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Weak Non-Symmetric Shocks in

One-Dimension

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Numercial Analysis Report 4/88

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Contents

Abstract

This paper contains a mathematical analysis of the structure of weak shocks with the presence of viscosity. A simple one-dimensional model is explored and an attempt is made to fit the structure of such shocks as the theoretical background to computational applications using an extension of the shock propogation conditions currently used.

1. Description

Before any theory, it is necessary to define the variables used and the concepts of a weak shock and a non-symmetric shock.

1.0 Preliminary Definitions

x - space co-ordinate.

u - fluid velocity.

ρ - density.

p - pressure.

a - sound speed.

ρ_o, p_o, a_o - still air values of ρ, p, a respectively.

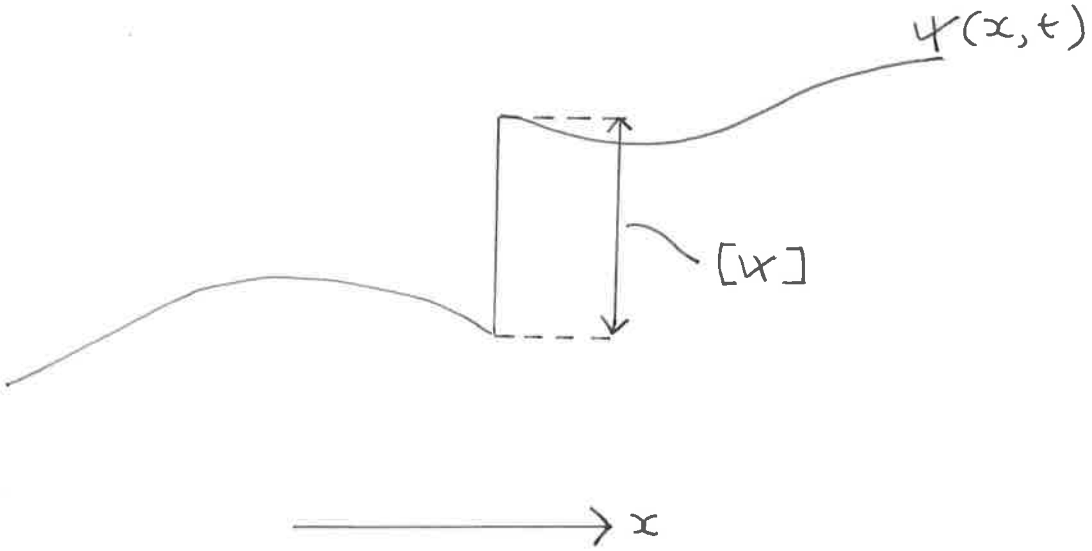
t - time.

δ - diffusivity of air[1].

γ - ratio of specific heats.

1.1 Introduction

Let ψ be a generalised thermodynamic quantity. Let $[\psi]$ be the jump in ψ over the discontinuity (a shock).



Let $\langle \psi \rangle$ be a scale value for ψ .

Hence

$$\langle u \rangle = a_o$$

$$\langle \rho \rangle = \rho_o$$

$$\langle p \rangle = p_o$$

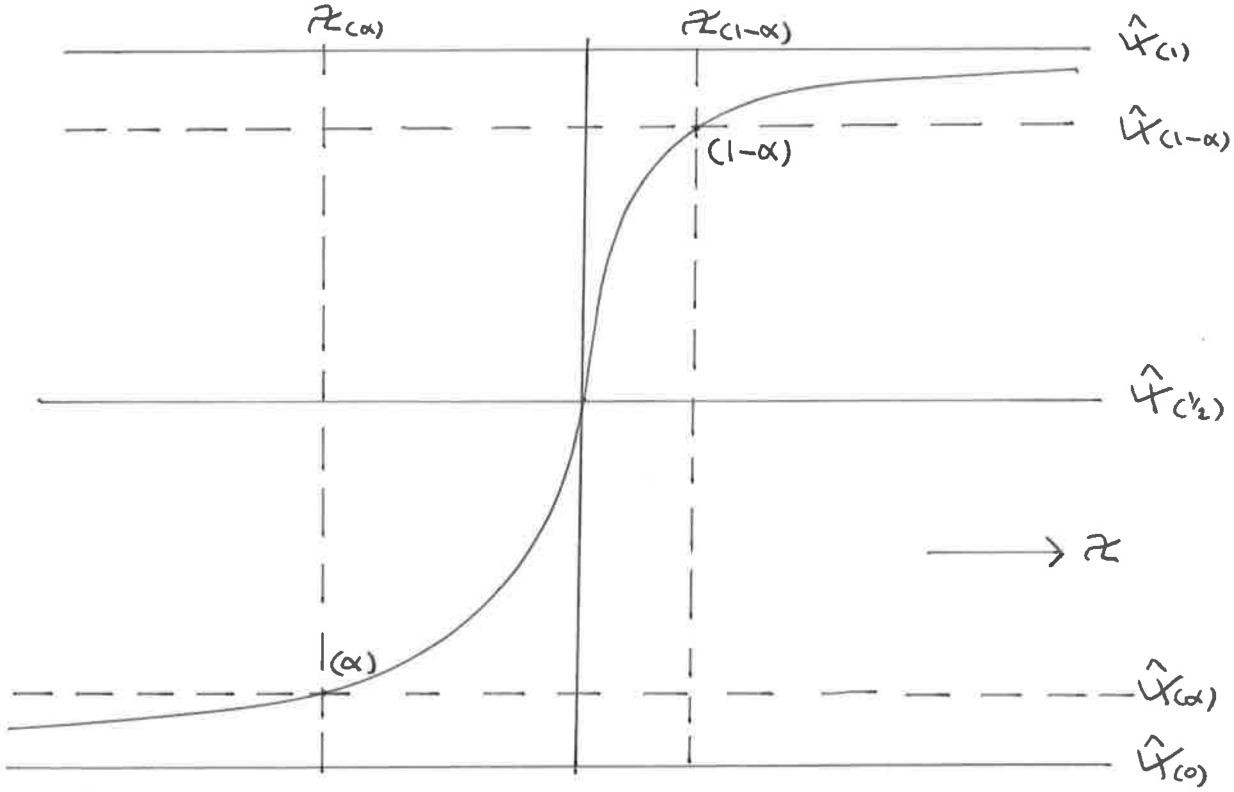
$$\langle a \rangle = a_o$$

Then, we define weak shocks as those with the property

$$\left|\frac{[\,\psi\,]}{\langle\,\psi\,\rangle}\right| << 1 \tag{1.1}$$

Now, this definition needs to be relaxed to include non-discontinuous, approximate shocks; and for this we need to introduce new notation.

1.2 Non-symmetric, Asymptotic Functions and (α), $(1-\alpha)$ Points.



Firstly, let $\hat{\psi}_{(0)}$, $\hat{\psi}_{(1)}$ be 2 values of $\hat{\psi}$ with $\hat{\psi}_{(0)} < \hat{\psi}_{(1)}$.

Define

$$\forall\ \lambda\epsilon(0,1)\ ,\quad \hat{\psi}_{(\lambda)} = \lambda\hat{\psi}_{(1)} + (1-\lambda)\,\hat{\psi}_{(0)} \tag{1.2}$$

Consider a function $\hat{\psi}$ of a variable χ. Let $\hat{\psi}(\chi)$ have the following properties:

(i) $\left.\frac{d\hat{\psi}}{d\chi} > 0 \;\forall\; \chi\right\} \Rightarrow \hat{\psi}(\chi)$ is strictly monotonic increasing (1.3)

(ii) $\lim_{\chi \to -\infty} \hat{\psi} = \hat{\psi}_{(0)}$

(iii) $\lim_{\chi \to +\infty} \hat{\psi} = \hat{\psi}_{(1)}$

$\Rightarrow \hat{\psi}(\chi)$ is asymptotic to $\hat{\psi} = \hat{\psi}_{(0)}$, $\hat{\psi}_{(1)}$ as $\chi \to -\infty$, ∞ respectively (1.4)

(iv) $\left.\chi = 0 \Leftrightarrow \hat{\psi} = \hat{\psi}_{(\frac{1}{2})} = \frac{1}{2}(\hat{\psi}_{(0)} + \hat{\psi}_{(1)})\right\} \Rightarrow \chi$ is normalised. (1.5)

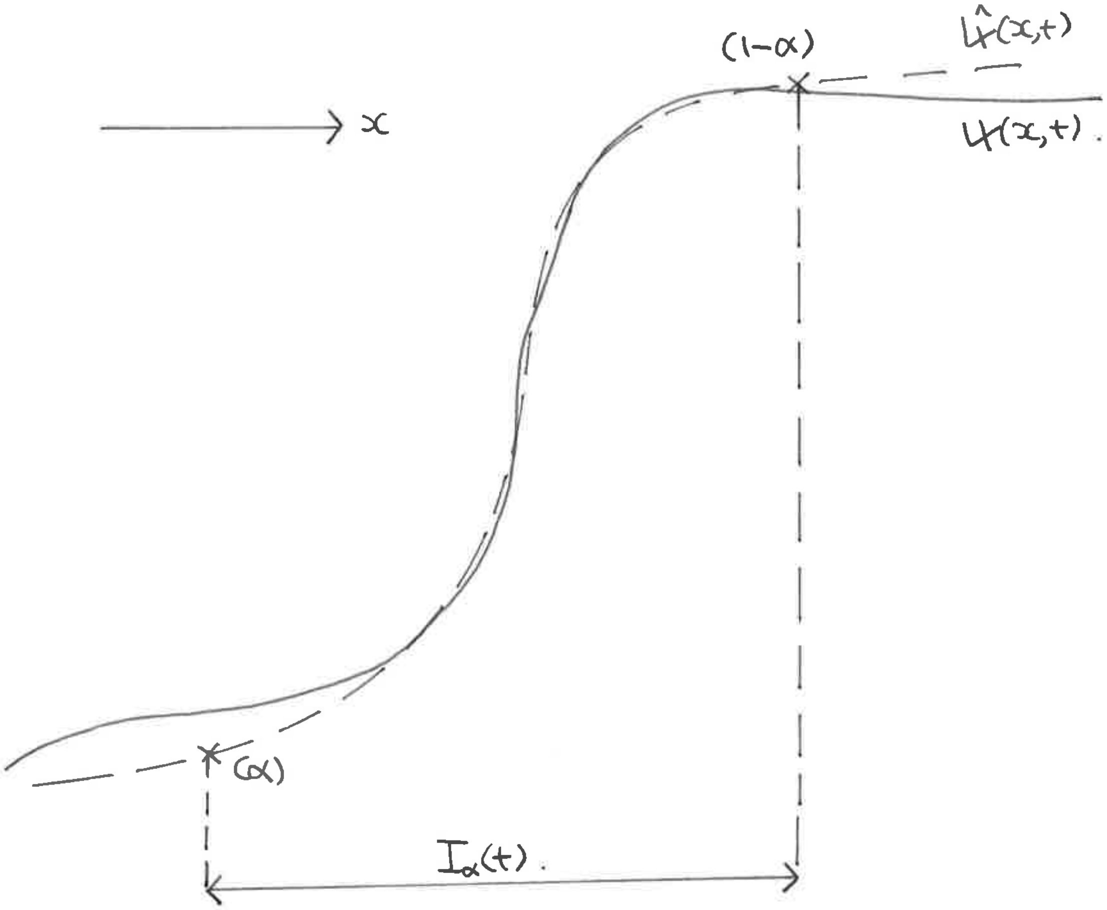
(v) $\forall\; \alpha \in (0,\frac{1}{2})$,

$$\left.\begin{array}{l} \hat{\psi} = \hat{\psi}_{(\alpha)} \Leftrightarrow \chi = \chi_{(\alpha)} \\ \hat{\psi} = \hat{\psi}_{(1-\alpha)} \Leftrightarrow \chi = \chi_{(1-\alpha)} \end{array}\right\}$$ - defines the (α), $(1-\alpha)$ points of $\hat{\psi}$ and χ. (1.6)

$\hat{\psi}(\chi)$ is a non-symmetric asymptotic function in the sense that

$|\chi_{(1-\alpha)}| \neq |\chi_{(\alpha)}|$ for a general point $\alpha \in (0,\frac{1}{2})$. (1.7)

1.3 Curve-fitting for Continuous Approximate Shocks



It seems sensible to fit $\psi(x,t)$ with our curve-fitting function $\hat{\psi}(x,t)$ within an interval $I_\alpha(t)$, where the ends of $I_\alpha(t)$ correspond to the (α), $(1-\alpha)$ points of $\hat{\psi}(\chi) \equiv \hat{\psi}(x,t)$. Introduce a mid-shock variable $\xi(t)$ defined by

$$x = \xi(t) \Leftrightarrow \hat{\psi} = \hat{\psi}_{(\frac{1}{2})} \quad \forall t \tag{1.8}$$

Let $\chi = x - \xi(t)$ (1.9)

Then $I_\alpha(t) = [\chi_{(\alpha)}, \chi_{(1-\alpha)}]$ (1.10)

Now this definition of $\hat{\psi}(\chi)$ includes the assumption that the approximate shock is fitted with a function with constant asymptotes $\hat{\psi}_{(0)}$, $\hat{\psi}_{(1)}$. Hence a more general definition of $\hat{\psi}$ is

$$\hat{\psi} = \hat{\psi}(\chi;\ \hat{\psi}_{(0)}(t),\ \hat{\psi}_{(1)}(t)) \tag{1.11}$$

From now on, the dependence of $\hat{\psi}_{(0)}$, $\hat{\psi}_{(1)}$ will be kept implicit until a distinction needs to be made.

1.4 Relaxation of the Weak Shock Condition to Continuous Approximate Shocks

Let $\psi(x,t)$ be fitted by $\hat{\psi}(\chi)$, corresponding to ψ having an approximate continuous shock (a-c shock). This a-c shock is considered weak if

$$\left.\begin{array}{l} 0 < \alpha << \tfrac{1}{2}\ , \quad \text{and} \\ \left| \dfrac{\hat{\psi}_{(1)} - \hat{\psi}_{(0)}}{\langle \psi \rangle} \right| << 1 \end{array}\right\} \tag{1.12}$$

The modulus sign here is unnecessary as $\hat{\psi}_{(1)} > \hat{\psi}_{(0)}$ ($\langle\psi\rangle > 0$ is assumed), but we can generalise the definition of the curve-fitting function $\hat{\psi}$ to strictly monotonic functions (i.e. not necessarily increasing), by making $\hat{\psi}_{(1)} < \hat{\psi}_{(0)}$ and $d\hat{\psi}/d\chi < 0$.

Also, $\hat{\psi}_{(1)} - \hat{\psi}_{(0)}$ can be represented by a jump notation with the definition

$$[\hat{\psi}]_{\alpha} = \hat{\psi}_{(1-\alpha)} - \hat{\psi}_{(\alpha)} \qquad \forall\ \alpha \in [0,\tfrac{1}{2}) \tag{1.13}$$

Then, as $\alpha \ll \frac{1}{2}$ in (1.12) we can write it as

$$\left.\begin{aligned} &0 < \alpha \ll \tfrac{1}{2}\,, \quad \text{and} \\ &\left|\frac{[\hat{\psi}]_\alpha}{\langle \psi \rangle}\right| \ll 1 \end{aligned}\right\} \qquad (1.14)$$

(c.f. equation (1.1)).

1.5 Definition of Shock Width

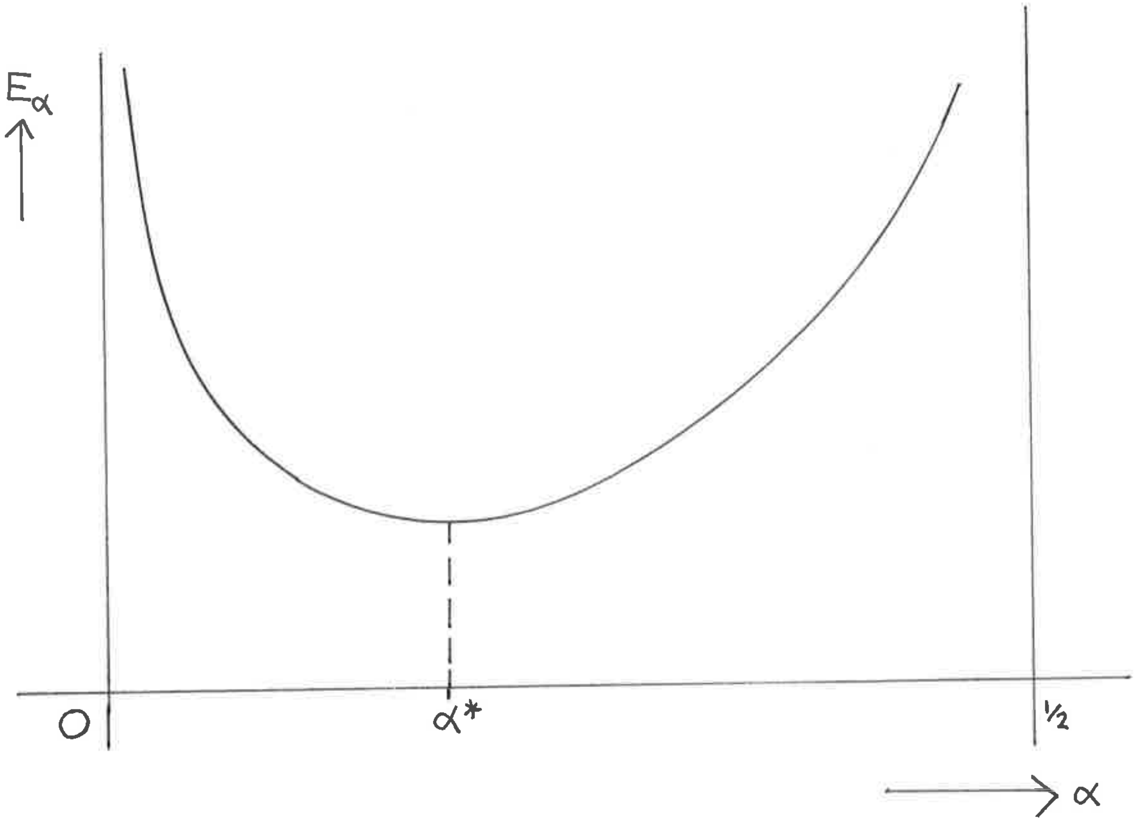
Given an a-c shock in $\psi(x,t)$, there exists a family of functions $\hat{\Psi} = \{\hat{\psi}^{(\alpha)} : \alpha \in (0,\frac{1}{2})\}$ which could be used to approximate ψ, where $\hat{\psi}$ is approximated over the region I_α by the function $\hat{\psi}^{(\alpha)}$. Within this family $\hat{\Psi}$, there exists a 'best-fit' function $\hat{\psi}^{(\alpha^*)}$, where 'best-fit' is defined in some way to make α^* unique. Then the shock-width of $\psi(x,t)$ is defined as λ_{α^*}, where

$$\lambda_{\alpha^*} = |x_{(1-\alpha^*)} - x_{(\alpha^*)}| \,. \qquad (1.15)$$

Notation for α^* :

Let $E(\psi, \hat{\psi}^{(\alpha)})$ be the error measure for $\hat{\psi}^{(\alpha)}$. Let $E_\alpha = E(\psi, \hat{\psi}^{(\alpha)})$. Then

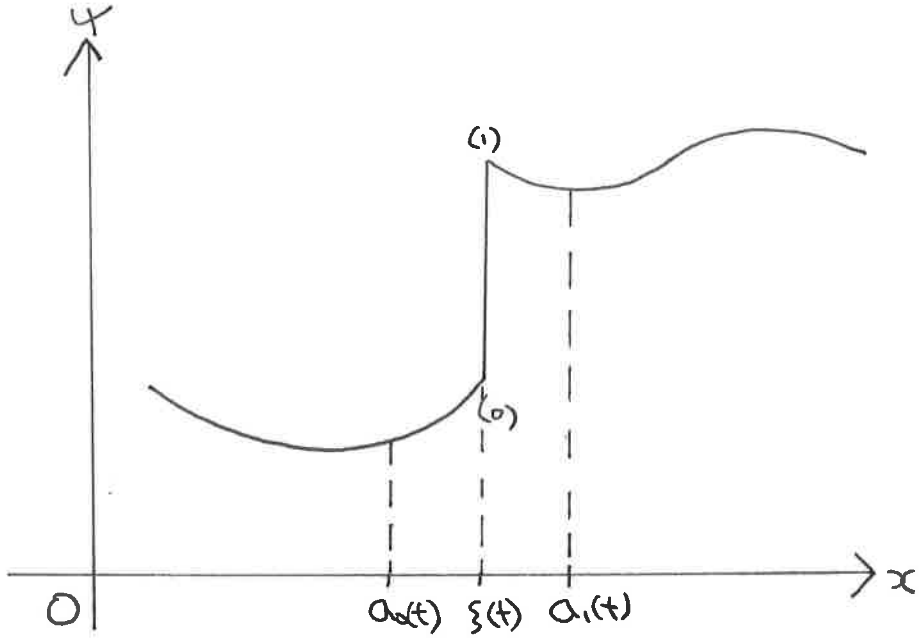
$$\alpha^* : E_{\alpha^*} = \min_{\alpha \in (0,\frac{1}{2})} \{E_\alpha\} \,. \qquad (1.16)$$



E_α
O
α^*
1/2
α

2. Generalisation of the Rankine-Hugoniot Jump Conditions to A-C Shocks.

2.1 Resumé of Jump Conditions for Discontinuous Shocks[2].



The conservation of mass, momentum and energy, and the increase or conservation of entropy can all be decribed by considering a jump quantity

$$J = \int_{a_o(t)}^{a_1(t)} \psi(x,t)\, dx \ , \tag{2.1}$$

where $\xi(t)$, the shock co-ordinate satisfies

$$\xi(t) \in [a_o(t) \ , \ a_1(t)] \tag{2.2}$$

$\psi(x,t)$ is again a generalised thermodynamic quantity.

$$\text{Then} \quad \frac{dJ}{dt} = \int_{a_o(t)}^{a_1(t)} \frac{\partial\psi}{\partial t}(x,t) + \psi_o\dot{\xi}(t) - \psi(a_o,t)\, u(a_o,t)$$

$$+ \psi(a_1,t)\, u(a_1,t) = \psi_1\dot{\xi}(t)$$

where $\psi_o = \psi(\xi^-,t)$

$\psi_1 = \psi(\xi^+,t)$

Let $\dot{\xi}(t) = U(t)$ – the shock speed (2.3)

Let $v_i = u_i - U \quad , \quad i = 0,1$ (2.4)

Write $\frac{dJ}{dt}$ as $\dot{J}(\psi)$

Then, $\lim_{(a_1-a_o)\to 0} \dot{J}(\psi) = \psi_1 v_1 - \psi_o v_o = [\psi v]$ (2.5)

The four thermodynamic laws apply $\forall\, a_o(t)\ ,\ a_1(t)$ provided (2.2) holds, and are

i) Conservation of mass:

$$\dot{J}(\rho) = 0 \tag{2.6}$$

ii) Conservation of momentum:

$$\dot{J}(\rho u) = -\, [p]_{x=a_o}^{x=a_1} \tag{2.7}$$

iii) Conservation of energy:

$$\dot{J}(\rho\{½u^2 + e\}) = -[pu]_{x=a_0}^{x=a_1} \tag{2.8}$$

iv) Increase or conservation of entropy:

$$\dot{J}(\rho S) \geq 0 \tag{2.9}$$

where S is entropy and e is internal energy.

2.2 Derivation of Shock Speed

From (2.5) and (2.6)

$$[\rho v] = 0 \Rightarrow [\rho(u-U)] = 0$$

$$\Rightarrow [\rho u] - U[\rho] = 0$$

$$\Rightarrow U = \frac{[\rho u]}{[\rho]} \tag{2.10}$$

There are also two other derivations using (2.7) and (2.8), but these do not hold for a viscous fluid; whereas (2.6) does.

2.3 The Generalised Jump Conditions

2.3a Steady Flow

As viscous affects are being considered, of the 4 relations in (2.1), only conservation of mass is important.

$$\left.\begin{array}{l} \text{steady flow} \Rightarrow \rho = \rho(\chi) \\ \qquad\qquad\qquad\quad u = (\chi) \end{array}\right\} . \tag{2.11}$$

The continuity equation is

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u)$$

So in steady state,

$$\frac{\partial\rho}{\partial t} = \frac{d\rho}{d\chi}\frac{\partial\chi}{\partial t} = -U\frac{d\rho}{d\chi} \qquad \text{(U must be constant).}$$

$$\frac{\partial}{\partial\chi}(\rho u) = \frac{d}{d\chi}(\rho u)\frac{\partial\chi}{\partial x} = \frac{d}{d\chi}(\rho u)$$

$$\Rightarrow \quad -U\frac{d\rho}{d\chi} = -\frac{d}{d\chi}(\rho u)$$

$$\Rightarrow \quad \frac{d}{d\chi}(\rho v) = 0 \qquad v = u-U$$

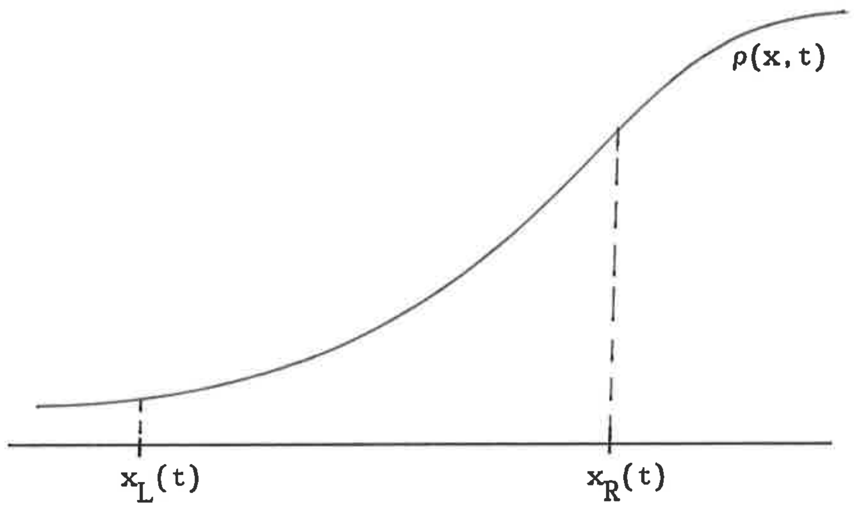
$$\left.\begin{array}{ll} \Rightarrow \quad \rho v = \rho_o v_o & (\rho_o,\ v_o \ \text{constants}) \\ \Rightarrow \quad \rho(u-U) = \rho_o(u_o - U) & (U_o \ \text{constant}) \end{array}\right\} \tag{2.12}$$

$$\Rightarrow \quad U = \frac{\rho_o u_o - \rho u}{\rho_o - \rho} \tag{2.13}$$

2.3b <u>Unsteady Flow</u>

Consider an interval $I_x(t)$ in x with the properties:

$$\left.\begin{array}{l} I_x(t) = [x_L(t)\ ,\ x_R(t)] \\ \dot{x}_L(t) = U(t) \\ \dot{x}_R(t) = U(t) \end{array}\right\} \tag{2.14}$$



Let M_I be the mass in this interval, so

$$M_I(t) = \int_{x_L}^{x_R} \rho \, dx \; . \tag{2.15}$$

By considering the mass flow at the two ends we obtain:

$$M_I(t + \delta t) - M_I(t) = (u_L - U)\delta t \; \rho_L - (u_R - U)\delta t \; \rho_R + O(\delta t^2)$$

$$\Rightarrow \quad \frac{dM_I}{dt} = -\,(v_R \rho_R - v_L \rho_L) = -\,[v\rho]_L^R \; , \quad \text{say} \tag{2.16}$$

But (2.15) can be written as

$$M_I(t) = \int_{\chi_L}^{\chi_R} \rho \, d\chi$$

$$\chi_R = x_R - \xi(t)$$

$$\chi_L = x_L - \xi(t) \; .$$

Noting $\dot{x}_R = \dot{x}_L = 0$, and writing $\rho = \rho(\chi, t)$, we obtain by differentiation;

$$\frac{dM_I}{dt} = \int_{x_L}^{x_R} \frac{\partial\rho}{\partial t}(\chi, t)\Big|_{\chi} d\chi .$$

Hence by (2.16) we obtain

$$\int_{x_L}^{x_R} \frac{\partial\rho}{\partial t}(\chi, t)\Big|_{\chi} d\chi = U(t)\ [\rho]_L^R - [u\rho]_L^R$$

$$\Rightarrow \quad U(t) = \frac{1}{[\rho]_L^R}\left\{[u\rho]_L^R + \int_{x_L}^{x_R} \frac{\partial\rho}{\partial t}(\chi, t)\Big|_{\chi} d\chi\right\} \tag{2.17}$$

This result is true $\forall(x_R(t), x_L(t)) \equiv \forall(x_R, x_L)$. As we approach steady state, the second term in the bracket tends to zero relative to the first.

This result has no meaning unless $U(t)$ is defined more clearly, i.e. the evaluation of U involves an arbitrary interval with ends that move with speed $U(t)$.

The t-dependence of ρ independent of χ may be expressed in a general form as

$$\rho = \rho(\chi, \underline{\phi}) ,$$

where $\underline{\phi} = (\phi_1, \ldots, \phi_n) = \underline{\phi}(t)$

The analogue of (2.17) is then clearly

$$U(t) = \frac{1}{[\rho]_L^R} \left\{ [u\rho]_L^R + \sum_{i=1}^{n} \dot{\phi}_i \int_{\chi_L}^{\chi_R} \frac{\partial \rho}{\partial \phi_i} (\chi, \underline{\phi}) \Big|_{\chi, \phi} d\chi \right\} \qquad (2.18)$$

This relates with (1.11) when

$$\phi_1(t) = \hat{\psi}_{(0)}(t)$$

$$\phi_2(t) = \hat{\psi}_{(1)}(t)$$

3. An Example of Weak Non-Symmetric Shock Theory.

3.1 Resumé of M. Lighthill's Work on Weak Shocks in 1D

After careful argument, Lighthill arrives at the system of differential equations:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{2}{\gamma-1}\, a\, \frac{\partial a}{\partial x} = \delta\, \frac{\partial^2 u}{\partial x^2} \\ &\frac{\partial a}{\partial t} + u\, \frac{\partial a}{\partial x} + \frac{\gamma-1}{2}\, a\, \frac{\partial u}{\partial x} = 0 \end{aligned}\right\}, \qquad (3.1)$$

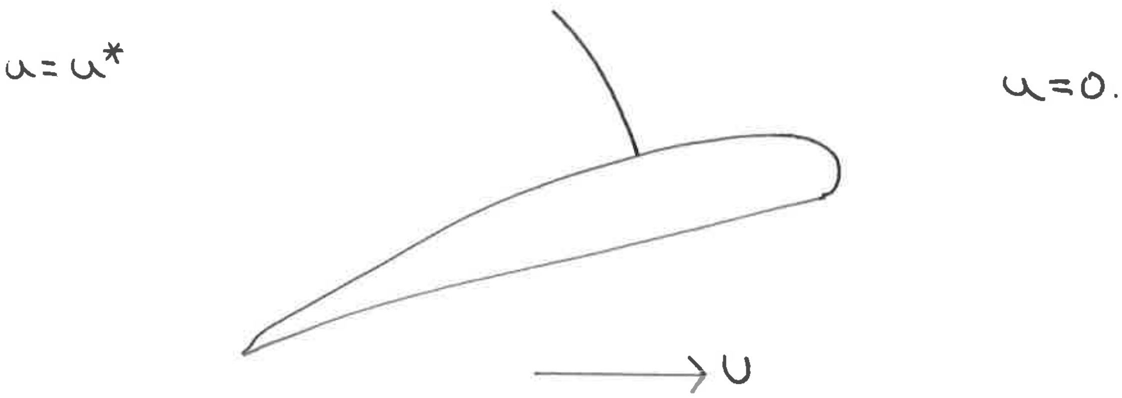
where

$a = a_o\left(\frac{\rho}{\rho_o}\right)^{\frac{\gamma-1}{2}}$ is the local sound speed

a_o is the still - air sound speed

ρ_o is the still - air density, and

$\delta = \frac{1}{\rho_o}\left\{\frac{4}{3}\mu_o + \mu_{vo} + \frac{(\gamma-1)k_o}{c_p}\right\}$ is the 'diffusivity' of sound, which resembles kinematic viscosity



For a steady shock, $\psi = \psi(\chi)$, where $\chi = x - Ut$ is the modified x co-ordinate and $\psi = u , a , \rho$ etc.

For a 1-D analogy to a steady shock on an aeroplane wing,

$$\left.\begin{array}{lll} \lim\limits_{\chi\to\infty} u = 0 & \Rightarrow & \lim\limits_{\chi\to\infty} a = a_o , \quad \text{etc.} \\ \lim\limits_{\chi\to-\infty} u = u^* & & \end{array}\right\} \tag{3.2}$$

Let $\psi' = \frac{d\psi}{d\chi}$, $\psi = u , a , \rho , \ldots$. Then (3.1) becomes

$$- Uu' + uu' + \frac{2}{\gamma-1} aa' = \delta u''$$

$$- Ua' + ua' + \frac{\gamma-1}{2} au' = 0$$

Define $v = U - u$, which leads to:

$$vv' + \frac{2}{\gamma-1} aa' = - \delta v'' \tag{3.3}$$

$$- va' - \frac{\gamma-1}{2} av' = 0 \tag{3.4}$$

In order to solve (3.4), the weak shock condition is introduced:

$$0 < u^* \ll U \tag{3.5}$$

So, $v > 0$ in (3.4). Hence

$$(3.4) \Rightarrow \quad - va' = \frac{\gamma-1}{2} av'$$

$$\Rightarrow \quad \frac{a'}{a} = \left[\frac{\gamma-1}{2}\right] \frac{v'}{v}$$

$$\Rightarrow \quad \ln a = - \left[\frac{\gamma-1}{2}\right] \ln v + A \qquad (\text{as } v > 0)$$

$$\chi \to \infty \Rightarrow a \to a_o , \; v \to U$$

Hence $\ln a_o = -\left(\frac{\gamma-1}{2}\right) \ln U + A$

$$\ln\left(\frac{a}{a_o}\right) = -\left(\frac{\gamma-1}{2}\right) \ln\left(\frac{v}{U}\right)$$

$$\Rightarrow \quad \left(\frac{a}{a_o}\right) = \left(\frac{v}{U}\right)^{-\left(\frac{\gamma-1}{2}\right)} . \qquad (3.6)$$

Substitute for aa' in (3.3):

$$\frac{a'}{a_o} = U^{\frac{\gamma-1}{2}} \left(-\frac{\gamma-1}{2}\right) v' \; v^{-\left(\frac{\gamma-1}{2}\right)-1}$$

$$= -\left(\frac{\gamma-1}{2}\right) U^{\frac{\gamma-1}{2}} v^{-\left(\frac{\gamma+1}{2}\right)} v'$$

$$\Rightarrow \quad aa' = - a_o^2 \, U^{\gamma-1} \left(\frac{\gamma-1}{2}\right) v^{-\gamma} v' .$$

Hence (3.3) $\Rightarrow$ $vv' - a_o^2 \, U^{\gamma-1} v^{-\gamma} v' = -\delta v''$

Integrating w.r.t. v gives

$$\tfrac{1}{2} v^2 + \frac{a_o^2}{\gamma-1} U^{\gamma-1} v^{-(\gamma-1)} = -\delta v' + B$$

$$\lim_{\chi \to \infty} v' = 0 \, , \quad \lim_{\chi \to \infty} v = U$$

$$\Rightarrow \; B = \tfrac{1}{2} U^2 - \frac{a_o^2}{\gamma-1}$$

$$\Rightarrow \; -\delta v' = \tfrac{1}{2}(v^2 - U^2) + \frac{a_o^2}{\gamma-1} \left[\left(\frac{U}{v}\right)^{\gamma-1} - 1\right]$$

Re-substitute u for v :

$$2\delta\, u' = (U-u)^2 - U^2 + \frac{2a_o^2}{\gamma-1}\left[\left(\frac{U}{U-u}\right)^{\gamma-1} - 1\right] \qquad (3.7)$$

$$\Rightarrow\ -2\delta\, u' = 2Uu - u^2 - \frac{2a_o^2}{\gamma-1}\left[\left(\frac{U}{U-u}\right)^{\gamma-1} - 1\right] = f(u)$$

Let $u = \frac{u^*}{2}$ when $\chi = 0$

(N.B. could also have used $v = a$ when $\chi = 0$, which yields the same first order approximation).

$$\Rightarrow\ -2\delta \frac{du}{f(u)} = d\chi$$

$$\Rightarrow\ -2\delta \int_{u^*/2}^{u} \frac{d\hat{u}}{f(\hat{u})} = \int_0^{\chi} d\hat{\chi} = \chi \ .$$

$$\left.\begin{aligned} &\text{i.e. } \chi = -2\delta \int_{u^*/2}^{u} \frac{d\hat{u}}{f(\hat{u})} \ , \\ &\text{where } f(u) = 2Uu - u^2 - \frac{2a_o^2}{\gamma-1}\left[\left(\frac{U}{U-u}\right)^{\gamma-1} - 1\right] \end{aligned}\right\} . \qquad (3.8)$$

Now $u' = 0$ when $u = 0$ and $u = u^*$, so (3.7) yields the equation

$$f(u^*) = 0 \ ,$$

for U, i.e.

$$2Uu^* - u^{*2} - \frac{2a_o^2}{\gamma-1}\left[\left(\frac{U}{U-u^*}\right)^{\gamma-1} - 1\right] = 0 \qquad (3.9)$$

As we shall see, $U \simeq a_o$ for small u^* . So, introducing small variables

$$\phi = \frac{u}{U} ,$$

with

$$\phi^* = \frac{u^*}{U} ,$$

a first order approximation to U can be found from (3.9):

$$(3.9) \Rightarrow 2\phi^* - \phi^{*2} - \frac{2a_o^2}{(\gamma-1)U^2}\left[(1-\phi^*)^{-(\gamma-1)} - 1\right] = 0 .$$

Now, $(1-\phi^*)^{-(\gamma-1)} = 1 + (\gamma-1)\phi^* + \frac{\gamma(\gamma-1)}{2}\phi^{*2} + 0(\phi^{*3})$.

$$\text{So} \quad 2 - \phi^* - \frac{2a_o^2}{U^2}\left[1 + \frac{\gamma}{2}\phi^* + 0(\phi^{*2})\right] = 0$$

$$\Rightarrow U^2 = \frac{2a_o^2(1 + \frac{\gamma}{2}\phi^* + 0(\phi^{*2}))}{2 - \phi^*}$$

$$= \frac{a_o^2(1 + \frac{\gamma}{2}\phi^* + 0(\phi^{*2}))}{1 - \phi^*/2}$$

$$\Rightarrow U^2 = a_o^2\left[1 + \left[\frac{\gamma}{2} + \frac{1}{2}\right]\phi^* + 0(\phi^{*2})\right]$$

$$\Rightarrow U = a_o\left[1 + \left[\frac{\gamma+1}{4}\right]\phi^* + 0(\phi^{*2})\right] \qquad (3.10)$$

Also, f(u) may be approximated as follows:

$$f(u) = U^2\left[2\phi - \phi^2 - \frac{2a_o^2}{U^2}\,\phi\,\left[1 + \frac{\gamma}{2}\,\phi + O(\phi^2)\right]\right]$$

$$= U^2\phi\left[2\left(1 - \frac{a_o^2}{U^2}\right) - \phi\left(1 + \frac{\gamma a_o^2}{U^2}\right) + O(\phi^2)\right]$$

$$= U^2\phi\left[2\left(1 - \{1 - \left(\frac{\gamma+1}{2}\right)\phi^* + O(\phi^{*2})\}\right) - \phi(1 + \gamma + O(\phi^*)\right.$$

$$\left. + O(\phi^2)\right]$$

$$= U^2\phi\left[(\gamma + 1)\phi^* - (\gamma + 1)\phi + O(\phi^{*2})\right]$$

$$= (\gamma + 1)\; u(u^* - u) + O(\phi^* u^{*2})$$

Let the superscript $^{(n)}$ correspond to the n^{th} order approximation, hence

$$f^{(1)}(u) = (\gamma+1)\; u(u^* - u) \tag{3.11}$$

from (3.8):

$$\chi^{(1)} = -\,2\delta \int_{u^*/2}^{u} \frac{d\hat{u}}{f^{(1)}(\hat{u})}\,,$$

$$= -\,\frac{2\delta}{(\gamma+1)u^*} \int_{u^*/2}^{u} \left[\frac{1}{\hat{u}} + \frac{1}{u^* - \hat{u}}\right] d\hat{u}$$

$$= -\,\frac{2\delta}{(\gamma+1)u^*}\left[\ln\,\hat{u} - \ln\,(u^* - \hat{u})\right]_{u^*/2}^{u}$$

$$= - \frac{2\delta}{(\gamma+1)u^*} \left[\ln \left(\frac{u}{u^*-u} \right) - \ln \frac{u^*}{2} + \ln \frac{u^*}{2} \right]$$

$$\Rightarrow \chi^{(1)} = - \frac{2\delta}{(\gamma+1)u^*} \ln \left(\frac{u}{u^*-u} \right) . \qquad (3.12)$$

This can be converted into an equation for $u^{(1)}(\chi)$.

Let $\chi_o = - \dfrac{2\delta}{(\gamma+1)u^*}$, then (3.13)

$$\frac{u^{(1)}}{u^*-u^{(1)}} = e^{-\chi/\chi_o}$$

$$\Rightarrow u^{(1)}(1 + e^{-\chi/\chi_o}) = u^* e^{-\chi/\chi_o}$$

$$\Rightarrow u^{(1)} = \frac{u^*}{1+e^{\chi/\chi_o}}$$

$$= \frac{u^*}{2} \left(1 - \tanh \frac{\chi}{\chi_o} \right) . \qquad (3.14)$$

The first order shock width $\lambda_\alpha^{(1)}$ can be easily found:

$\lambda_\alpha^{(1)} = |\chi_{(1-\alpha)}^{(1)} - \chi_{(\alpha)}^{(1)}|$, as in (1.15), where

$$\chi_{(\mu)}^{(1)} = \chi^{(1)}(\mu u^*) \quad \forall \ \mu \in [0,1]$$

$$\lambda_\alpha^{(1)} = |\chi^{(1)}([1-\alpha]u^*) - \chi^{(1)} (\alpha u^*)|$$

$$= \chi_o \left| \ln\left[\frac{1-\alpha}{\alpha}\right] - \ln\left[\frac{\alpha}{1-\alpha}\right] \right|$$

$$= 2 \chi_o \ln\left[\frac{1-\alpha}{\alpha}\right]$$

$$= \frac{4\delta}{(\gamma+1)u^*} \ln\left[\frac{1-\alpha}{\alpha}\right] \qquad (3.15)$$

3.2) Calculation of $U^{(2)}$, $f^{(2)}$, $\chi^{(2)}$ and $\lambda_\alpha^{(2)}$

The 2^{nd} order calculations are important because they lead to non-symmetric curves and can be calculated analytically.

It is not necessary to calculate $U^{(2)}$ explicitly as the condition $f^{(2)}(u^*) = 0$ will yield the unknown coefficient. A more suitable variable change here is

$$\left.\begin{aligned} &\epsilon = \frac{u^*}{a_o} \;, \quad \text{and} \\ &\theta = \frac{u}{u^*} \quad \Rightarrow \quad \theta \in [o,1] \end{aligned}\right\} \tag{3.16}$$

Let $$U = a_o\left[1 + \frac{1}{4}(\gamma + 1)\epsilon + A\epsilon^2 + O(\epsilon^3)\right] \tag{3.17}$$

From now on, terms in ϵ^3 will be omitted, and the superfix $^{(2)}$ used. (3.17) is obtained from (3.10).

Now, $$\left[\left[\frac{U}{U-u}\right]^{(\gamma-1)} - 1\right] = (\gamma - 1)\phi\left[1 + \frac{\gamma}{2}\,\phi + \frac{\gamma(\gamma+1)}{6}\,\phi^2\right]$$

So the last term of $f^{(2)}$ is

$$-\frac{2a_o^2}{(\gamma-1)}\left[\left[\frac{U}{U-u}\right]^{(\gamma-1)} - 1\right] = -2\phi\, a_o^2\left[1 + \frac{\gamma}{2}\phi + \frac{\gamma(\gamma+1)}{6}\,\phi^2\right]. \tag{3.18}$$

But $\phi = \frac{u}{U} = \frac{u}{a_o}\left[1 + \frac{1}{4}(\gamma + 1)\epsilon + A\epsilon^2\right]^{-1}$.

Therefore the right hand side of (3.18) is

$$- 2a_o u\left[1 + \tfrac{1}{4}(\gamma + 1)\epsilon + A\epsilon^2\right]^{-1} \left[1 + \tfrac{\gamma}{2}\theta\ \epsilon\left\{1 - \tfrac{1}{4}(\gamma + 1)\epsilon\right\} + \frac{\gamma(\gamma+1)}{6}\theta^2\epsilon^2\right]$$

$$= - 2a_o u\left[1 - \tfrac{1}{4}(\gamma + 1)\epsilon + \left[\tfrac{1}{16}(\gamma + 1)^2 - A\right]\epsilon^2\right] \left[1 + \frac{\gamma\theta}{2}\epsilon + \left[\frac{\gamma(\gamma+1)}{6}\theta^2 - \frac{\gamma(\gamma+1)}{8}\theta\right]\epsilon^2\right]$$

$$= - 2a_o u\left[1 - \tfrac{1}{4}(\gamma + 1)\epsilon + \left\{\frac{(\gamma+1)^2}{16} - A\right\} \epsilon^2\right] \left[1 + \frac{\gamma\theta\epsilon}{2} + \frac{\gamma(\gamma+1)}{24}\theta\ (4\theta - 3)\epsilon^2\right]$$

$$= - 2a_o u\left[1 + \left\{- \tfrac{1}{4}(\gamma + 1) + \frac{\gamma\theta}{2}\right\} \epsilon + \left\{- \frac{\gamma(\gamma+1)}{8}\theta + \frac{(\gamma+1)^2}{16} - A + \frac{\gamma(\gamma+1)}{24}\ \theta\ (4\theta - 3)\right\} \epsilon^2\right]$$

Let $x = - \frac{\gamma(\gamma+1)}{8}\theta + \frac{(\gamma+1)^2}{16} - A + \frac{\gamma(\gamma+1)}{24}\theta\ (4\theta - 3)$, then (3.19)

$$f^{(2)}(u) = 2Uu - u^2 - 2a_o u\left[1 + \epsilon\left\{\frac{\gamma\theta}{2} - \frac{(\gamma+1)}{4}\right\} + x\epsilon^2\right]$$

$$= 2a_o u\left[1 + \frac{(\gamma+1)}{4}\epsilon + A\epsilon^2\right] - u^2 - 2a_o u\left[1 + \epsilon\left\{\frac{\gamma\theta}{2} - \frac{(\gamma+1)}{4}\right\} + x\ \epsilon^2\right]$$

$$= \tfrac{1}{2}(\gamma + 1)\epsilon a_o u - u^2 + \tfrac{1}{2}(\gamma + 1)\epsilon a_o u - \gamma a_o u\epsilon\theta + 2a_o u\epsilon^2(A - x)\ .$$

But $a_o\epsilon\theta = a_o \frac{u^*}{a_o}\frac{u}{u^*} = u$,

and $a_o\epsilon = u^*$.

Therefore $f^{(2)}(u) = (\gamma + 1)uu^* - (\gamma + 1)u^2 + 2uu^*\epsilon(A - x)$

$$= (\gamma + 1)u(u^* - u) + 2uu^*(A - x)\epsilon \qquad (3.20)$$

$$f(u^*) = 0 \Rightarrow f^{(2)}(u^*) = 0 \text{ , } u = u^* \Rightarrow \theta = 1 \text{ .}$$

Hence $x|_{\theta=1} = A$,

ie $\quad -\frac{\gamma(\gamma+1)}{8} + \frac{(\gamma+1)^2}{16} - A + \frac{\gamma(\gamma+1)}{24} = A$

$$\Rightarrow 2A = \frac{(\gamma+1)^2}{16} - \frac{\gamma(\gamma+1)}{24}(3 - 1) = \frac{(\gamma+1)^2}{16} - \frac{\gamma(\gamma+1)}{12}$$

$$\Rightarrow A = \frac{(3-\gamma)(\gamma+1)}{96} \tag{3.21}$$

$$\Rightarrow U^{(2)} = a_o \left[1 + \frac{1}{4}(\gamma + 1)\epsilon + \frac{1}{96}(\gamma + 1)(3 - \gamma)\epsilon^2 + 0(\epsilon^3)\right] \tag{3.22}$$

$$A - x = 2A - \frac{(\gamma+1)^2}{16} + \frac{\gamma(\gamma+1)}{8}\theta - \frac{\gamma(\gamma+1)}{24}\theta\ (4\theta - 3)$$

$$= 2A - \frac{(\gamma+1)^2}{16} + \gamma(\gamma+1)\theta \left[\frac{1}{8} + \frac{1}{8}\right] - (\gamma+1)\ \frac{\theta^2}{6}$$

$$= \frac{(3-\gamma)(\gamma+1)}{48} - \frac{(\gamma+1)^2}{16} + \frac{\gamma(\gamma+1)}{4}\theta - \frac{\gamma(\gamma+1)}{6}\theta^2$$

$$= \frac{\gamma(\gamma+1)}{12}\left[- 1 + 3\theta - 2\theta^2\right]$$

$$= - \frac{\gamma(\gamma+1)}{12}\ (2\theta - 1)\ (\theta - 1)$$

therefore

$$f^{(2)}(u) = (\gamma+1)u \left[(u^* - u) - \frac{\gamma}{6}u^*\ (2\theta - 1)(\theta-1)\epsilon\right] \text{ , } u^*\theta = u$$

$$= (\gamma+1)u(u^* - u)\left[1 - \frac{\gamma}{6}\epsilon\ (1 - 2\theta)\right]$$

$$= (\gamma+1)u(u^* - u)\left[1 - \frac{\gamma}{6a_o}(u^* - 2u)\right] .$$

Let $V = \frac{6a_o}{\gamma}$, then

$$f^{(2)}(u) = \left[\frac{\gamma+1}{V}\right] u(u^* - u)(V - u^* + 2u) \tag{3.23}$$

Let $\frac{1}{u(u^*-u)(V-u^*+2u)} \equiv \frac{a}{u} + \frac{b}{u^*-u} + \frac{c}{V-u^*+2u}$

whence $\quad 1 \equiv (u^* - u)(V - u^* + 2u)a + (V - u^* + 2u)ub + u(u^* - u)c$

$$\text{set} \quad u = 0] \quad 1 = u^*(V - u^*)a \quad \Rightarrow a = \frac{1}{u^*(V-u^*)} \tag{3.24}$$

$$\text{set} \quad u = u^*] \quad 1 = u^*(V - u^*)b \quad \Rightarrow b = \frac{1}{u^*(V-u^*)} \tag{3.25}$$

set $\quad V - u^* + 2u = 0] \quad 1 = u(u^* - u)c$

$$u = \frac{u^*-V}{2} \quad \text{here}$$

$$\Rightarrow u^* - u = u^* - \frac{u^*-V}{2}$$

$$= \frac{u^*+V}{2}$$

$$\Rightarrow u(u^* - u) = \frac{u^{*2}-V^2}{4}$$

$$\Rightarrow c = \frac{4}{u^{*2}-V^2} . \tag{3.26}$$

Let $I(u) = \int_{u^*/2}^{u} \frac{d\hat{u}}{\hat{u}(u^*-\hat{u})(V-u^*+2\hat{u})}$ (3.27)

$$= \int_{u^*/2}^{u} \left[\frac{a}{\hat{u}} + \frac{b}{u^*-\hat{u}} + \frac{c}{V-u^*+2\hat{u}}\right] d\hat{u}$$

$u^* \ll a_o < V \Rightarrow V - u^* + 2\hat{u} > 0$.

Then $I(u) = \left[a \ln \hat{u}\, \phi - b \ln (u^* - \hat{u}) + \frac{c}{2} \ln \left(\hat{u} + \frac{V-u^*}{2}\right)\right]_{u^*/2}^{u}$

$$= a \ln \left(\frac{2u}{u^*}\right) - b \ln \left[\frac{2(u^*-u)}{u^*}\right] + \frac{c}{2} \ln \left(\frac{2u+V-u^*}{V}\right) . \qquad (3.28)$$

From (3.8), (3.23) and (3.28):

$$\chi^{(2)} = - 2\delta \int_{u^*/2}^{u} \frac{d\hat{u}}{f^{(2)}(\hat{u})}$$

$$= - \frac{2\delta\ V}{(\gamma+1)} I(u)$$

$$= -\frac{2\delta\ V}{(\gamma+1)u^*(V+u^*)(V-u^*)}\left\{(V+u^*)\ln\left(\frac{2u}{u^*}\right)-(V-u^*)\ln\left[\frac{2(u^*-u)}{u^*}\right]-2u^*\ln\left(\frac{V+2u-u^*}{V}\right)\right\} .$$

(3.29)

Note that in the limit $u^* \to 0$, $\chi^{(1)}(u) \equiv \chi^{(2)}(u)$ as expected. The behaviour of $\chi^{(2)}(u)$ for $u^* > V$ is interesting but unimportant as the 2nd order approximation has no meaning in this region (it corresponds to $\epsilon > {}^6/\gamma$) .

Calculation of $\lambda_\alpha^{(2)}$:

$$\lambda_\alpha^{(2)} = |\chi^{(2)}_{(1-\alpha)} - \chi^{(2)}_{(\alpha)}|$$

Let $K = -\dfrac{2\delta V}{(\gamma+1)u^*(V+u^*)(V-u^*)}$

$$\chi^{(2)}_{(1-\alpha)} = -K\left\{(V+u^*)\ln 2(1-\alpha) - (V-u^*)\ln 2\alpha - 2u^*\ln\left[1 + (1-2\alpha)\frac{u^*}{V}\right]\right\}$$

$$\chi^{(2)}_{(\alpha)} = -K\left\{(V+u^*)\ln 2\alpha - (V-u^*)\ln 2(1-\alpha) - 2u^*\ln\left[1 - (1-2\alpha)\frac{u^*}{V}\right]\right\}$$

therefore

$$\lambda^{(2)}_{(\alpha)} = K\left|(V+u^*)\ln\left[\frac{1-\alpha}{\alpha}\right] - (V-u^*)\ln\left[\frac{\alpha}{1-\alpha}\right] - 2u^*\left\{\ln\left[1 + (1-2\alpha)\frac{u^*}{V}\right]\right.\right.$$

$$\left.\left. - \ln\left[1 - (1-2\alpha)\frac{u^*}{V}\right]\right\}\right|$$

$$= K\left|2V\ln\left[\frac{1-\alpha}{\alpha}\right] - 2u^*\left[2(1-2\alpha)\frac{u^*}{V} + 0\left(\left[\frac{u^*}{V}\right]^3\right)\right]\right|$$

$$= 2KV\left\{\ln\left[\frac{1-\alpha}{\alpha}\right] - 2(1-2\alpha)\frac{u^{*2}}{V^2} + 0\left(\left[\frac{u^*}{V}\right]^4\right)\right\}$$

$$= \frac{4\delta}{(\gamma+1)u^*}\left[1 - \frac{u^{*2}}{V^2}\right]^{-1}\left\{\ln\left[\frac{1-\alpha}{\alpha}\right] - 2(1-2\alpha)\frac{u^{*2}}{V^2} + 0\left(\left[\frac{u^*}{V}\right]^4\right)\right\}$$

$$= \frac{4\delta}{(\gamma+1)u^*}\left\{1 + \frac{u^{*2}}{V^2} + 0\left[\frac{u^{*4}}{V^2}\right]\right\}\left\{\ln\left[\frac{1-\alpha}{\alpha}\right] - 2(1-2\alpha)\frac{u^{*2}}{V^2} + 0\left[\frac{u^{*4}}{V^4}\right]\right\}$$

$$r^{+}\phi^{+} \quad \lambda_{\alpha}^{(2)} = \frac{4\delta}{(\gamma+1)u^{*}}\left\{\ln\left[\frac{1-\alpha}{\alpha}\right] + \left[\ln\left[\frac{1-\alpha}{\alpha}\right] - 2(1-2\alpha)\right]\frac{u^{*2}}{V^{2}} + 0\left[\frac{u^{*4}}{V^{4}}\right]\right\}$$

(3.30)

So, $\alpha \simeq ½ \Rightarrow \lambda_{\alpha}^{(2)} < \lambda_{\alpha}^{(1)}$

$\alpha \simeq 0 \Rightarrow \lambda_{\alpha}^{(2)} > \lambda_{\alpha}^{(1)}$.

3.3 Observations

An important feature of this analysis is

$\lambda_{\alpha}^{(n)} = 0(^{1}/u^{*})$ as $u^{*} \to 0$ for $n = 1,2$, and presumably for all n . If the superscript (∞) represents the exact solution then we assert

$$\lambda_{\alpha}^{(\infty)} = 0(^{1}/u^{*}) \qquad (3.31)$$

i.e. the shock width is asymptotically inversely proportional to the velocity jump. The order approximations require progressively more work and yield progressively less knowledge.

Equation (3.6) may be written

$$\left[\frac{a}{a_{o}}\right]^{-\frac{2}{\gamma-1}} = \frac{U-u}{U}$$

$$\Rightarrow U = \frac{u}{1-\left[\frac{a}{a_{o}}\right]^{-2/_{\gamma-1}}} .$$

But $a = a_{o}\left[\frac{\rho}{\rho_{o}}\right]^{\frac{\gamma-1}{2}}$, therefore

$$\left[\frac{a}{a_o}\right]^{-\frac{2}{\gamma-1}} = \frac{\rho_o}{\rho} ,$$

so

$$U = \frac{\rho u}{\rho-\rho_o} = \frac{\rho_o u_o - \rho u}{\rho_o - \rho} , \quad \text{where} \quad u_o = 0 .$$

This is result (2.12), which means there is a constant mass flux in the frame moving with the shock.

3.4 Relationship Between Lighthill's Work and the Polytropic Navier-Stokes Equations

The 1-D Navier-Stokes equations are :

$$\left.\begin{aligned} &\text{mass conservation:} && \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \\ &\text{momentum conservation:} && \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} \end{aligned}\right\} \tag{3.31}$$

The polytropic assumption is

$e \propto T$, and leads to the equation

$p\rho^{-\gamma}$ = constant.

This can be written as

$$p\rho^{-\gamma} = p_o\rho_o^{-\gamma} \tag{3.32}$$

Now, $\rho = \rho_o\left[\frac{a}{a_o}\right]^{\frac{2}{\gamma-1}}$

$$\Rightarrow \qquad p = p_o \left(\frac{\rho}{\rho_o}\right)^{\gamma}$$

$$= p_o \left(\frac{a}{a_o}\right)^{\frac{2\gamma}{\gamma-1}} .$$

Therefore $\dfrac{\partial \rho}{\partial x} = p_o \, a_o^{-\left(\frac{2\gamma}{\gamma-1}\right)} \left(\dfrac{2\gamma}{\gamma-1}\right) a^{\left(\frac{2\gamma}{\gamma-1} - 1\right)} \dfrac{\partial a}{\partial x}$

$$\Rightarrow \qquad \frac{1}{\rho}\frac{d\rho}{dx} = \frac{\gamma p_o}{\rho_o} \, \frac{2}{\gamma-1} \, a_o^{\left(\frac{2}{\gamma-1} - \frac{2\gamma}{\gamma-1}\right)} \, a^{\left(-\frac{2}{\gamma-1} + \frac{\gamma+1}{\gamma-1}\right)} \frac{da}{dx}$$

$$= \frac{2}{\gamma-1} \, \frac{\gamma p_o}{\rho_o} \, a_o^{-2} \, a \, \frac{\partial a}{\partial x} .$$

Hence $\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} = \dfrac{2}{\gamma-1} \, a \, \dfrac{\partial a}{\partial x}$ when

$$\frac{\gamma p_o}{\rho_o} = a_o^2 .$$

This is the condition for the 1-D Polytropic Navier-Stokes equations to be identical to the original 1-D Lighthill system.

But $a^2 = \dfrac{dp}{d\rho}$ (by definition)

$$= p_o \rho_o^{-\gamma} \quad \gamma \rho^{\gamma-1}$$

therefore

$$a_o^2 = \frac{\gamma p_o}{\rho_o} .$$

Thus the two systems of equations are identical, apart for the definition of the diffusive constant - being either ν or δ .

4. Curve Fitting Theory

4.1 General Theory

In §1.2, a function $\hat{\psi}(\chi)$ was defined by

$$\left.\begin{aligned} \lim_{\chi\to-\infty} \hat{\psi} &= \hat{\psi}_{(0)} \\ \lim_{\chi\to\infty} \hat{\psi} &= \hat{\psi}_{(1)} \end{aligned}\right\} \tag{1.4}$$

$$\hat{\psi}\big|_{\chi=0} = \hat{\psi}_{(\frac{1}{2})} \tag{1.5}$$

$$\left.\begin{aligned} \hat{\psi} = \hat{\psi}_{(\alpha)} &\Leftrightarrow \chi = \chi_{(1-\alpha)} \\ \hat{\psi} = \hat{\psi}_{(1-\alpha)} &\Leftrightarrow \chi = \chi_{(1-\alpha)} \end{aligned}\right\} . \tag{1.6}$$

The α in (1.6) is assumed fixed, and $(\chi_{(\alpha)} , \hat{\psi}_{(\alpha)})$ is known as the (α) point, (similarly for the $(1-\alpha)$ point).

Now, if a function $Z(\chi)$ is introduced with the property:

$$\hat{\psi}(\chi) = \frac{\hat{\psi}_{(1)}}{Z+1} + \frac{Z}{Z+1}\,\hat{\psi}_{(0)} \tag{4.1}$$

then (1.4) $\Rightarrow$

$$\left.\begin{aligned} &\lim_{\chi\to\infty} Z = \infty \quad , \\ &\lim_{\chi\to-\infty} Z = 0 \quad , \quad \text{and (1.5) gives} \\ &Z\big|_{\chi=0} = 1 \end{aligned}\right\} . \tag{4.2}$$

So $Z(\chi)$ is similar to e^{χ} in some sense. In the case where $|\chi_{(\alpha)}| = |\chi_{(1-\alpha)}|$, i.e. the curve is symmetric, $Z(\chi)$ is exactly

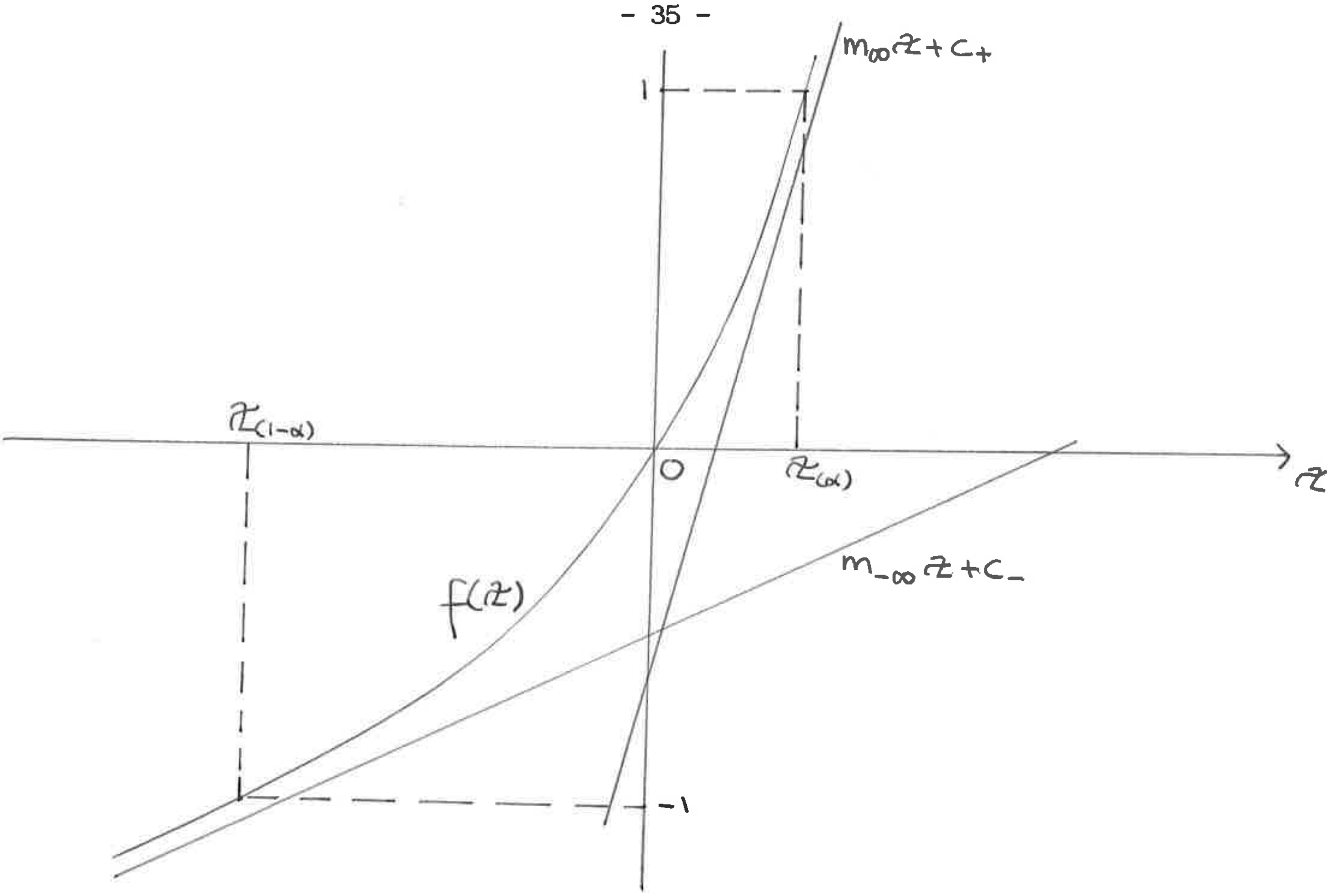
$$\left.\begin{aligned} &Z = e^{\theta\bar{\chi}} , \\ &\text{where} \\ &\theta = \ln\left[\frac{1-\alpha}{\alpha}\right] \\ &\text{and} \\ &\bar{\chi} = \frac{\chi}{|\chi_{(\alpha)}|} = \frac{\chi}{|\chi_{(1-\alpha)}|} \end{aligned}\right\} . \qquad (4.3)$$

Now, a sensible generalisation of this function to the non-symmetric use is

$$Z = e^{\theta f(\chi)} \qquad (4.4)$$

where $f(\chi)$ behaves like χ as $\chi \to \pm\infty$. The (α) , $(1-\alpha)$ conditions become

$$\left.\begin{aligned} &f(\chi_{(1-\alpha)}) = -1 \\ &f(\chi_{(\alpha)}) = 1 \\ \text{Also,}\quad &Z(0) = 1 \Rightarrow f(0) = 0 \end{aligned}\right\} . \qquad (4.5)$$



A sensible general definition of f is then

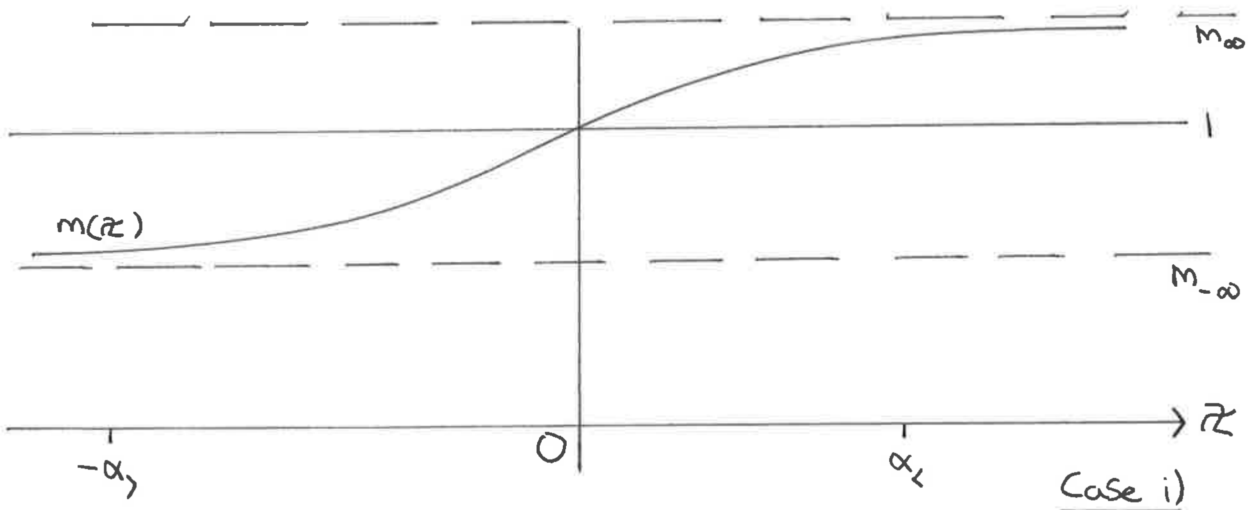
$$\left.\begin{aligned} & f(\chi) = \chi m(\chi)\ , \\ & \text{where} \\ & \lim_{\chi\to\infty} m(\chi) = m_\infty \\ & \lim_{\chi\to-\infty} m(\chi) = m_{-\infty} \end{aligned}\right\} . \qquad (4.6)$$

Clearly, from the diagram, in this case

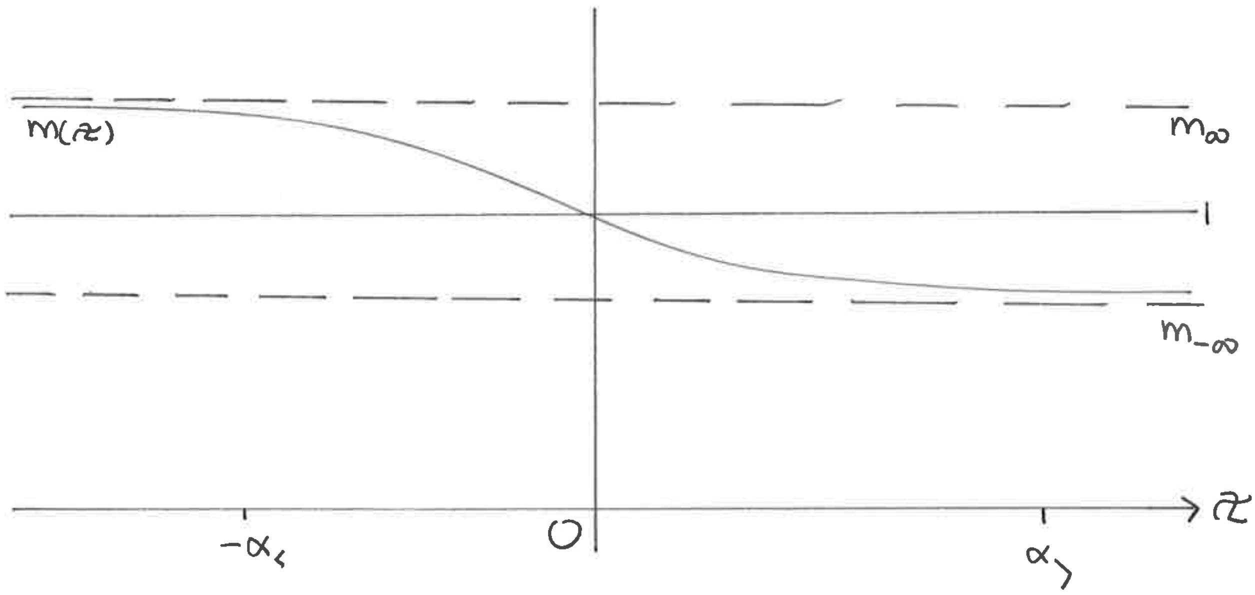
$$|\chi_{(1-\alpha)}| > |\chi_{(\alpha)}|\ ,\quad m_\infty > 1\ ,\quad m_{-\infty} < 1\ .$$

But the converse is true. To represent this, the following notation is introduced:

$$\left.\begin{aligned} \alpha_{>} &= \max\,\{|x_{(1-\alpha)}|\;,\;|x_{(\alpha)}|\} \\ \alpha_{<} &= \min\,\{|x_{(1-\alpha)}|\;,\;|x_{(\alpha)}|\} \end{aligned}\right\} \qquad (4.7)$$



Case i)



Case ii)

Here we have case i). In both cases, $m(x)$ appears to have a similar non-symmetric asymptotic behaviour as that of the $\hat{\psi}(x)$ it is dependent on.

This leads to the definition of a new function $\hat{\psi}(\chi)$. In its definition there is the implicit assumption that

$$m_{\infty} + m_{-\infty} = 2 \tag{4.8}$$

Define $\tilde{\psi}$ by:

$$\left.\begin{array}{ll} \tilde{\psi}(\chi) = \beta \left[\frac{1-Z(\chi)}{1+Z(\chi)}\right] , & \text{where} \\ 0 < \beta \leq 1 , & \text{and} \\ Z = e^{\phi\chi(1\pm\tilde{\psi})} , & \text{with the condition that} \\ \text{'+' used for } |\chi_{(1-\alpha)}| , = \alpha_{<} & \text{ie left hand end sharper,} \\ \text{'−' used for } |\chi_{(1-\alpha)}| , = \alpha_{>} & \text{ie left hand end shallower,} \\ \text{and } \phi \text{ is a constant } > 0 . & \end{array}\right\} \tag{4.9}$$

$\tilde{\psi} \in (-\beta,\beta)$ so $\beta \neq 1 \Rightarrow Z \rightarrow \infty$ as $\chi \rightarrow \infty$

$\Rightarrow \tilde{\psi} \rightarrow -\beta$ as $\chi \rightarrow \infty$,

$\tilde{\psi} \rightarrow \beta$ as $\chi \rightarrow -\infty$.

Let $\tilde{\psi}(\chi_{(1-\alpha)}) = \tilde{\psi}_{(1-\alpha)} = (1 - 2\alpha)\beta$

$\tilde{\psi}(\chi_{(\alpha)}) = \tilde{\psi}_{(\alpha)} = -(1 - 2\alpha)\beta$.

The motivation for this is to linearise $\tilde{\psi}(\chi)$ to form $\hat{\psi}(\chi)$ in such a way that the (α) , $(1 - \alpha)$ points are preserved. Then

$$Z(\chi_{(1-\alpha)}) = Z_{(1-\alpha)} = \exp\left\{\phi\chi_{(1-\alpha)}[1\pm(1-2\alpha)\beta]\right\}$$

$$Z(\chi_{(\alpha)}) = Z_{(\alpha)} = \exp\left\{\phi\chi_{(\alpha)}[1\mp(1-2\alpha)\beta]\right\}$$

$$(4.9) \Rightarrow \quad Z = \frac{\beta-\tilde{\psi}}{\beta+\tilde{\psi}} \text{ so that} \tag{4.10}$$

$$Z_{(1-\alpha)} = \frac{\alpha}{1-\alpha} = e^{-\theta} \quad ,$$

and $Z_{(\alpha)} = \frac{1-\alpha}{\alpha} = e^{\theta}$.

Therefore

$$\phi\, \chi_{(1-\alpha)}\, [1\pm(1-2\alpha)\beta] = -\,\theta \quad ,$$

$$\phi\, \chi_{(\alpha)}\, [1 \mp (1-2\alpha)\beta] = \theta \quad ,$$

$$\Rightarrow \quad \left.\begin{aligned} \phi\alpha_{<}\, [1 + (1-2\alpha)\beta] &= \theta \quad , \\ \phi\alpha_{>}\, [1 - (1-2\alpha)\beta] &= \theta \quad , \end{aligned}\right\} \tag{4.11}$$

Hence

$$\alpha_{<}\, [1 + (1-2\alpha)\beta] = \alpha_{>}\, [1 - (1-2\alpha)\beta]$$

$$\Rightarrow \quad (1-2\alpha)\beta = \frac{\alpha_{>}-\alpha_{<}}{\alpha_{>}+\alpha_{<}}$$

$$\Rightarrow \quad \beta = \frac{1}{(1-2\alpha)}\, \frac{\alpha_{>}-\alpha_{<}}{\alpha_{>}+\alpha_{<}} \, . \tag{4.12}$$

$$(4.11) \Rightarrow \phi\, [1 + (1-2\alpha)\beta] = \frac{\theta}{\alpha_{<}}$$

$$\phi\, [1 - (1-2\alpha)\beta] = \frac{\theta}{\alpha_{>}}$$

$$\Rightarrow \qquad 2\phi = \theta\left[\frac{1}{\alpha_<} + \frac{1}{\alpha_>}\right]$$

$$\Rightarrow \qquad \phi = \frac{\theta}{2}\left[\frac{1}{\alpha_<} + \frac{1}{\alpha_>}\right]$$

$\tilde{\psi}(\chi)$ cannot be found analytically, but $\chi(\tilde{\psi})$ can:

$$(4.9), (4.10) \Rightarrow \phi\,\chi\,(1 \pm \tilde{\psi}) = \ln\left[\frac{\beta-\tilde{\psi}}{\beta+\tilde{\psi}}\right]$$

$$\Rightarrow \chi(\tilde{\psi}) = \frac{1}{\phi(1\pm\tilde{\psi})} \ln\left[\frac{\beta-\tilde{\psi}}{\beta+\tilde{\psi}}\right] \tag{4.14}$$

From $\tilde{\psi}$, $\hat{\psi}$ may be generated as follows:

$$\left.\begin{aligned} \hat{\psi}(\chi) &= \hat{\psi}_{(\frac{1}{2})} + \frac{[\hat{\psi}]}{2\beta}\tilde{\psi}(\chi) , \\ \text{where} \quad [\hat{\psi}] &= \hat{\psi}_{(1)} - \hat{\psi}_{(0)} \end{aligned}\right\} . \tag{4.15}$$

$\tilde{\psi}$ can be seen as parameterising χ and $\hat{\psi}$ over the range of their values.

Naturally, by the definition of $f(\chi)$ in (4.6), the fitting function $\hat{\psi}(\chi)$ can only be expected to behave well for approximating functions with the same symptotic behaviour.

Note also that (4.15) $\Rightarrow$

$$\hat{\psi}(\chi) = \frac{\hat{\psi}_{(o)}Z}{Z+1} + \frac{\hat{\psi}_{(1)}}{Z+1} \qquad \text{as before}$$

So in the limit $\alpha_> \to \chi_o^+/\theta$

$\alpha_< \to \chi_o^-/\theta$.

$$\phi \to \frac{1}{\chi_o}$$

$$\beta \to 0 \quad \Rightarrow \quad \tilde{\psi} \to 0$$

$$\Rightarrow \quad Z \in e^{\chi/\chi_o} \ .$$

So we recover the tanh-curve shape in the limit $\beta \to 0$.

It is also possible to write χ as a function of $\hat{\psi}$:

$$(4.15) \Rightarrow \hat{\psi} = \frac{1}{2}\left[\hat{\psi}_{(0)} + \hat{\psi}_{(1)} + \frac{\tilde{\psi}}{\beta}(\hat{\psi}_{(1)} - \hat{\psi}_{(0)})\right]$$

$$\Rightarrow \quad \frac{\tilde{\psi}}{\beta} = \frac{2\hat{\psi} - \hat{\psi}_{(0)} - \hat{\psi}_{(1)}}{\hat{\psi}_{(1)} - \hat{\psi}_{(0)}}$$

$$\Rightarrow \quad 1 - \frac{\tilde{\psi}}{\beta} = \frac{\hat{\psi}_{(1)} - \hat{\psi}_{(0)} - 2\hat{\psi} + \hat{\psi}_{(0)} + \hat{\psi}_{(1)}}{\hat{\psi}_{(1)} - \hat{\psi}_{(0)}}$$

$$= \frac{2(\hat{\psi}_{(1)} - \hat{\psi})}{\hat{\psi}_{(1)} - \hat{\psi}_{(0)}}$$

$$\Rightarrow \quad 1 + \frac{\tilde{\psi}}{\beta} = \frac{2(\hat{\psi} - \hat{\psi}_{(0)})}{\hat{\psi}_{(1)} - \hat{\psi}_{(0)}} \ .$$

Therefore

$$(4.14) \quad \Rightarrow \quad \chi(\hat{\psi}) = \frac{1}{\beta\phi}\left[\frac{1}{\beta} \pm \frac{2\hat{\psi} - \hat{\psi}_{(0)} - \hat{\psi}_{(1)}}{\hat{\psi}_{(1)} - \hat{\psi}_{(0)}}\right]^{-1} \ln\left(\frac{\hat{\psi}_{(1)} - \hat{\psi}}{\hat{\psi} - \hat{\psi}_{(0)}}\right) \qquad (4.16)$$

4.2 Relationship with 2nd Order Solution in §3.2

Here, ψ , the generalised thermodynamic quantity, is just the fluid velocity u :

$$\psi = u ,$$

$$\hat{\psi}_{(1)} = u^* ,$$

$$\hat{\psi}_{(0)} = 0 .$$

Let $\hat{\psi} = \hat{u}$ – the fitted fluid velocity. (4.16) gives χ as a function of $\hat{\psi}$. We are assuming the velocity bounds (u^* and 0), so it is consistent to write

$$\hat{\chi}(u) = \frac{1}{\beta\phi}\left[\frac{1}{\beta} \pm \frac{2u - u^*}{u^*}\right]^{-1} \ln\left[\frac{u^* - u}{u}\right] , \quad \text{where}$$

$$\beta = \frac{1}{(1-2\alpha)}\,\frac{\delta_\alpha}{\lambda_\alpha} , \quad \text{and}$$

$$\phi = \frac{\theta}{2}\left[\frac{4\lambda_\alpha}{\lambda_\alpha^2 - \delta_\alpha^2}\right] , \quad \text{where}$$

$$\delta_\alpha = |\chi([1-\alpha]u^*) + \chi(\alpha u^*)| .$$

Here, we are fitting the 2nd order solution, so we are concerned with

$$\lambda_\alpha = \lambda_\alpha^{(2)}$$

$$\delta_\alpha = \delta_\alpha^{(2)} .$$

It is possible to show that for $u^* \ll V$,

$$\hat{\chi}(u) \sim \chi^{(2)}(u) \quad \text{as} \quad \frac{u}{u^*} \to 0^+$$

and $$\hat{\chi}(u) \sim \chi^{(2)}(u) \quad \text{as} \quad \frac{u}{u^*} \to 1^- \quad .$$

This is because both curves are tending to the same tanh curve.

(Here $\sim$ represents approximate asymptotic behaviour.)

5. Conclusion

This paper ties together a number of ideas concerning the structure of weak shocks in one-dimension. The theory produced needs to be backed up by experimental results, numerical application and thermodynamical theory in order to be more convincing. An extension into higher-dimensional flows is also required.

6. References

1. Equation (19) of Lighthill, M.J., Viscosity effects in Soundwaves of Finite Amplitude. In Survey's in Mechanics by G.K. Batchelor & R.M. Davies Cambridge Univ. Press, Cambridge (1956), pp. 250-351.

2. See Section 54 of Supersonic Flow and Shock Waves by R. Courant & K. Friedrichs. Springer-Verlag, New York (1976).

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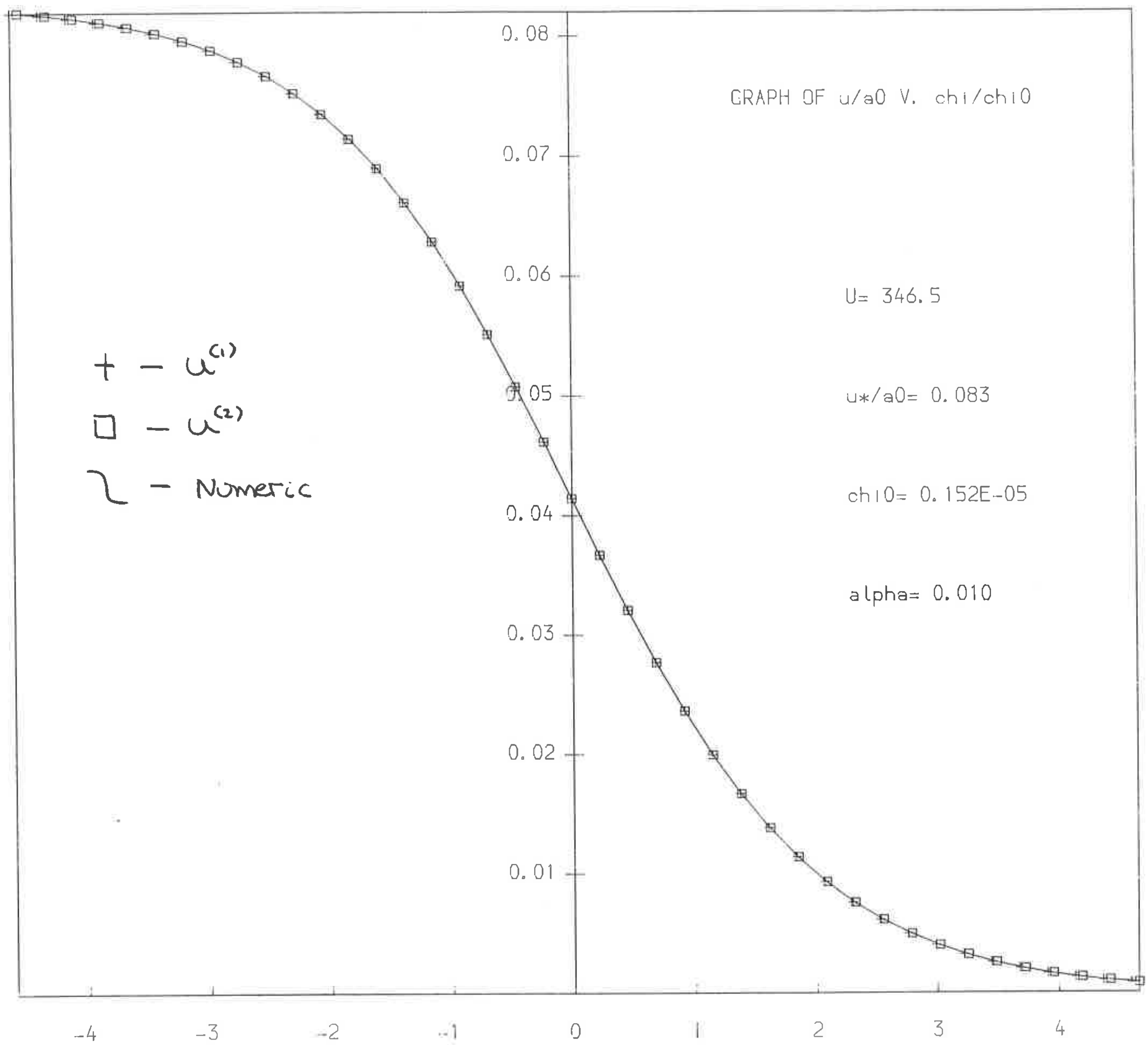
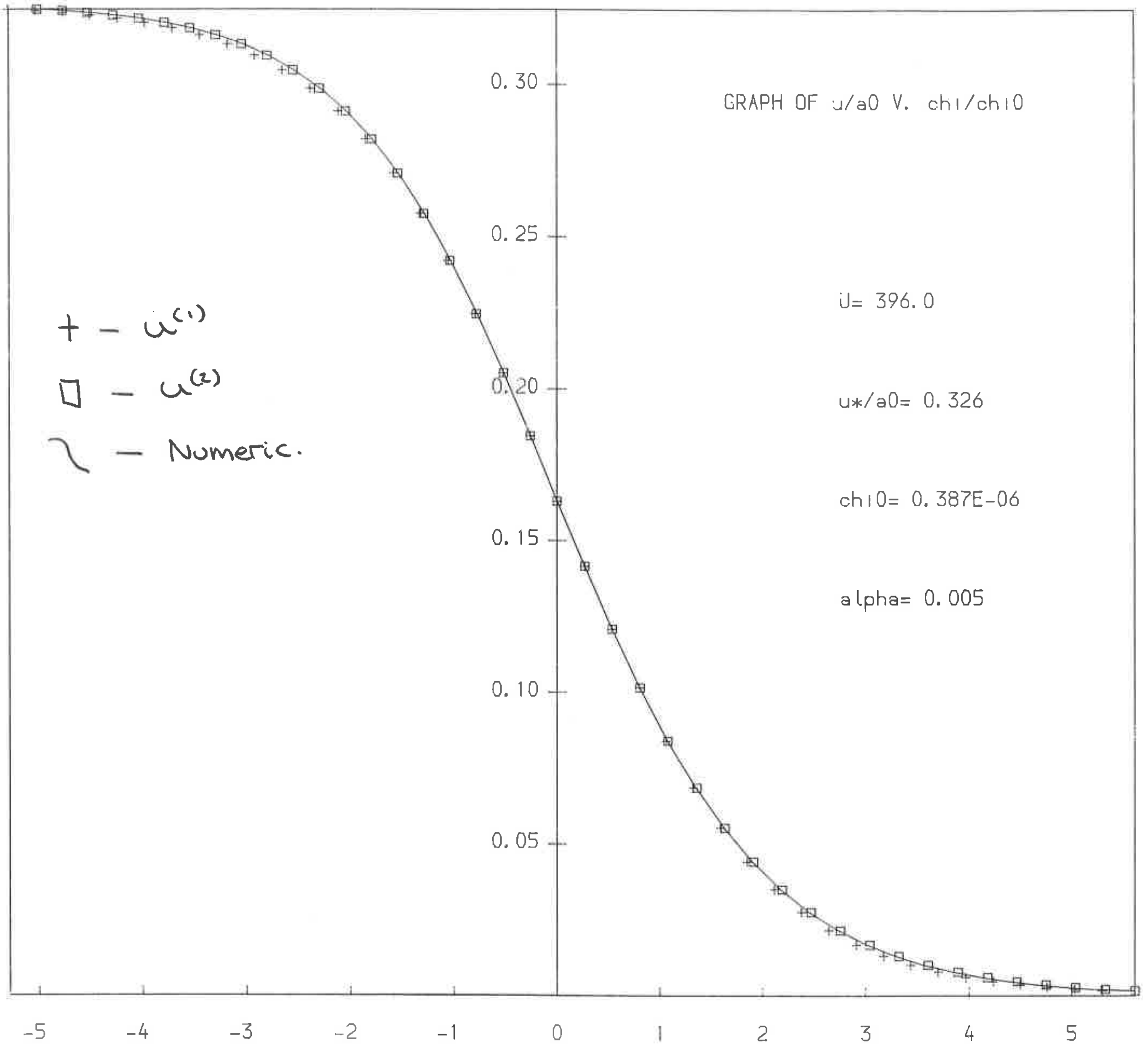
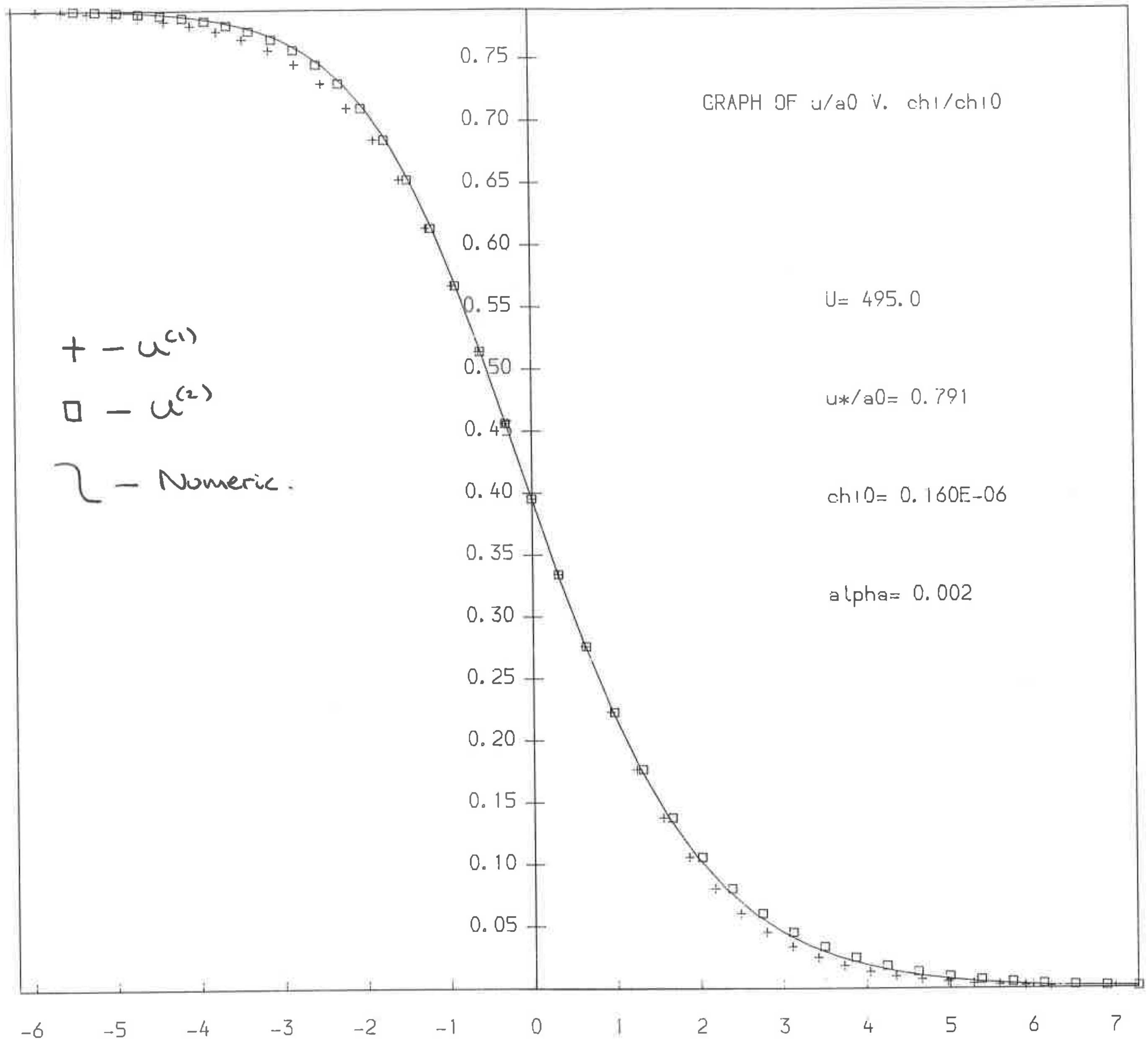
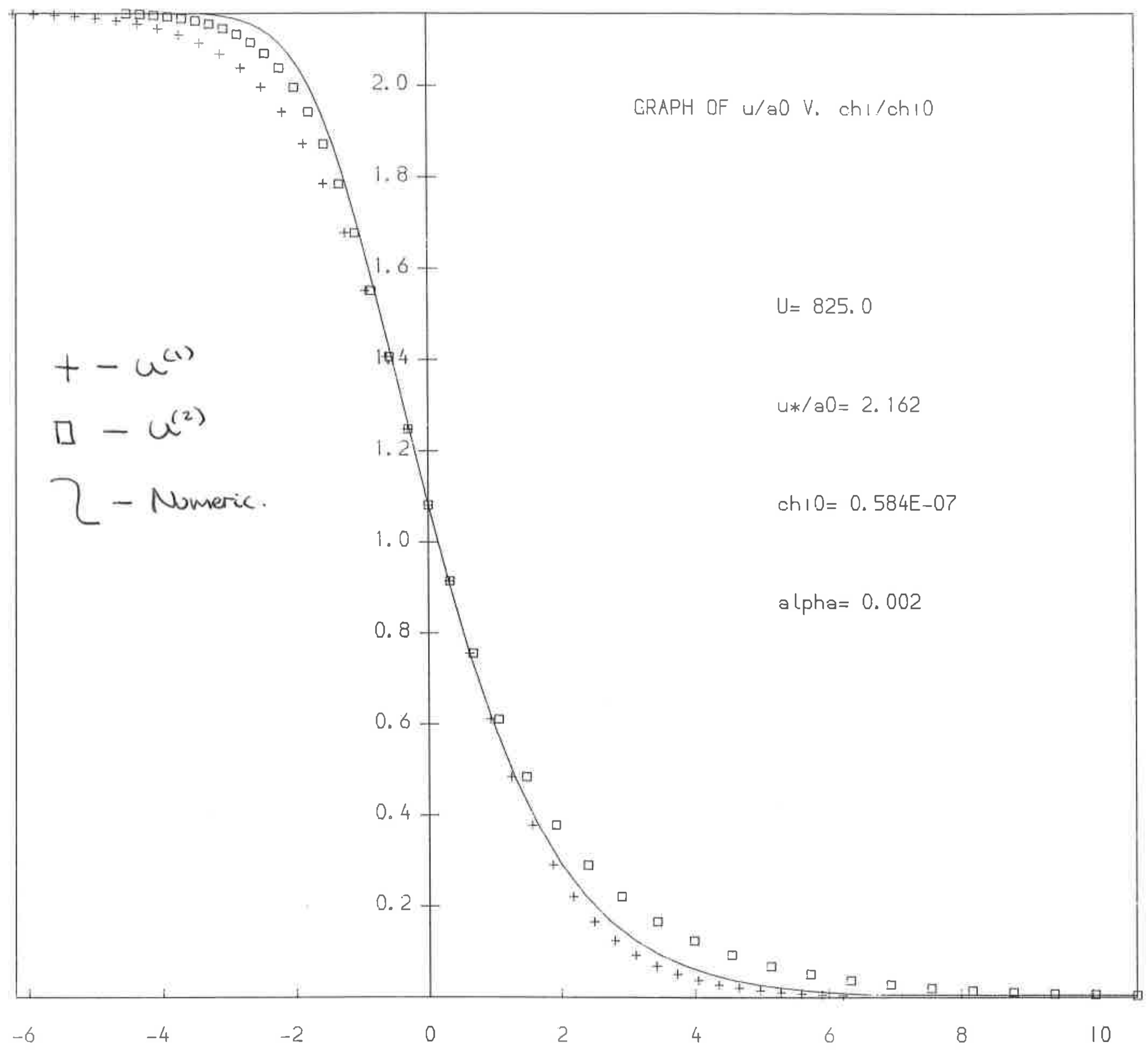
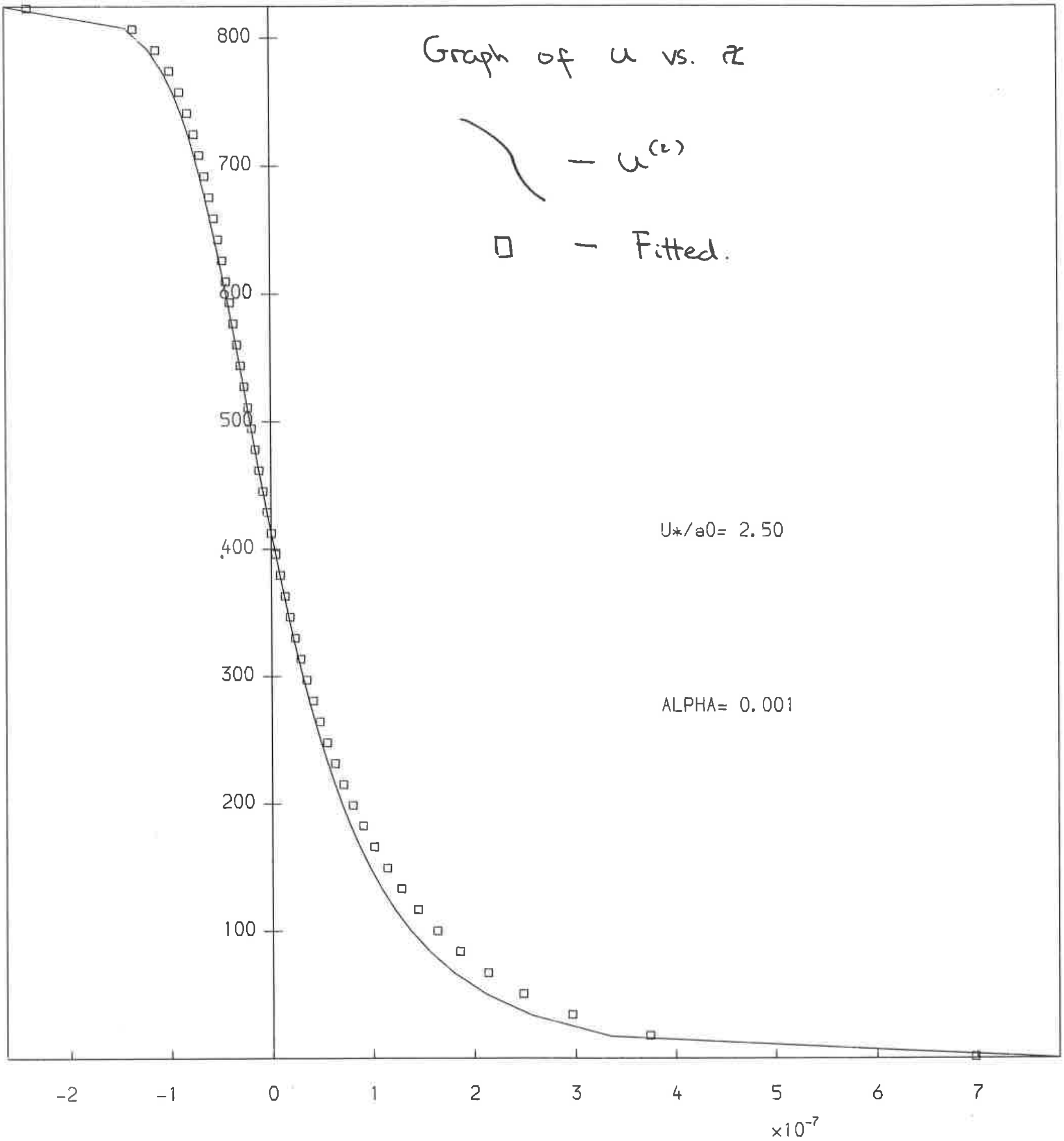

GRAPH OF u/a0 V. chi/chi0
U= 346.5
u*/a0= 0.083
chi0= 0.152E-05
alpha= 0.010
+ – u(1)
□ – u(2)
– Numeric
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
-4
-3
-2
-1
0
1
2
3
4

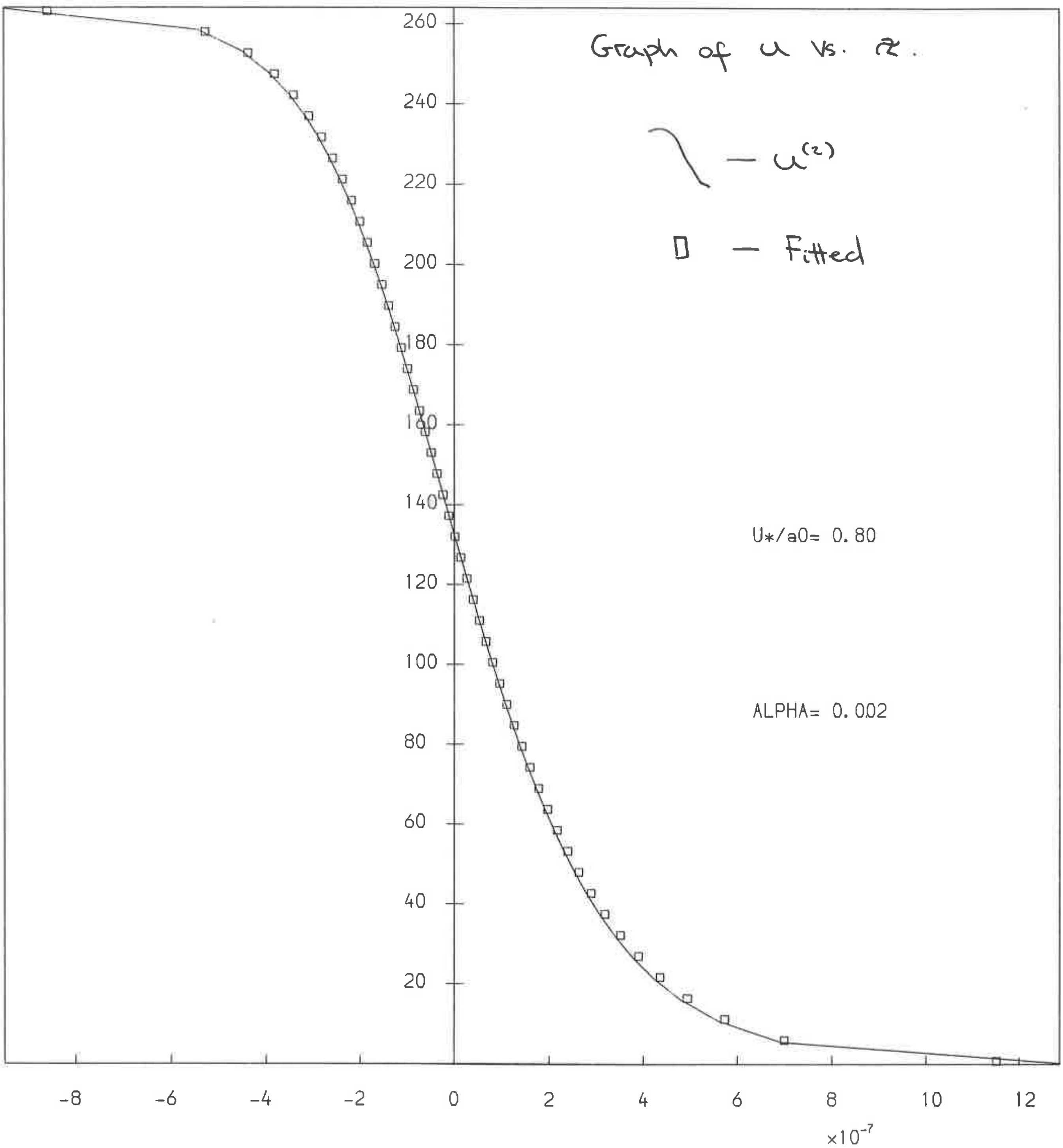

GRAPH OF u/a0 V. chi/chi0
U= 396.0
u*/a0= 0.326
chi0= 0.387E-06
alpha= 0.005
+ – u(1)
□ – u(2)
– Numeric.
0.30
0.25
0.20
0.15
0.10
0.05
-5
-4
-3
-2
-1
0
1
2
3
4
5


GRAPH OF u/a0 V. chi/chi0
U= 495.0
u*/a0= 0.791
chi0= 0.160E-06
alpha= 0.002
+ – $u^{(1)}$
□ – $u^{(2)}$
– Numeric.
0.75
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7

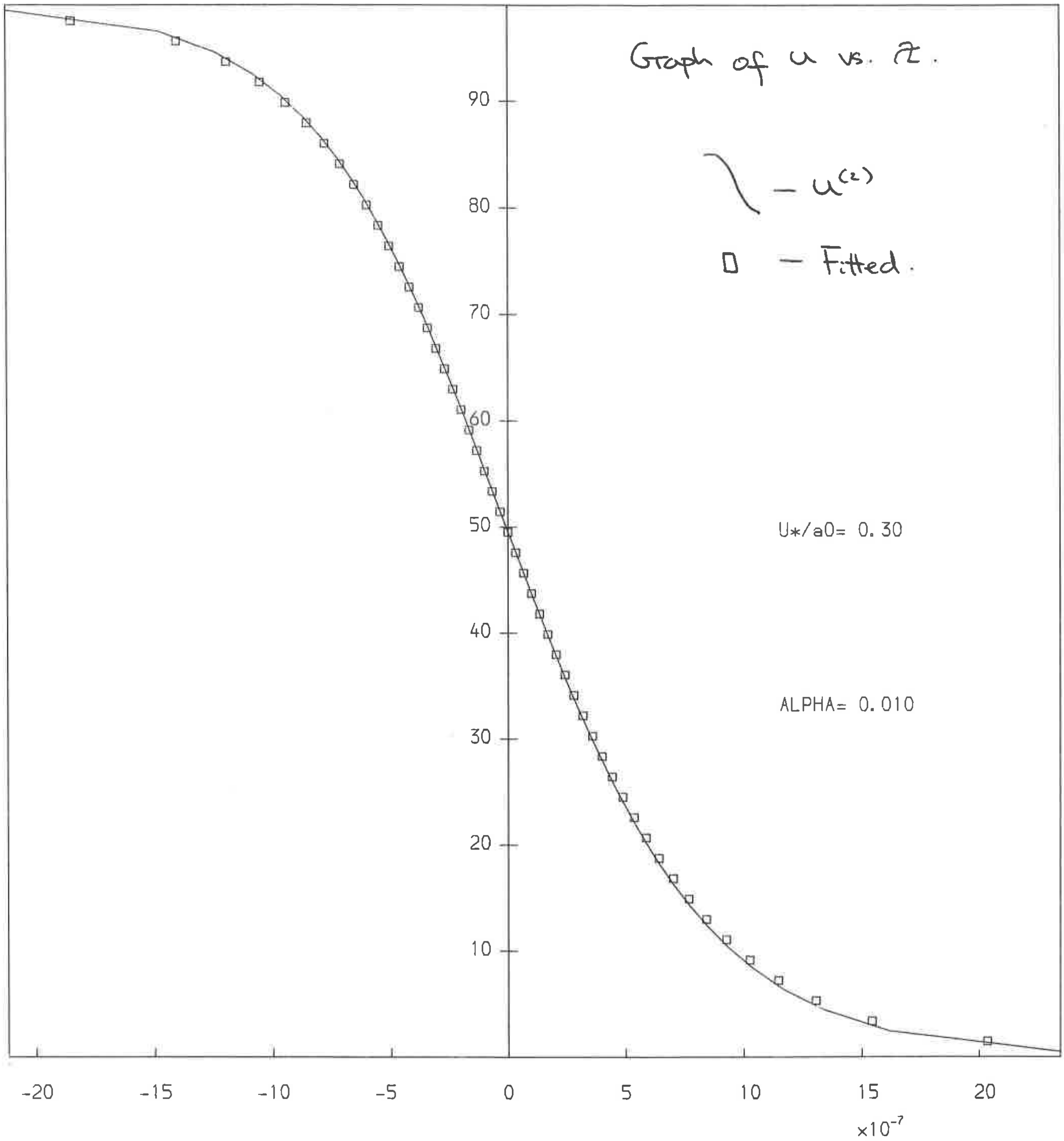

GRAPH OF u/a0 V. chi/chi0
U= 825.0
u*/a0= 2.162
chi0= 0.584E-07
alpha= 0.002
+ – u(1)
□ – u(2)
~ – Numeric.
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
-6
-4
-2
0
2
4
6
8
10



Graph of u vs. z
— u^(2)
□ — Fitted.
U*/a0= 2.50
ALPHA= 0.001
800
700
600
500
400
300
200
100
-2
-1
0
1
2
3
4
5
6
7
×10^-7



Graph of u vs. z.
— u(z)
□ — Fitted
U*/a0= 0.80
ALPHA= 0.002
260
240
220
200
180
160
140
120
100
80
60
40
20
-8
-6
-4
-2
0
2
4
6
8
10
12
×10⁻⁷



Graph of u vs. z.
— $u^{(2)}$
□ — Fitted.
U*/a0= 0.30
ALPHA= 0.010
90
80
70
60
50
40
30
20
10
-20
-15
-10
-5
0
5
10
15
20
$\times 10^{-7}$