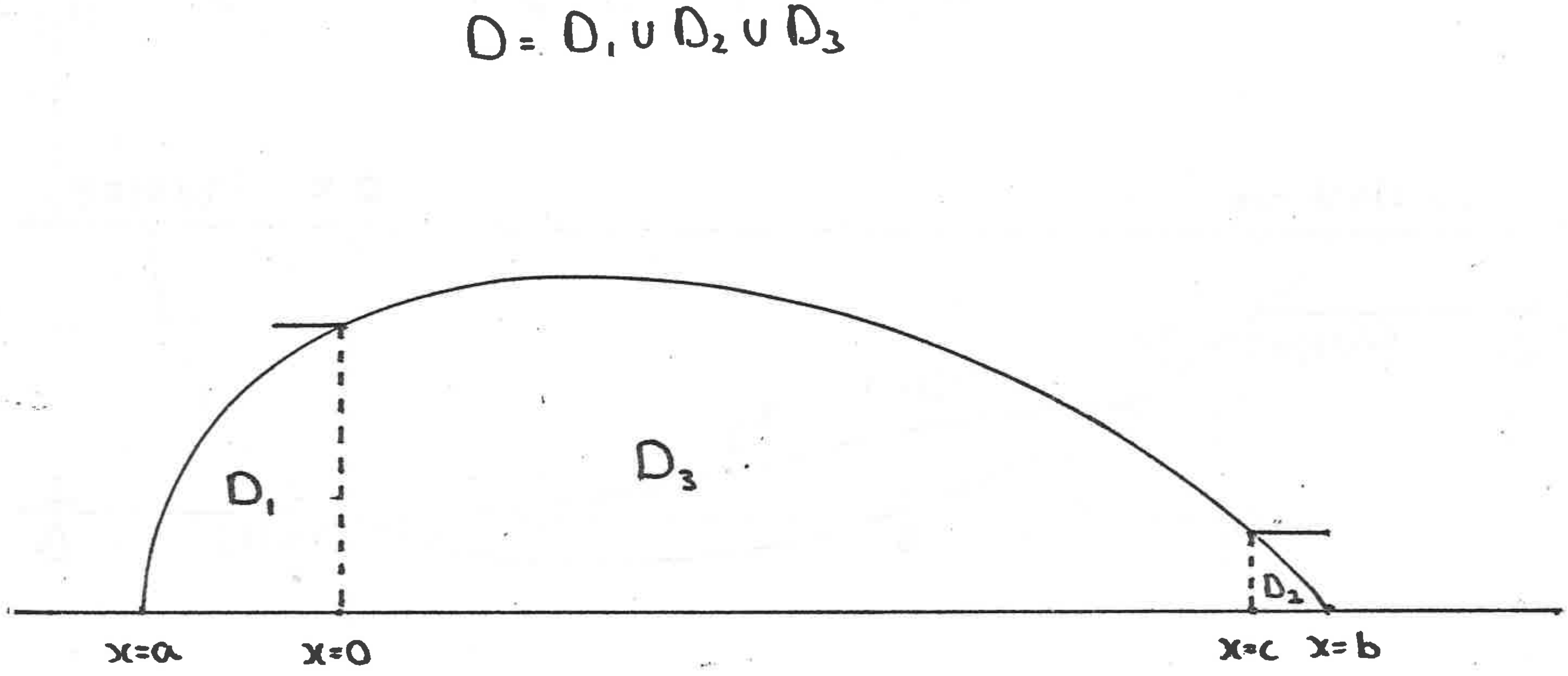
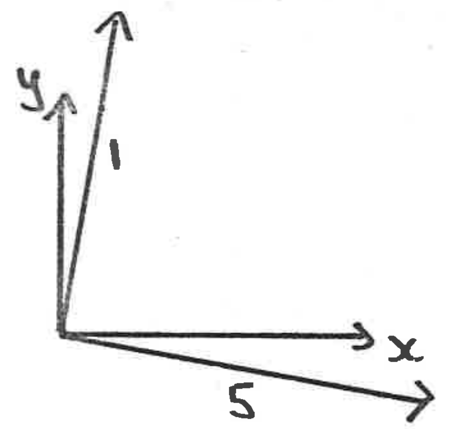


Fig 2. Flow Regions

Principal axes at $-\pi/18$ to horizontal, ratio of permeabilities = 5 : 1

$$K \equiv \begin{bmatrix} 1 & -0.140 \\ -0.140 & 0.230 \end{bmatrix}$$



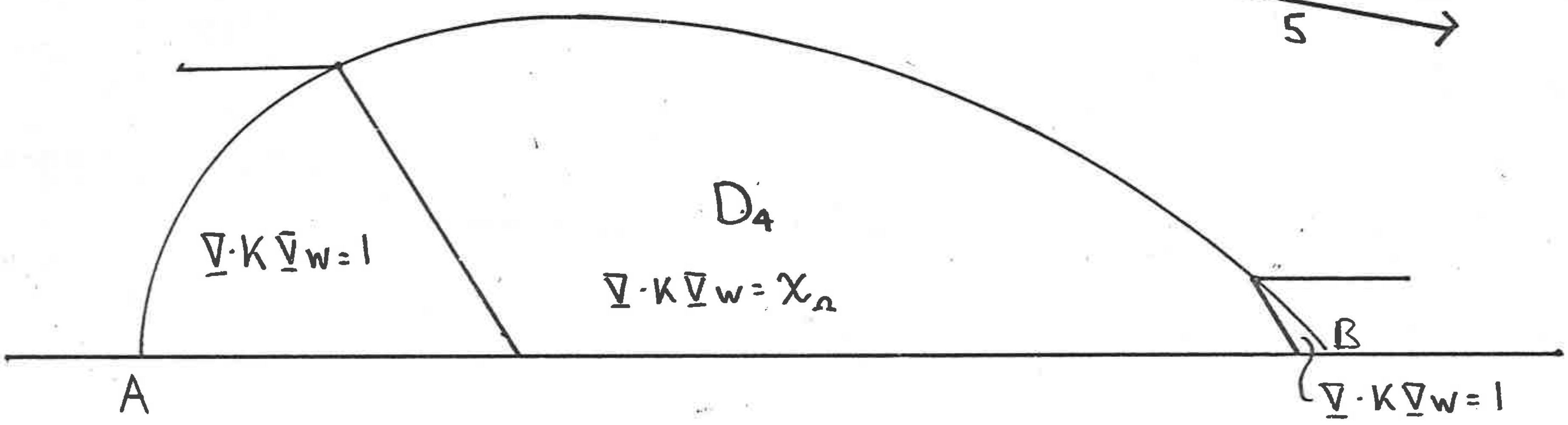


Fig 3 Region D_4 for given data

Principal axes at $-\pi/18$ to horizontal, ratio of permeabilities = 1:5

$$K \equiv \begin{bmatrix} 0.230 & 0.440 \\ 0.140 & 1 \end{bmatrix}$$

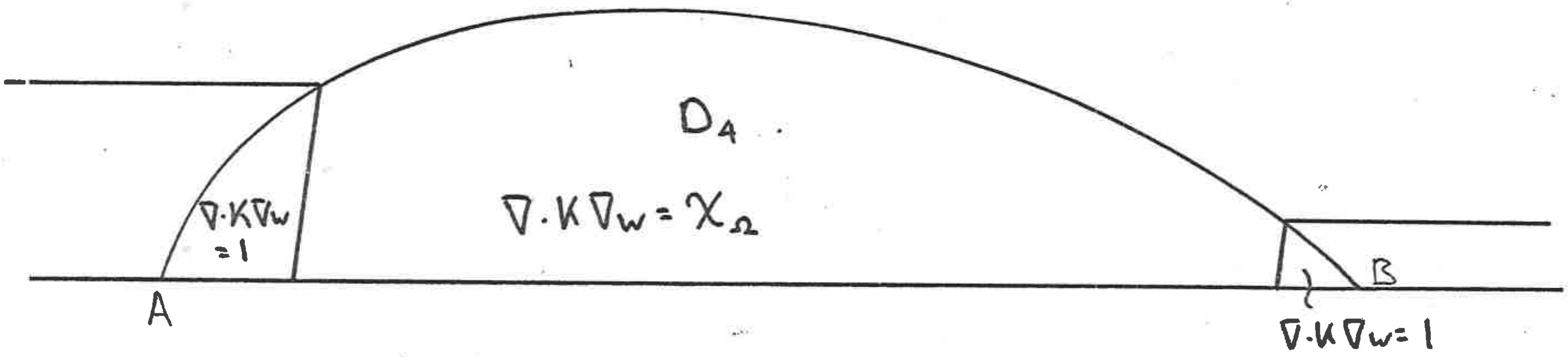


Fig 4 Region D_4 for given data

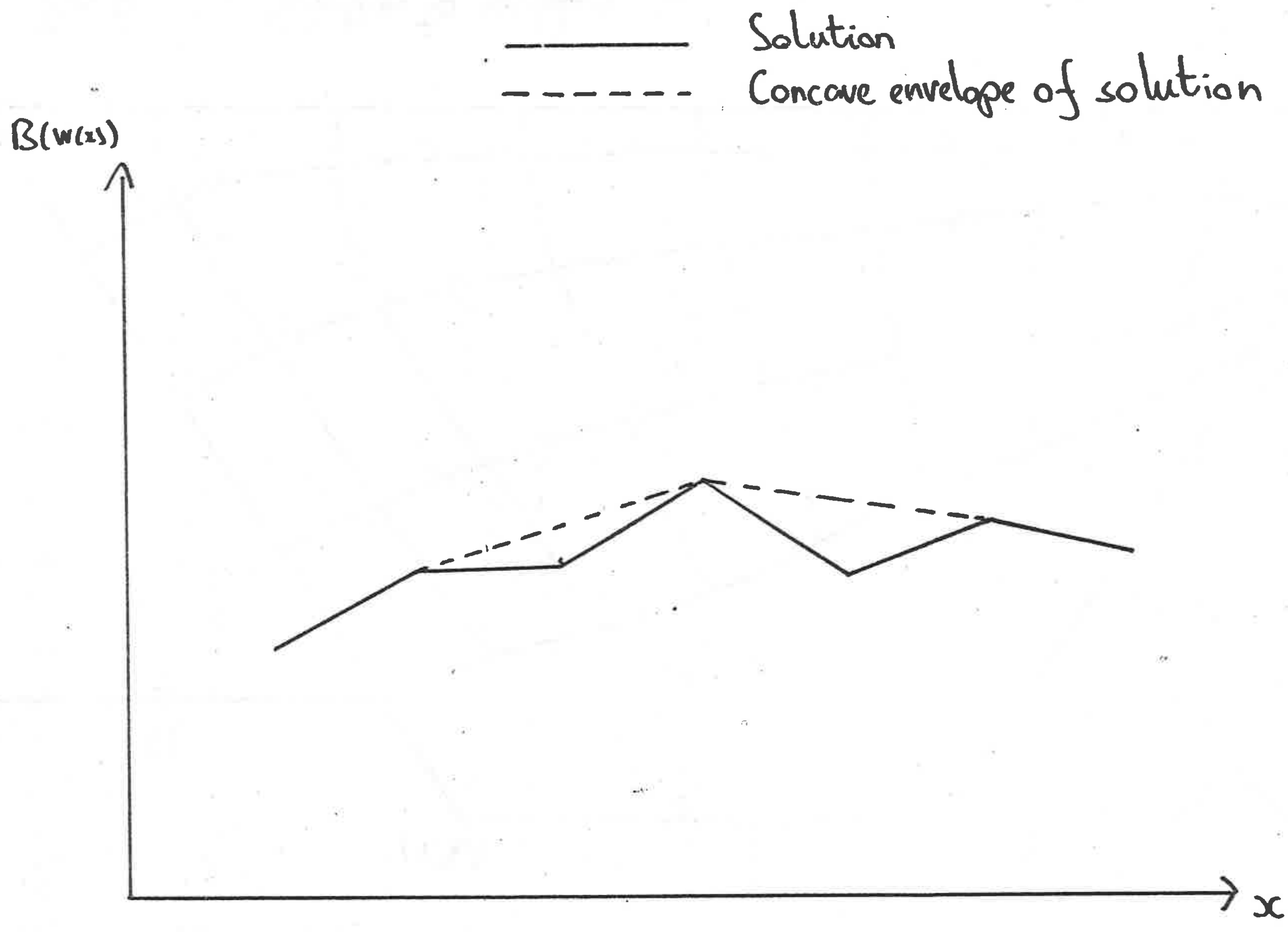


Fig 5 Numerical Concavization Procedure

Problem 1, $K = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ corresponding to principal directions at $\pi/4$ to horizontal,

ratio of permeabilities 5:1 , $y_1 = 1{\cdot}25$; $y_2 = 0{\cdot}25$

Number of Unknowns = 20 (5×5 elements)

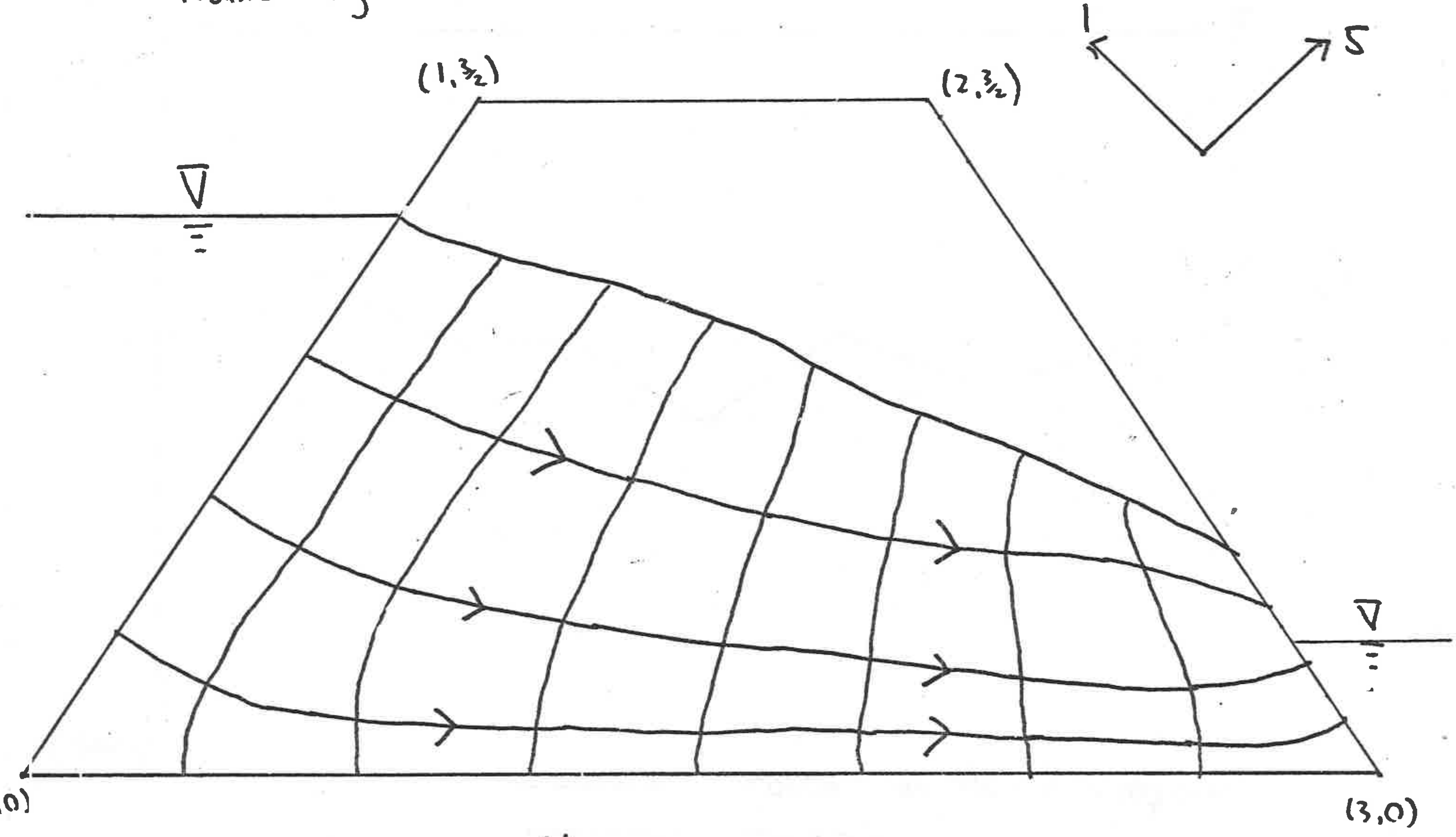


Fig 6 Solution to problem 1

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