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EQUIDISTRIBUTION AND THE LEGENDRE TRANSFORMATION

by

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U N I V E R S I T Y   O F   R E A D I N G

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§1. INTRODUCTION

In many computational situations an irregular grid is desirable. This is partly because resolution of local features of a solution can be obtained more economically with such a grid and partly because a regular grid for a complicated 3-D problem may be prohibitively expensive.

Many attempts at adapting the grid have been made. A favourite device is to ensure that a measure of the solution is equidistributed in each irregular interval. In particular, the second derivative of the solution is often chosen as the measure, on the grounds that if the approximation is piecewise linear, the local approximation error is a function of the second derivative.

In this report we investigate a method of generating an equidistributed second derivative by considering the Legendre Transformation of the underlying approximated function, as explained in §2 below. The benefit is that the corresponding equidistribution in the dual space is relatively trivial. Thus, as in other applications of this transformation, the problem is solved more easily in the dual space and the only difficulty is in transforming back.

In §2 we describe the procedure in detail and extend it to a more general equidistributing principle. In §3 the resulting approximations are compared with each other and with the results from equispaced grids. A number of different approximations arise from different ways of treating the dual function. We consider chordal, tangential and least squares approximations.

§2. EQUIDISTRIBUTION AND USE OF THE LEGENDRE TRANSFORMATION

Given a function u of x, it is possible to approximate the function by linear interpolation between a finite number of points of its graph in the x,u plane. For example this can be done with the x coordinates of the points equally spaced - an equispaced approximation. We seek a better approximation to the function by equidistributing the abscissae, in accordance with some rule, to give a new linear interpolation between points on the graph of the function at these abscissae.

It can be seen that an equispaced approximation tends to represent a function more accurately in regions of the graph where the function has smaller slope or smaller curvature. Concentrating abscissae in regions where the function has higher slope or higher curvature may create a better approximation to the function in terms of reducing some error measurement.

There are several ways to equidistribute points. The first method used here is, given a function, make the second derivative of the function integrated between each pair of adjacent abscissae equal a constant, i.e.

$$\int_{x_{i-1}}^{x_i} u''(x)dx = \text{constant} \qquad (i = 1,2,\ldots,n) \tag{2.1}$$

where $u(x)$ is the function to be approximated and the x_i $(i = 0,1,\ldots,n)$ are to be the abscissae of the interpolation points. This should cause a clustering of the x_i in regions where $u''(x)$ is large.

Solving for the x_i $(i = 0,1,\ldots,n)$ is done using the Legendre

Transform. Consider the space with coordinates (m,v) - dual to the space with coordinates (x,u) , defined as follows. Let $u(x)$ be the function to be approximated in the (x,u) plane. In the dual space this becomes $v(m)$ where, if

$$m = u'(x) \tag{2.2}$$

can be solved for x as a function of m ,

$$v(m) = m\,x(m) - u(x(m)) \tag{2.3}$$

is a function of m in the m,v plane and is the dual of the function $u(x)$. Using $m = u'(x)$ equation (2.1) becomes

$$\text{constant} = \int_{x_{i-1}}^{x_i} u''(x)dx = \int_{x_{i-1}}^{x_i} m'(x)dx = m(x_i) - m(x_{i-1}) \quad (i = 1,2,\dots,n). \tag{2.4}$$

So, to equidistribute the abscissae x_i $(i = 0,1,\dots,n)$ as in equation (2.1) it is simply necessary to equispace the m coordinate in the m,v plane, then transform back to give the x_i .

The trial functions are all to be approximated on the interval [0,1] and are strictly increasing functions of x so that

$$m_i = m_o + \frac{i}{n}(m_n - m_o) \qquad (i = 0,1,\dots,n) \tag{2.5}$$

where

$$m_o = \min_{x\epsilon[0,1]} u'(x) \tag{2.6}$$

$$m_n = \max_{x\epsilon[0,1]} u'(x) . \tag{2.7}$$

Note that the m_i $(i = 0,1,\ldots,n)$ in (2.5) is forced to be an increasing sequence in i .

It is a property of the Legendre Transform that the tangent to a point on a curve in one space transforms to a point on the transformed curve in the dual space. The tangents to the graph of $v(m)$ at the $n + 1$ points given by (2.5), transform to $n + 1$ points in the x,u plane as follows

$$x_i = v'(m_i) \tag{2.8}$$

$$u_i = x_i m_i - v(m_i) . \tag{2.9}$$

This yields a chordal type or interpolation approximation (see fig. 1).

The approximation to the function, $u(x)$, in the interval $[x_{i-1},x_i]$ $(i = 1,2,\ldots,n)$ is given by

$$u_i^h(x) = \frac{u_i-u_{i-1}}{x_i-x_{i-1}} (x - x_i) + u_i \qquad x_{i-1} \leq x \leq x_i . \tag{2.10}$$

Using this method it is also possible to obtain another, tangential, type approximation to $u(x)$ where the approximation meets the function at a tangent. To generate the $n+1$ abscissae x_i $(i = 0,1,\ldots,n)$ it is necessary to equispace the m coordinates:

$$m_i = m_o + \frac{i}{n-1}(m_n - m_o) \quad (i = 0,1,\ldots,n-1) \tag{2.11}$$

with m_o, m_n as given in (2.6) and (2.7), and to form

$$v_i = v(m_i) \quad (i = 0,1,\ldots,n-1). \tag{2.12}$$

Then the approximation points (x_i, u_i) $(i = 1,2,\ldots,n-1)$ may be found by making a chordal approximation to the function $v(m)$ at the points (m_i, v_i) $(i = 0,1,\ldots,n-1)$. In fact x_i is the slope of the chord in the interval $[m_{i-1}, m_i]$ and u_i is the negative of the intercept of this chord with the line $m = 0$,

$$x_i = \frac{v_i - v_{i-1}}{m_i - m_{i-1}} \quad (i = 1,2,\ldots,n-1) \tag{2.13}$$

$$u_i = x_i m_i - v_i \quad (i = 1,2,\ldots,n-1). \tag{2.14}$$

To ensure that $u(x)$ is approximated over the whole interval $[0,1]$, the points (x_o, u_o), (x_n, u_n) are defined as follows:

$$x_o = 0 \text{ , } u_o = u(x_o) \tag{2.15}$$

$$x_n = 1 \text{ , } u_n = u(x_n) \text{ .} \tag{2.16}$$

Then, with these x_i, u_i $(i = 0,1,\ldots,n)$, a tangential approximation to $u(x)$ may be created as in (2.10) (see fig. 2).

A further possibility is to generate a best least squares linear approximation

$$v^h(m) = x_i m - u_i \tag{2.17}$$

to the transformed function $v(m)$ in each equispaced interval (m_{i-1}, m_i) by minimising

$$||v(m) - v^h(m)||_2 \tag{2.18}$$

where x_i, u_i are constants (eventually the required points in the x,u plane). To obtain x_i and u_i, minimise the element error

$$||v(m) - x_i m + u_i||_2 \tag{2.19}$$

over x_i and u_i, giving the normal equations

$$\left.\begin{aligned} \int_{m_{i-1}}^{m_i} (v(m) - x_i\, m + u_i)dm = 0 \\ \int_{m_{i-1}}^{m_i} (v(m) - x_i\, m + u_i)mdm = 0 \end{aligned}\right\} \tag{2.20}$$

Thus x_i, u_i satisfy

$$\begin{bmatrix} \frac{1}{3}(m_i^3 - m_{i-1}^3) & -\frac{1}{2}(m_i^2 - m_{i-1}^2) \\ \\ \frac{1}{2}(m_i^2 - m_{i-1}^2) & -(m_i - m_{i-1}) \end{bmatrix} \begin{bmatrix} x_i \\ \\ u_i \end{bmatrix} = \begin{bmatrix} \int_{m_{i-1}}^{m_i} mv(m)dm \\ \int_{m_{i-1}}^{m_i} v(m)dm \end{bmatrix} \qquad (2.21)$$

The line $v^h(m)$ derived in this manner is a better local fit to $v(m)$ than either the chordal approximation in the same interval or a tangential approximation based on that interval and therefore is expected to give a better x_i than either of these (see fig. 3).

However, this last approximation is discontinuous and does not preserve the equispaced m values. The intersections of the approximating lines give new m values, even though they are close to the old ones.

A best least squares approximation which preserves the m values is a continuous piecewise linear approximation, of the form

$$v^h(m) = \sum v_i^h \phi_i(m) \qquad (2.22)$$

where

$$\phi_i(m) = \begin{cases} \dfrac{m_{i+1} - m}{m_{i+1} - m_i} & m_i \leq m \leq m_{i+1} \\ \dfrac{m - m_{i-1}}{m_i - m_{i-1}} & m_{i-1} \leq m \leq m_i \end{cases} \qquad (2.23)$$

and v_i^h are coefficients determined by minimising

$$||v(m) - v^h(m)||_2 \qquad (2.24)$$

over the v_i^h . This gives the normal equations

$$\sum_j \left[\int \phi_i(m)\ \phi_j(m)\ dm \right] v_j^h = \int \phi_i(m)\ v(m)\ dm \qquad \forall_i\ . \qquad (2.25)$$

The boundary conditions are $v'(m_o) = x_o$, $v'(m_n) = x_n$. When these equations have been solved for v_i^h the x-coordinates are given by

$$x_i = \frac{v_{i+1}^h - v_i^h}{m_{i+1} - m_i}\ . \qquad (2.26)$$

The above procedures give four possible equidistributions of a finite number of abscissae derived from the belief that if equation (2.1) is true for all x_i $(i = 0,1,\ldots,n)$ then this will yield a more accurate approximation to a function than merely equispacing the abscissae $\hat{x}_i$ and interpolating.

Carey and Dinh [1] showed that the criterion for equidistributing abscissae to minimise the L_2 error between a function and its linear interpolation is, asymptotically,

$$\int_{\hat{x}_{i-1}}^{\hat{x}_i} [u'']^{2/5}\ dx = \text{constant} \qquad (i = 1,2,\ldots,n). \qquad (2.27)$$

This yields a different set of abscissae $\hat{x}_i$ $(i = 0,1,\ldots,n)$ but we can construct them as follows.

Again, put $\hat{x}_o = 0$. The $\hat{x}_i$ $(i = 1,2,\ldots,n)$ are found by altering the length of each interval $[x_{i-1}, x_i]$ $(i = 1,2,\ldots,n)$ where the x_i are one of the sets calculated above. This is done by taking

$(x_i - x_{i-1})^{2/5}$ and scaling it by a factor α so that $\hat{x}_n = 1$,

$$\hat{x}_i = \alpha(x_i - x_{i-1})^{2/5} + \hat{x}_{i-1} \qquad (i = 1,2,\ldots,n) \ . \qquad (2.28)$$

So there are four more possible equidistributions of abscissae to use in the search for a good linear approximation to a function using a given, finite number of nodes.

§3. RESULTS AND DISCUSSION

Sets of points (x_i, u_i) $(i = 0,1,\ldots,n)$ corresponding to linear approximations of several trial functions are calculated as in §2. In particular those points giving both chordal and tangential approximations in the following categories of abscissae distribution are calculated: i) equispaced, ii) equidistributed as in equation (2.1) and iii) equidistributed as in equation (2.27). In addition, sets of points equidistributed using continuous and discontinuous best fit approximations in the dual space are also found. These give linear approximations in the form of equation (2.10).

The accuracy of these approximations may be measured using the L_2 error norm. If $u(x)$ is the function to approximated and $u^h(x)$ is the linear approximation the L_2 error is given by:

$$\text{error} = \left\{\int_0^1 (u(x) - u^h(x))^2 \, dx\right\}^{1/2} = \left\{ \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (u(x) - u_i^h(x))^2 dx\right\}^{1/2} . \qquad (3.1)$$

This may be calculated for various numbers of abscissae $n + 1$.

To provide a comparison of the overall accuracy of the approximations one further approximation is considered. This is the best least squares fit using piecewise linear continuous functions and is given by equation (2.25) with v,m and u,x interchanged, the x abscissae being equally spaced. By its derivation this approximation will have the minimum L_2 error measurement of any continuous linear approximation on the equally spaced grid. Comparison with the L_2 error measurements of the other approximations will give an indication of any improvement made by equidistributing the abscissae.

Results are shown in tables 1 and 2 for the trial functions

$u(x) = e^{-5x}$, $u(x) = e^{-8x}$ with up to 40 points.

Comparison of L_2 errors for various approximations

KEY to Tables 1 and 2

a) 'equispaced' tangent approximation

b) equidistributed tangent approximation :

$$\int_{x_{i-1}}^{x_i} u'' dx = \text{constant} \quad (i = 1,2,\ldots,n)$$

c) equispaced chordal approximation

d) equidistributed chordal approximation :

$$\int_{x_{i-1}}^{x_i} u'' dx = \text{constant} \quad (i = 1,2,\ldots,n)$$

e) equispaced local best fit approximation

f) equidistributed chordal approximation :

$$\int_{x_{i-1}}^{x_i} (u'')^{2/5} dx = \text{constant} \quad (i = 1,2,\ldots,n)$$

g) equidistributed tangent approximation :

$$\int_{x_{i-1}}^{x_i} (u'')^{2/5} dx = \text{constant} \quad (i = 1,2,\ldots,n)$$

h) transform of local best fit approximation

i) transform of global best fit approximation

j) equispaced global best fit approximation.

n	a	b	c	d	e
5	1.58×10^{-2}	2.68×10^{-2}	2.73×10^{-2}	4.40×10^{-2}	1.11×10^{-2}
10	4.26×10^{-3}	9.81×10^{-3}	7.11×10^{-3}	1.68×10^{-2}	2.90×10^{-3}
20	1.09×10^{-3}	3.24×10^{-3}	1.80×10^{-3}	5.99×10^{-3}	7.34×10^{-4}
40	2.75×10^{-4}	1.17×10^{-3}	4.51×10^{-4}	1.95×10^{-3}	1.84×10^{-4}

n	f	g	h	i	j
5	1.31×10^{-2}	1.34×10^{-2}	1.43×10^{-2}	1.59×10^{-2}	1.18×10^{-2}
10	3.37×10^{-3}	2.61×10^{-3}	4.82×10^{-3}	5.33×10^{-3}	2.97×10^{-3}
20	8.57×10^{-4}	5.91×10^{-4}	1.74×10^{-3}	1.90×10^{-3}	7.39×10^{-4}
40	2.16×10^{-4}	1.41×10^{-4}	6.38×10^{-4}	6.78×10^{-4}	1.84×10^{-4}

Table 1

L_2 errors for $u(x) = e^{-5x}$

n	a	b	c	d	e
5	9.24×10^{-2}	2.70×10^{-2}	5.12×10^{-2}	6.90×10^{-2}	2.07×10^{-2}
10	2.88×10^{-2}	1.17×10^{-2}	1.41×10^{-2}	3.09×10^{-2}	5.74×10^{-3}
20	6.94×10^{-3}	5.34×10^{-3}	3.62×10^{-3}	1.35×10^{-2}	1.48×10^{-3}
40	1.47×10^{-3}	2.44×10^{-3}	9.11×10^{-4}	5.77×10^{-3}	3.72×10^{-4}

n	f	g	h	i	j
5	1.91×10^{-2}	1.51×10^{-2}	2.45×10^{-2}	2.69×10^{-2}	2.29×10^{-2}
10	5.27×10^{-3}	3.45×10^{-3}	9.28×10^{-3}	1.02×10^{-2}	6.02×10^{-3}
20	1.39×10^{-3}	8.80×10^{-4}	3.69×10^{-3}	4.08×10^{-3}	1.50×10^{-3}
40	3.58×10^{-4}	2.22×10^{-4}	1.47×10^{-3}	1.63×10^{-3}	3.74×10^{-4}

Table 2

L_2 errors for $u(x) = e^{-8x}$

The chordal and tangential approximations with abscissae equidistributed as in equation (2.1) have larger L_2 errors than the corresponding approximations with equispaced abscissae. Thus, for these functions at least, equidistribution according to equation (2.1) does not improve the approximation to the function as measured by this error norm. Indeed, there is some evidence that this equidistribution "overdoes" the distortion since taking an average between these equidistributed points and the equispaced points does quite well.

However, the approximations using equation (2.27) to equidistribute the abscissae have smaller L_2 errors than the equispaced approximations. For the two functions considered these errors are close to those of the continuous best fit approximation, and indeed the tangential approximation has smaller errors in both cases. This may, in some way, be as a result of the trial functions used. For the most extreme function, $u(x) = e^{-8x}$, the chordal approximation equidistributed in this way also has smaller errors than the continuous best fit approximation.

Equidistribution by performing Legendre Transformations on the continuous and discontinuous linear best fits in the dual space also leads to approximations with L_2 errors larger than those of the approximations with abscissae equidistributed as in equation (2.27).

Thus, for the functions considered here, the tangential approximation with abscissae equidistributed using equation (2.27) gives the smallest L_2 error of all the approximations calculated. It may be that one of the other approximations gives a better representation of a function by reducing the error as measured using a different norm.

We have also calculated the convergence rates of the approximations. The convergence rate for equidistributed points based on the Legendre Transformation is only about 2½ , whereas the rate for equispaced points and equidistributed points using the "⅖ rule" is nearer to 4, as expected.

§4 CONCLUSIONS

It is possible to find a linear approximation to a function with a smaller L_2 error than that of a simple chordal or tangential approximation between points on the graph of the function at equally spaced abscissae. This may be done by equidistributing the abscissae - using the Legendre Transformation and equation (2.27) - then forming a linear approximation. It appears that, contrary to widely held belief, an equidistribution using equation (2.1) does not reduce the error of the linear approximation as measured by the L_2 norm.

§5. ACKNOWLEDGEMENT

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§6. REFERENCES

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