Least Squares, Equidistribution and Conservation

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Abstract

Let $F$ be a vector valued function which is piecewise conforming on a partition of a polygonal region $\Omega$ into triangles, with a prescribed flux across the boundary $\partial \Omega$. Then minimisation of the $l_2$ norm of the fluctuation of $F$ over the internal nodes of the partition is equivalent to minimisation of the $l_2$ norm of the differences in the fluctuations of $F$ over all the elements of the partition. In this sense $l_2$ minimisation is equivalent to an $l_2$ measure of equidistribution. The result is also true for the $l_2$ norm of the local average residuals, i.e. the fluctuations weighted by the reciprocal areas of the elements.

The particular case $F = U a$, where $U$ is a piecewise linear function prescribed on $\partial \Omega$ and $a$ is divergence-free, is considered in detail.

The result extends to the local average vorticity as well as to norms combining both the residual and the vorticity, as for example in the case of the Cauchy-Riemann equations.
1 Introduction

There are two main approaches to the problem of generating an irregular mesh on which to approximately solve a differential equation. A popular criterion has been that of equidistribution, in which the grid is determined by a monitor function whose integral is the same in each interval [1], [2], [3]. Another approach is to use direct minimisation of a suitable functional of the equation [3], [4], [5] in which the grid participates in the minimisation. Each of these methods has advantages and disadvantages but comparisons have been impeded by the absence of any link between them. In this report we demonstrate that, provided that the boundary values are fixed, direct minimisation of a discrete least squares error over internal variations of the function and the nodes is equivalent to minimising a corresponding measure of equidistribution.

2 Fluctuations and Residuals

Consider the first order conservation law

\[ \text{div} f = 0 \]  

(1)

in a polygonal region \( \Omega \) and let \( f \) be approximated by a conforming approximation \( F \) on a triangulation \{T\} of \( \Omega \). Then, following Roe [7], on each triangle \( T \) we may define the fluctuation

\[ \phi_T = - \int_T \text{div} F d\Omega. \]  

(2)

We may also define an local average residual on the triangle \( T \)

\[ \bar{R}_T = \frac{1}{S_T} \int_T \text{div} F d\Omega, \]  

(3)

where \( S_T \) is the area of the triangle, so that

\[ \bar{R}_T = - \frac{\phi_T}{S_T}. \]  

(4)

If \( F = U a \) where \( U \) is piecewise linear and \( a \) is divergence-free we have

\[ \text{div} F = a \cdot \nabla U \]  

(5)

and the fluctuation is

\[ \phi_T = - \int_T a \cdot \nabla U d\Omega = - \left( \int_T \text{ad} \Omega \right) (\nabla U)_T = - S_T (\bar{a} \cdot \nabla U)_T \]  

(6)

where \( \bar{a} \) is the centroid value of \( a \). In that case we may define the average residual to be

\[ \bar{R}_T = (\bar{a} \cdot \nabla U)_T = - \frac{\phi_T}{S_T}. \]  

(7)
in each triangle, using (6). If \( a \) is constant we may define the unique residual

\[
R_T = (\text{div}F)_T = (a \nabla U)_T.
\]  

These definitions also hold in higher dimensions with \( S_T \) replaced by the volume of the appropriate simplex and \( \Omega \) by the union of the simplexes.

3 A Zero Residual Property

In the case where \( a \) is constant it has been pointed out in [7] that if two of the vertices of a triangle \( T \) lie on a characteristic, i.e. a line in the direction of \( a \) on which \( U \) is constant, then the fluctuation \( \phi_T \) (and therefore the residual \( R_T \)) vanishes. This follows from (5) since any line joining two vertices of \( T \) on which \( U \) is constant is a level line of the locally linear function \( U \) and therefore perpendicular to \( \nabla U \). Similarly, in higher dimensions, if any two vertices of a simplex lie on a characteristic then the fluctuation \( \phi_T \) and the residual \( R_T \) vanish by the same argument.

We give an algebraic proof of this result in two dimensions which also serves to introduce some notation. Let the vertices \((X_i, Y_i)\) of the triangle \( T \) be numbered \( i = 1, 2, 3 \) in an anticlockwise sense. Then in triangle \( T \)

\[
\nabla U = \frac{1}{S_T} \left( \sum Y_1(U_2 - U_3), \ - \sum X_1(U_2 - U_3) \right)
\]  

where the sum is taken cyclically over the vertices of the triangle. In the same notation the area \( S_T \) of the triangle is

\[
S_T = \sum X_1(Y_2 - Y_3) = - \sum Y_1(X_2 - X_3).
\]  

It follows that

\[
\phi_T = -S_T a \cdot \nabla U = - \sum (a Y_1 - b X_1) (U_2 - U_3)
\]  

where \( a = (a, b) \) which, if \( U_2 = U_3 \), reduces to

\[
\phi_T = -(a Y_2 - b X_2) (U_3 - U_1) - (a Y_3 - b X_3) (U_1 - U_2)
\]

\[
= (-a(Y_2 - Y_3) + b(X_2 - X_3))(U_2 - U_1).
\]  

The right hand side vanishes when the vector \((X_2 - X_3, Y_2 - Y_3)\) is in the direction of \( a \).

4 Fluctuation Equidistribution

Consider now the identity

\[
\phi_1^2 + \phi_2^2 \equiv \frac{1}{2} (\phi_1 + \phi_2)^2 + \frac{1}{2} (\phi_1 - \phi_2)^2
\]
which may readily be generalised to
\[
\sum_{i=1}^{N} \phi_i^2 = \frac{1}{N} \left( \sum_{i=1}^{N} \phi_i \right)^2 + \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} (\phi_i - \phi_j)^2.
\] (14)

Let \( \phi_i \) be the fluctuation \( \phi_T \) in triangle \( T_i \) as defined in section 1 and let \( N \) be the number of triangles in \( \Omega \). The first term in brackets on the right hand side of (14) may then be written
\[
\sum_{i=1}^{N} \phi_i = -\sum_{i=1}^{N} \int_{T_i} \text{div} F d\Omega = -\sum_{i=1}^{N} \int_{\partial T_i} F \cdot ds
\] (15)
where \( \partial T_i \) is the boundary of \( T_i \). Since \( F \) is conforming (15) reduces to
\[
-\int_{\partial \Omega} F \cdot ds
\] (16)
over the boundary \( \partial \Omega \) of \( \Omega \). The quantity in (16) is fixed when the flux of \( F \) across the outer boundary is preserved. In the original conservation law problem only the contribution to (16) from the inflow is prescribed but the least squares minimisation procedure described below also demands an outflow condition, in which case (16) is fully prescribed.

It then follows from (14) that, if (16) is fixed, then under variations of the internal values of \( F \) and the internal grid points, the \( l_2 \) norm of \( \phi_T \) is least when the final term in (14) vanishes. If the \( \phi_{T_i} \) are equal for all \( i \) then the minimum is achieved and equidistribution is equivalent to least squares minimisation. If this condition is unattainable the result is restricted to the observation that the two norms
\[
\sum_{i=1}^{N} \phi_i^2 \quad \text{and} \quad \sum_{i=1}^{N} \sum_{j=1}^{N} (\phi_i - \phi_j)^2
\] (17)
are minimised simultaneously. The second of these is a measure of the equidistribution of \( \phi_T \) over the triangles.

The result generalises to any number of dimensions.

When \( F = U \mathbf{a} \) with \( U \) piecewise linear and \( \mathbf{a} \) divergence-free \( \phi_T \) is given by
\[
\phi_T = -S_T \mathbf{a} \cdot \nabla U
\] (18)
(cf. (11)) which in two dimensions has the form
\[
\phi_T = -\sum (\overline{a}Y_1 - \overline{b}X_1)(U_2 - U_3).
\] (19)

In that case minimisation of the norm
\[
\sum_{i=1}^{N} \phi_i^2 = \sum_{i=1}^{N} (S_T \mathbf{a} \cdot \nabla U)^2 = \sum_{i=1}^{N} \left( \sum (\overline{a}Y_1 - \overline{b}X_1)(U_2 - U_3) \right)^2
\] (20)
is equivalent to minimisation of the \( l_2 \) norm of the differences between the \( \phi_T \)'s of (19) provided that the boundary quantity in (16) is held constant.
5 Residual Equidistribution

From (7) we have
\[ \phi_T = -S_T \overline{R}_T \] (21)
so that the norms in (17) become
\[ \sum_{i=1}^{N} (S_T \overline{R}_T)_i \text{ and } \sum_{i=1}^{N} \sum_{j=1}^{N} (S_T \overline{R}_T)_i - (S_T \overline{R}_T)_j \] (22)

Another useful norm, used in [7], is the \( l_2 \) norm of the residual \( \overline{R}_T \) weighted by the triangle area, i.e.
\[ \sum_{i=1}^{N} S_T \overline{R}_T^2 \] (23)
(cf. (22)). In that case we may consider a generalisation of the identity (14), in the form
\[ \sum_{i=1}^{N} S_i \sum_{i=1}^{N} \left( \frac{\phi_i}{S_i} \right)^2 = \left( \sum_{i=1}^{N} \phi_i \right)^2 + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} S_i S_j \left( \frac{\phi_i}{S_i} - \frac{\phi_j}{S_j} \right)^2 \] (24)
where \( S_i = S_{T_i} \) or, equivalently,
\[ \sum_{i=1}^{N} S_i \sum_{i=1}^{N} \overline{R}_i \equiv \left( \sum_{i=1}^{N} \phi_{T_i} \right)^2 + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} S_i S_j (\overline{R}_i - \overline{R}_j)^2 \] (25)
(see (4)). Clearly
\[ \sum_{i=1}^{N} S_i = \int_{\Omega} d\Omega_{T_i} = \Omega \] (26)
is a constant equal to the total area of the domain while
\[ \sum_{i=1}^{N} \phi_i = -\sum_{i=1}^{N} \int_{\partial T_i} \text{div} F d\Omega = -\int_{\partial \Omega} F.d\Omega, \] (27)
independent of the internal values of \( F \) and the internal grid locations as before.

We may thus write (25) as
\[ \sum_{i=1}^{N} S_i \overline{R}_i^2 \equiv \frac{1}{\Omega} \left( \int_{\partial \Omega} F.d\Omega \right)^2 + \frac{1}{2\Omega} \sum_{i=1}^{N} \sum_{j=1}^{N} S_i S_j (\overline{R}_i - \overline{R}_j)^2 \] (28)
from which it follows that, provided (27) is held constant, the weighted \( l_2 \) norm (23), when minimised over internal values of \( U \) and the internal grid points, is
least when the average residual $\overline{R}_T$ is equidistributed. If this property is not attainable we still have the result that the two norms

$$\sum_{i=1}^{N} S_i \overline{R}_i^2 \quad \text{and} \quad \sum_{i=1}^{N} \sum_{j=1}^{N} S_i S_j \left( \overline{R}_i - \overline{R}_j \right)^2$$ (29)

are minimised simultaneously.

Once again the result hold in higher dimensions.

When $F = U a$ where $U$ is piecewise linear and $a$ is divergence-free $\overline{R}_T$ is given by

$$\overline{R}_T = \overline{a} \nabla U$$ (30)

(cf. (11)) which in two dimensions takes the form

$$\overline{R}_T = \frac{\sum (\overline{a} Y_1 - \overline{b} X_1)(U_2 - U_3)}{S_T}.$$ (31)

In that case minimisation of the norm

$$\sum_{i=1}^{N} S_i \overline{R}_i^2 = \sum_{i=1}^{N} S_i (\overline{a} \nabla U)^2_i = \sum_{i=1}^{N} \left( \frac{\sum (\overline{a} Y_1 - \overline{b} X_1)(U_2 - U_3)}{S_T} \right)^2$$ (32)

is equivalent to minimisation of the norm of the difference in the residuals

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \left( \overline{R}_i - \overline{R}_j \right)^2,$$ (33)

again provided that (16) is held constant.

6 An Example of Residual Minimisation

An example in two dimensions, quoted in [7], for which $\overline{R}_{T_i}$ and $\overline{R}_{T_j}$ are equal to a high degree of approximation, is as follows. Let $\Omega$ be the rectangular box $|x| < 1, y > 1$ and let $a = (-y, x)$ so that $\text{div} a = 0$. Then from (6) and (7)

$$\phi_T = -S_T (-\overline{\nabla} U_x + \overline{X} U_y)_T$$ (34)

and

$$\overline{R}_T = -(-\overline{\nabla} U_x + \overline{X} U_y)_T$$ (35)

where $\overline{x}, \overline{y}$ are the centroid values of $x, y$. Let the inflow conditions be prescribed as zero except at two points on the boundary where $U$ takes the value 1 (see figure 1) and suppose that outflow conditions are also prescribed as zero except at the two 'mirror' points on the boundary where $U$ is also taken to be 1.
The weighted $l_2$ norm of the residual is now minimised over both nodal $U$ values and nodal locations, giving the variational equations

$$
\sum_{i=1}^{N} \frac{1}{2} R_i \left( -Y (Y_3 - Y_2) + X (X_3 - X_2) \right) = 0,
$$

$$
\sum_{i=1}^{N} \left( -\frac{b}{2} R_i (U_3 - U_2) + \frac{1}{4} R_i^2 (Y_3 - Y_2) \right) = 0
$$

and

$$
\sum_{i=1}^{N} \left( \frac{a}{2} R_i (U_3 - U_2) - \frac{1}{4} R_i^2 (X_3 - X_2) \right) = 0
$$

where node $i$ is also node 1 of the cyclic trio of nodes 1, 2, 3 and $R_T$ is given by (35).

The resulting grid is shown in figure 1. The nodes move into positions for which the residuals are small and, in accordance with the discussion in section 4, the sides of the triangles attempt to line up with the characteristics. Moreover, by the result in section 5, the $l_2$ norm of the differences in the residuals is also minimised, leading to approximate equidistribution.

7 Systems

A generalisation of the underlying identity (25) to systems has been given by Roe [8]. If $g_i$ is a column vector and $Q$ is a matrix of constant values we have the identity

$$
\sum_{i=1}^{N} S_i \sum_{i=1}^{N} S_i g_i^t Q g_i

\equiv \left( \sum_{i=1}^{N} S_i g_i \right)^t Q \left( \sum_{i=1}^{N} S_i g_i \right) + \sum_{i=1}^{N} \sum_{j=1}^{N} S_i S_j (g_i - g_j)^t Q (g_i - g_j).
$$

With $g_i = R_i$ the first sum in brackets on the right hand side is

$$
\sum_{i=1}^{N} S_i R_i = \sum_{i=1}^{N} \int_{\Omega_i} \text{div} F d\Omega = \sum_{i=1}^{N} \int_{\partial \Omega_i} F \cdot ds = \int_{\partial \Omega} F \cdot ds,
$$

(independent of internal values of $F$ or internal grid locations). Hence, from (39), provided that (40) is held fixed the minimum of the weighted least squares norm of the residuals

$$
\sum_{i=1}^{N} S_i R_i^t Q R_i
$$

is achieved when the weighted norm of the residual differences

$$
\sum_{i=1}^{N} \sum_{j=1}^{N} S_i S_j (R_i - R_j)^t Q (R_i - R_j)
$$

(42)
is also minimised. The result is true in any number of dimensions.

Although the problem of solving $\vec{R}_i = \vec{R}_j$ is overdetermined in general the result shows, in the case of a vector residual for a conservative system, that there is a close connection between equidistribution and minimisation of the weighted least squares norm of the average residual in the sense that a weighted least squares norm of the average residual differences is minimised.

8 Vorticity

The self-cancelling property of (15) applies also to

$$\int_{\partial T_i} \vec{F} \times d\vec{s} = \int_{\partial T_i} \text{curl} \vec{F} d\Omega$$

so that defining

$$\overline{\omega} = \frac{1}{S_i} \int \text{curl} \vec{F} d\Omega$$

as the average local vorticity in triangle $T_i$, we have, using (39) with $g = \overline{\omega}$ and $Q = I$ that

$$\sum_{i=1}^{N} S_i \sum_{i=1}^{N} S_i \left| \overline{\omega}_i \right|^2$$

$$= \left( \sum_{i=1}^{N} S_i \overline{\omega}_i \right)^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} S_i S_j \left| \overline{\omega}_i - \overline{\omega}_j \right|^2.$$  \hspace{1cm} (45)

Now

$$\sum_{i=1}^{N} S_i \overline{\omega}_i = \sum_{i=1}^{N} \int_{\partial T_i} \text{curl} \vec{F} d\Omega = \sum_{i=1}^{N} \int_{\partial T_i} \vec{F} \times d\vec{s} = \int_{\partial \Omega} \vec{F} \times d\vec{s}$$

which is independent of internal values of $\vec{F}$ and internal grid points. Hence if (46) is held fixed we have from (45) that the weighted $l^2$ norm of $\overline{\omega}$ is minimised when the weighted $l^2$ norm of the differences in $\overline{\omega}$ is also minimised.

Combining (45) with (28) we have

$$\sum_{i=1}^{N} S_i \left( \sum_{i=1}^{N} S_i \left( \overline{R}_i^2 + \gamma \left| \overline{\omega}_i \right|^2 \right) \right)$$

$$\equiv \left( \int_{\partial \Omega} \vec{F} \cdot d\vec{s} \right)^2 + \gamma \left| \int_{\partial \Omega} \vec{F} \times d\vec{s} \right|^2$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{N} S_i S_j \left( \left( \overline{R}_i - \overline{R}_j \right)^2 + \gamma \left| \overline{\omega}_i - \overline{\omega}_j \right|^2 \right)$$

for any constant $\gamma$. Then if the boundary values of $\vec{F}$ and its grid are held fixed it follows that the norms

$$\sum_{i=1}^{N} S_i \left( \overline{R}_i^2 + \gamma \left| \overline{\omega}_i \right|^2 \right)$$

(48)
and
\[ \sum_{i=1}^{N} \sum_{j=1}^{N} S_i S_j \left( (\bar{R}_i - \bar{R}_j)^2 + \gamma |\bar{\omega}_i - \bar{\omega}_j|^2 \right) = 0 \]  
are minimised simultaneously. Hence (48) is least when both the average residuals and vorticities are equidistributed.

If \( \text{curl} \mathbf{F} = 0 \) and there exists a potential function \( \psi \) such that \( \text{div} \mathbf{F} = \nabla^2 \psi \) and
\[ \bar{R}_i = \frac{1}{S_i} \int_{\Omega_i} \nabla^2 \psi \, d\Omega = \frac{1}{S_i} \int_{\partial \Omega_i} \frac{\partial \psi}{\partial n} \, ds \]  
in two dimensions. Of course if \( \text{div} \mathbf{F} \) is also zero then \( \psi \) is harmonic.

\section{Cauchy-Riemann Equations}

An application of the result of the previous section is to the Cauchy-Riemann equations
\[ \text{div} \mathbf{F} = 0, \quad \text{curl} \mathbf{F} = 0 \]  
in two dimensions, i.e.
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0. \]  

Defining
\[ \bar{R}_i = \frac{1}{S_i} \int_{\Omega_i} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \, d\Omega \]  
\[ \bar{\omega}_i = \frac{1}{S_i} \int_{\Omega_i} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \, d\Omega, \]
approximate solutions may be obtained by minimising the least squares norm
\[ \sum_{i=1}^{N} S_i \left( \bar{R}_i^2 + \bar{\omega}_i^2 \right) \]  
over internal values of \( u, v \) and \( x, y \). By the result in (48), (49) the norm
\[ \sum_{i=1}^{N} \sum_{j=1}^{N} S_i S_j \left( (\bar{R}_i - \bar{R}_j)^2 + (\bar{\omega}_i - \bar{\omega}_j)^2 \right) \]
is also minimised, indicating approximate equidistribution of both \( \bar{R}_i \) and \( \bar{\omega}_i \).

In [7] Roe also considers the system
\[ (1 - M^2) \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \]  

9
where $M^2 > 1$ corresponds to a hyperbolic system and $M^2 < 1$ to an elliptic system (see also [9]). The norm minimised is

$$
\sum_{i=1}^{N} S_i \left( \overline{R}_i^2 + \left| M^2 - 1 \right| \overline{w}_i^2 \right)
$$

(57)

corresponding to (47) with $\gamma = |M^2 - 1|$. Minimisation of (57) over internal parameters thus implies minimisation of

$$
\sum_{i=1}^{N} \sum_{j=1}^{N} S_i S_j \left( (\overline{R}_i - \overline{R}_j)^2 + |M^2 - 1| (\overline{w}_i - \overline{w}_j)^2 \right),
$$

(58)

which indicates the extent to which $\overline{R}_i$ and $\overline{w}_i$ are approximately equidistributed.

10 Acknowledgement

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11 References


