

The Boundary Integral Method For Steady
Compressible Flow In Two Dimensions.

by

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Abstract. This is a preliminary report on the application of the boundary integral method to steady compressible flow past a body. Such problems are typical of many for which boundary integral methods would seem to be advantageous but for which a simple fundamental solution (corresponding to $\log R$ for potential problems in two dimensions) is not readily available. Various approaches to obtaining fundamental solutions of the equations are considered, as well as methods of approximating the field integral which will remain if these cannot be found. Some simple numerical experiments are reported which help to identify the most promising approaches to be adopted in future work.

1 General Description of Problem

1.1 Introduction

When fluid flow past a body, such as an aerofoil, is being considered, the values of the pressure, velocity etc. are seldom required some distance away from the body. Indeed it often happens that such values are required only on the body, and then when feasible a boundary integral method is well known to be best suited for their evaluation. Since this method involves only an integral around the boundary it effectively reduces the problem by one dimension, and this may give a substantial saving on computational time.

Moreover, if we apply a field method to such problems, some technique has to be devised to cope with the infinite domain. Although the boundary integral method produces an integral at infinity this can be dealt with very easily whereas field methods need to employ special techniques such as i) a simple truncation of the domain, ii) infinite elements or an infinite sequence of elements, iii) use of an asymptotic form of the solution far away from the body or iv) use of an inversion mapping procedure like that of Garabedian and Korn [1971], or Sellis [1968].

Most field methods, too, have more difficulty in applying the boundary conditions than the boundary integral method. In recent years various boundary-fitted coordinate systems (e.g. Thames et al. [1977]) have been used to produce a system in which the boundary conditions can be more easily applied, but even this approach requires more computational work than just a discretization of the body, which is all that is needed in the boundary integral method.

With these advantages of the boundary integral method in mind, we shall be considering in this report how to extend its application from

incompressible flows where it is well established to two dimensional compressible flow. Our objective is to consider possible ways of obtaining a simple fundamental solution, to indicate the difficulties which arise and to propose some ideas on how these difficulties might be overcome. Our prime aim is to explore the situation when compressibility is not only too important to be completely ignored, but also when the Prandtl-Glauert equation, which has been successfully used by Hunt [1980], is inadequate.

1.2 The Boundary Integral Method

If we consider steady, irrotational, two-dimensional subsonic motion, then the equation of continuity is

$$\nabla \cdot (\rho \nabla \phi) = 0, \quad (1.1a)$$

where ρ is the density and ϕ the velocity potential. The density is obtained from Bernoulli's equation

$$dp = -\rho q dq, \quad (1.1b)$$

where p is the pressure and q the speed. For incompressible flow the equation of continuity reduces to $\nabla^2 \phi = 0$ and the boundary integral technique is readily applicable.

For two suitably chosen functions ϕ and ψ the divergence theorem gives

$$\int_D (\psi \nabla^2 \phi - \phi \nabla^2 \psi) d\Omega = \int_C \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right), \quad (1.2)$$

where D is a domain bounded by the union of closed curves C and $\partial/\partial n$

is the inward normal derivative. The approach being adopted here is described in many texts (see, for example, Garabedian [1964]). Let $\psi = -\log R$, where $R = |\underline{x} - \underline{\xi}|$ ($\underline{\xi}$ being a fixed point), so that $\nabla^2\psi = 0$ provided $R \neq 0$. If $R = 0$ is a point in D then surround it by a circle of radius ϵ , apply (1.2) to the punctured domain and consider the limit as $\epsilon \rightarrow 0$. Now take ϕ to be the perturbation velocity potential of an incompressible fluid; then $\nabla^2\phi = 0$ gives

$$2\pi\phi(\underline{\xi}) = \int_C \left(\log R \cdot \frac{\partial\phi}{\partial n} - \phi \frac{\partial}{\partial n} \log R \right) ds. \quad \underline{\xi} \in D. \quad (1.3)$$

If $\underline{\xi}$ is on C , then by a similar method one obtains the boundary integral equation

$$\pi\phi(\underline{\xi}) = \int_C \left(\log R \frac{\partial\phi}{\partial n} - \phi \frac{\partial}{\partial n} \log R \right) ds. \quad \underline{\xi} \in C. \quad (1.4)$$

From the integral equation (1.4) the value of ϕ can be obtained everywhere on C if $\partial\phi/\partial n$ is specified on each point of C : then ϕ is given throughout D by (1.3). Equation (1.4) cannot be solved analytically in general and so some form of approximation has to be used. One of the standard approaches for dealing with it is given by Jaswon and Symm [1977] and is described below.

For incompressible flow past a body, D is now the infinite domain exterior to the body. Equation (1.2) is then applied in some annular region bounded by the body and some outer boundary which is allowed to go to infinity. Taking ϕ to be the perturbation velocity potential, which tends to zero as R tends to infinity, causes the contribution from the outer boundary to vanish. Then C is the body surface on which

$\partial\phi/\partial n$ is known. Now discretise the boundary into elements and approximate ϕ by Φ and $\partial\phi/\partial n$ by $\Phi^{(n)}$, where $\Phi, \Phi^{(n)}$ are constant in each element with the values supposed to be those at the mid points of the element. Let the body consist of N elements and denote an element by C_j , where $j = 1, \dots, N$, and the corresponding values of $\Phi, \Phi^{(n)}$ by $\Phi_j, \Phi_j^{(n)}$. Then the following system is obtained from (1.4) by taking ξ as the mid point of each element.

$$\pi\Phi_i = \sum_{j=1}^N \Phi_j^{(n)} \int_{C_j} \log R_i ds - \sum_{j=1}^N \Phi_j \int_{C_j} \frac{\partial}{\partial n} \log R_i ds$$

$$i = 1, 2, \dots, N \quad (1.5)$$

Here $R_i^2 = (x - \xi_i)^2 + (y - \eta_i)^2$, with (ξ_i, η_i) being the fixed point positioned at the centre of the element C_i , and $ds \equiv ds(x, y)$. These integrals can be approximated by a suitable approximating technique, e.g. Simpson's rule using the end points and mid-points of each element, except for the element containing the point (ξ_i, η_i) . For this we use the divergence theorem which gives the identity

$$\int_C \frac{\partial}{\partial n} \log R_i = -\pi.$$

Thus for the singular integral in the second sum we have

$$\int_{C_i} \frac{\partial}{\partial n} \log R_i ds = -\pi - \sum_{\substack{j=1 \\ j \neq i}}^N \int_{C_j} \frac{\partial}{\partial n} \log R_i ds.$$

Now consider the first sum in (1.5). For its singular component the corresponding element is taken as two straight lines (FIG 1) and then use is made of the fact that

$$\int_0^h \log |s| \cdot ds = h(\log h - 1) ,$$

to give the approximation

$$\int_{C_i} \log R_i \, ds = \hat{R}_i (\log \hat{R}_i - 1) + \hat{R}_{i+1} (\log \hat{R}_{i+1} - 1) ,$$

with \hat{R}_i, \hat{R}_{i+1} as shown in FIG 1.

In this way we obtain from (1.5) a system of equations for Φ in the form

$$A\Phi = \underline{b} ,$$

where A is a full, non-singular matrix and the known vector \underline{b} derives from the integral involving the $\Phi^{(n)}$'s ; Φ is the solution vector of the Φ 's . Several variants of this basic method are possible but this is sufficient for our present purpose.

1.3 Compressible Flow Equations

Whilst it is acceptable to assume that the density does not change for low subsonic motion we cannot make this assumption at higher speeds. For example, Serrin [1959] points out that if the maximum local mach number is 0.3 then the density variation is 5% . Thus for higher speeds some account of variation in density is required. Combining (1.1a) and (1.1b) gives the non-linear full potential equation

$$(c^2 - u^2)\phi_{xx} - 2uv\phi_{xy} + (c^2 - v^2)\phi_{yy} = 0 ,$$

where u, v are the velocity components in the x and y directions and c is the local speed of sound.

Another form of this equation which may be useful is that given by Von Mises [1958]

$$\frac{\partial^2 \phi}{\partial n^2} = (M^2 - 1) \frac{\partial^2 \phi}{\partial s^2} \quad (1.6)$$

where M is the local mach number, $\partial/\partial s$ is differentiation in the direction of the flow and $\partial/\partial n$ is differentiation in the direction normal to the flow. In general it is difficult to obtain the directions of n and s , but in the case of a uniform stream having a velocity U in the direction of the positive x -axis with velocity components at any point of the disturbed stream given by $(U + u, v)$, where $u/U, v/U \ll 1$, then writing the velocity potential in the form $Ux + \phi(x, y)$ reduces (1.6) to

$$(1 - M_\infty^2) \phi_{xx} + \phi_{yy} = 0 \quad ,$$

where M_∞ is the free stream mach number. This is the Prandtl-Glauert equation and by the change of variable $x = \bar{x}(1 - M_\infty^2)^{\frac{1}{2}}$, $y = \bar{y}$, it becomes

$$\phi_{\bar{x}\bar{x}} + \phi_{\bar{y}\bar{y}} = 0 \quad ,$$

and so can be dealt with as before.

Returning to the general case we note that for the boundary integral method to be applied directly to an elliptic problem, the governing equation needs to be linear. So consider solving system (1.1) by iterating between (1.1a) and (1.1b). From (1.1b) ρ can be obtained as a function of the velocity; thus one form of linearization is to take ρ as being a known quantity in (1.1a). Then for two suitable functions ϕ and ψ the divergence theorem gives

$$\int_D (\psi \nabla \cdot (\rho \nabla \phi) - \phi \nabla \cdot (\rho \nabla \psi)) d\Omega = \int_C \rho \left\{ \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right\} ds, \quad (1.7)$$

Thus we require a solution of $\nabla \cdot (\rho \nabla \psi) = 0$ which has a logarithmic singularity at $R = 0$. A solution with this property is known as a fundamental solution and its existence is guaranteed by the following theorem due to Miranda [1970].

Theorem If in a region Ω , ρ is continuous and has continuous first and second derivatives, then the equation $\nabla \cdot (\rho \nabla \phi) = 0$ admits a fundamental solution in every bounded domain contained in Ω .

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In the following section various analytical techniques are considered for obtaining an insight into the structure of the fundamental solution for a linearized form of (1.1a). Although we show that an integral representation for such a fundamental solution always exists, it is only for analytically known forms of ρ that the integrals can be evaluated exactly.

2 Methods For Obtaining A Fundamental Solution

2.1 Parametrix Approach

Consider the linear equation $L[\phi] \equiv \nabla \cdot (\rho \nabla \phi) = 0$ in two dimensions, where ρ is a known function. Then its parametrix is defined as a function $P(x, y; \xi, \eta)$ such that

$$P = O(\log R), \quad P' = O(R^{-1}), \quad P'' = O(R^{-2})$$

and $\rho \nabla^2 P = 0$, where P' represents the first derivative of P with respect to x or y , P'' represents the second derivatives of P , $R^2 = (x - \xi)^2 + (y - \eta)^2$ and (ξ, η) is a fixed point. Such a function is given by

$$\pi P(\underline{x}, \underline{\xi}) = \frac{1}{2} \rho(\underline{\xi}) \log\{\rho^{\frac{1}{2}}(\underline{\xi})/R\}, \quad (2.1)$$

where $\underline{x} = (x, y)$, $\underline{\xi} = (\xi, \eta)$. Now suppose we define the function

$$S(\underline{x}, \underline{\eta}) = P(\underline{x}, \underline{\eta}) + \int_D \lambda(\underline{\xi}) P(\underline{x}, \underline{\xi}) d\underline{\xi}, \quad (2.2)$$

where D is some domain over which we are considering $L[\phi] = 0$. Let L operate on P with respect to \underline{x} and the operator adjoint to L operate on P with respect to $\underline{\xi}$. Then by using the generalized Poisson equation, it can be shown that (Garabedian [1964]) if λ satisfies

$$\lambda(\underline{x}) = L[P(\underline{x}, \underline{\eta})] + \int_D L[P(\underline{x}, \underline{\xi})] \lambda(\underline{\xi}) d\underline{\xi}, \quad (2.3)$$

then $S(\underline{x}, \underline{\eta})$ is a fundamental solution of $L[S] = 0$. If the equation for $\lambda(\underline{x})$ is to be solved by a successive iteration process then convergence depends on $\|L[P]\| < 1$, where $\|\cdot\|$ is the L_2 norm (L_2 being the space of all square integrable functions in the usual way).

If $L[P]$ has a singularity of the form $R^{-\alpha}$ with $\alpha > \frac{1}{2}$, then this inequality will not hold, which will in turn mean that an iterative solution will not converge in any sense. So a method for finding $\lambda(\underline{x})$ which guarantees convergence is needed.

2.2 Vekua's Method

Vekua [1967] considered the solution of elliptic systems by transforming them to hyperbolic systems. Following his idea, consider the linear equation

$$L_1[\phi] \equiv \nabla^2 \phi + \hat{a}(x, y)\phi_x + \hat{b}(x, y)\phi_y + \hat{c}(x, y)\phi = 0,$$

where $\hat{a}, \hat{b}, \hat{c}$ are real analytic functions of their arguments. Make the change of variable $z = x + iy$, $z^* = x - iy$ and thereby introduce the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial z^*} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Then $L_1[\phi]$ can be re-written as

$$L'_1[\phi] \equiv \phi_{zz^*} + a(z, z^*)\phi_z + b(z, z^*)\phi_{z^*} + c(z, z^*)\phi = 0, \quad (2.4)$$

where $a(z, z^*) = \frac{1}{4} \left\{ \hat{a}(\frac{1}{2}z + z^*, \frac{1}{2}z^* - z) - i\hat{b}(\frac{1}{2}z + z^*, \frac{1}{2}z^* - z) \right\}$,

$$b(z, z^*) = \frac{1}{4} \left\{ \hat{a}(\frac{1}{2}z + z^*, \frac{1}{2}z^* - z) - i\hat{b}(\frac{1}{2}z + z^*, \frac{1}{2}z^* - z) \right\},$$

$$c(z, z^*) = \frac{1}{4} \hat{c}(\frac{1}{2}z + z^*, \frac{1}{2}z^* - z).$$

Assuming that ρ has an analytic continuation into the complex plane, then using the above transformation $L[\phi] \equiv \rho \nabla^2 \phi + \nabla \rho \cdot \nabla \phi = 0$ becomes

$$L'[\phi] \equiv \rho \phi_{zz^*} + \frac{1}{2} \rho_z \phi_{z^*} + \frac{1}{2} \rho_{z^*} \phi_z = 0. \quad (2.5)$$

Now seek a fundamental solution of $L'[\phi]$ in the form

$$S(z, z^* ; \delta, \delta^*) = -A(z, z^* ; \delta, \delta^*) \log R + B(z, z^* ; \delta, \delta^*)$$

where the singular point is $\delta = \xi + i\eta$, $\delta^* = \xi - i\eta$ and

$R^2 = (z - \delta)(z^* - \delta^*)$. The functions A and B are regular functions of z, z^*, δ, δ^* . The function S satisfies $L'[S] = 0$ i.e.

$$L'[A] \log R + \frac{1}{2} \frac{(2\rho_A{}_z + \rho_z A)}{(z^* - \delta^*)} + \frac{1}{2} \frac{(2\rho_A{}_{z^*} + \rho_{z^*} A)}{(z - \delta)} - L'[B] = 0 \quad (2.6)$$

Examining the singularities we see that since the $\log R$ singularity is multiple-valued it cannot be removed by contributions from poles alone and so it must be annihilated by setting $L'[A] = 0$. Removal of the other singularities is assured by setting

$$2\rho_A{}_z + \rho_z A = 0 \quad \text{when } z^* = \delta^*,$$

$$2\rho_A{}_{z^*} + \rho_{z^*} A = 0 \quad \text{when } z = \delta.$$

Setting $A(\delta, \delta^* ; \delta, \delta^*) = 1$ and integrating the last two equations gives

$$\left. \begin{aligned} A(z, \delta^* ; \delta, \delta^*) &= [\rho(\delta, \delta^*)/\rho(z, \delta^*)]^{\frac{1}{2}}, \\ A(\delta, z^* ; \delta, \delta^*) &= [\rho(\delta, \delta^*)/(\delta, z^*)]^{\frac{1}{2}}. \end{aligned} \right\} \quad (2.7)$$

It can be noted that A is the Riemann function of L' .

Now $L'[A]$ can be re-written as

$$(A_\rho)_{z^*} + (A_{z^*}\rho)_z = 0 \quad (2.8)$$

and integrating with respect to z and then z^* and integrating the resulting equation by parts, one obtains, on using (2.7),

$$\begin{aligned}
 A(z, z^* ; \delta, \delta^*) \rho(z, z^*) &= \rho(\delta, \delta^*) - \frac{1}{2} \int_z^\delta A(\sigma, z^* ; \delta, \delta^*) \rho_\sigma(\sigma, z^*) d\sigma \\
 &- \frac{1}{2} \int_{z^*}^{\delta^*} A(z, \tau ; \delta, \delta^*) \rho_\tau(z, \tau) d\tau . \quad (2.9)
 \end{aligned}$$

Since A satisfies a Volterra integral equation, if it is solved by a successive iteration process, then the resulting series solution will converge for all $\rho > 0$. Choosing $A^{(0)}(z, z^* ; \delta, \delta^*) = (\rho(\delta, \delta^*) / (z, z^*))^{\frac{1}{2}}$ gives, as the next iteration

$$\begin{aligned}
 A^{(1)}(z, z^* ; \delta, \delta^*) &= (\rho(\delta, \delta^*))^{\frac{1}{2}} \{ (\rho(\delta, \delta^*))^{\frac{1}{2}} - (\rho(\delta, z^*))^{\frac{1}{2}} \\
 &- (\rho(z, \delta^*))^{\frac{1}{2}} + 2(\rho(z, z^*))^{\frac{1}{2}} \} / \rho(z, z^*) .
 \end{aligned}$$

To carry out the next iteration the structure of $\rho(\delta, z^*)$ and $\rho(z, \delta^*)$ is needed, but this implies that ρ must be analytically continued into a new space, in such a way that $\rho(a + ib, c + id)$ is meaningful. This approach will not be taken further here but it looks promising.

2.3 Bergman's Method

Another way in which the function $A(z, z^* ; \delta, \delta^*)$ can be obtained, in principle, is by using a method due to Bergman [1969] which also results in an integral representation of a solution of $L'[A] = 0$. So consider the general linear elliptic equation in the variables $z = x + iy$, $z^* = x - iy$, given by (2.4) and make the change of dependent variable to

$$v(z, z^*) = \phi(z, z^*) \exp \left\{ \int_0^{z^*} a(z, \delta^*) d\delta^* - n(z) \right\} ,$$

where $n(z)$ is an arbitrary analytic function. Then (2.4) becomes

$$V_{zz^*} + D(z, z^*)V_{z^*} + F(z, z^*)V = 0, \quad (2.10)$$

where

$$D(z, z^*) = n'(z) + b(z, z^*) - \int_0^{z^*} a_z(z, \delta^*) d\delta^*,$$

$$F(z, z^*) = -\{a_z(z, z^*) + a(z, z^*)b(z, z^*) - c(z, z^*)\}.$$

Now we seek a solution of (2.10) in the form

$$V(z, z^*) = \int_{\mathcal{L}} E(z, z^*, t) f(\frac{1}{2}z(1-t^2)) \frac{dt}{(1-t^2)^{\frac{1}{2}}}, \quad (2.11)$$

where \mathcal{L} is a path in the complex t -plane which connects the point -1 to $+1$ avoiding the point $t = 0$, and $f(z)$ is an arbitrary analytic function, regular at the origin. The function $E(z, z^*, t)$ is taken to be an analytic function of t for $|t| \leq 1$ and of z, z^* in some neighbourhood of the origin in \mathbb{C}^2 . This form of operator, (2.11), was suggested by the study of the equation $\nabla^2 u + u = 0$, the details of which can be found in two of Bergman's papers [1930], [1945]. Substituting (2.11) into (2.10), and using $f_z = -f_t(1-t^2)/2zt$ we obtain, after integrating by parts,

$$V_{zz^*} + DV_{z^*} + FV = \int_{\mathcal{L}} \frac{\{(1-t^2)E_{tz^*} - t^{-1}E_{z^*} + 2tz(E_{zz^*} + DE_{z^*} + FE)\}}{2zt(1-t^2)^{\frac{1}{2}}} f dt. \quad (2.12)$$

Thus if E satisfies

$$(1-t^2)E_{tz^*} - t^{-1}E_{z^*} + 2tz(E_{zz^*} + DE_{z^*} + FE) = 0, \quad (2.13)$$

equation (2.11) gives a solution of (2.10). Existence and regularity

of $E(z, z^*, t)$ have been established by Colton [1976] by considering a solution of (2.13) in the form

$$E(z, z^*, t) = 1 + \sum_{n=1}^{\infty} t^{2n} z^n \int_0^{z^*} K^{(2n)}(z, z^*) dz^* \quad (2.14)$$

Substituting (2.14) into (2.13) gives the recursion formula

$$\left. \begin{aligned} K^{(2)} &= -2F, \\ (2n+1) K^{(2n+2)} &= -2 \left\{ K_z^{(2n)} + DK^{(2n)} + F \int_0^{z^*} K^{(2n)} dz^* \right\}, \quad n \geq 1 \end{aligned} \right\} (2.15)$$

from which it can be shown (Colton [1976]) that the series (2.14) converges absolutely and uniformly for t, z, z^* on compact subsets of \mathbb{C}^3 .

In the case of compressible flow the transformed continuity equation is given by (2.5), leading to

$$V(z, z^*) = \phi(z, z^*) (\rho(z, z^*))^{\frac{1}{2}} \text{Exp}(-n(z))$$

with

$$F(z, z^*) = -\rho^{\frac{1}{2}}(z, z^*) [\rho^{\frac{1}{2}}(z, z^*)]_{zz^*}, \quad D(z, z^*) = n'(z)$$

By taking $n(z) \equiv 0$ we get

$$\rho^{\frac{1}{2}} V_{zz^*} - V(\rho^{\frac{1}{2}})_{zz^*} = 0 \quad (2.16)$$

From (2.15) $\rho^{\frac{1}{2}} K^{(2)} = 2(\rho^{\frac{1}{2}})_{zz^*}$ and therefore if $K^{(4)}$ is to be found the value of

$$\int_0^{z^*} \rho^{-\frac{1}{2}}(z, z^*) [\rho^{\frac{1}{2}}(z, z^*)]_{zz^*} dz^*$$

is needed. Since ρ is not known analytically, explicit evaluation of

the integral is not possible at this stage. On the other hand we can deduce that for three different hypothetical forms of ρ , (2.16) has the following fundamental solutions:

$$1) \nabla^2 \rho^{\frac{1}{2}} = 0 \Rightarrow \text{a fundamental solution of (2.16) is } S = -\log R,$$

$$2) \nabla^2 \rho^{\frac{1}{2}} = c^2 \rho^{\frac{1}{2}}, \quad (c = \text{constant}) \Rightarrow S = K_0(cR),$$

$$3) \nabla^2 \rho^{\frac{1}{2}} = -c^2 \rho^{\frac{1}{2}}, \quad (c = \text{constant}) \Rightarrow S = -Y_0(cR),$$

where $K_0(cR)$ and $Y_0(cR)$ are the Bessel functions of the second kind.

From the first case we obtain the following theorem.

Theorem A necessary and sufficient condition for $-\rho^{-\frac{1}{2}} \log R$ to be a fundamental solution of $\nabla \cdot (\rho \nabla \phi) = 0$ is that $\nabla^2 \rho^{\frac{1}{2}} = 0$.

Proof. That the condition is necessary comes from substituting $A \log R$ into the complex form of the differential equation and then assuming $A\rho^{\frac{1}{2}} \neq \text{constant}$ to get a contradiction. The condition is sufficient by (2.16). ////

2.4 Bäcklund Transformation

In this section we consider obtaining the Riemann function of (2.16) by firstly introducing an auxiliary equation, and then transforming both equations into a form which is easier to handle.

We write (2.16) as

$$\gamma V_{zz^*} - V\gamma_{zz^*} = 0, \quad (2.17a)$$

where $\gamma^2(z, z^*) = \rho(z, z^*)$. Then following an idea by Bauer [1976] we introduce the auxiliary equation

$$\gamma^2 W_{zz^*} + \{\gamma \gamma_{zz^*} - 2\gamma_z \gamma_{z^*}\} W = 0, \quad (2.17b)$$

and transform these equations into the form

$$(V - W)_z = f(z, z^*, V, W, \gamma),$$

$$(V + W)_{z^*} = g(z, z^*, V, W, \gamma).$$

The integrability requirement $U_{zz^*} = U_{z^*z}$ generates the relations

$$V_{zz^*} = \frac{1}{2}(g_z + f_{z^*}), \quad (2.18a)$$

$$W_{zz^*} = \frac{1}{2}(g_z - f_{z^*}). \quad (2.18b)$$

Writing (2.17a) as $V = \gamma V_{zz^*} / \gamma_{zz^*}$ and differentiating once with respect to z and once with respect to z^* and using (2.18a) will produce a form of V_{zz^*} , in terms of derivatives of f and g , which can be equated to (2.18a). A similar process is carried out with (2.17b) and (2.18b). Then from the combination of these two new equations together with (2.17) it is found that a choice of f and g which satisfies all the necessary equations is

$$f(z, z^*, V, W, \gamma) = (V + W)\gamma_z / \gamma,$$

$$g(z, z^*, V, W, \gamma) = (V - W)\gamma_{z^*} / \gamma.$$

This gives the transformed equations

$$\left. \begin{aligned} \gamma(V - W)_z &= (V + W)\gamma_z, \\ \gamma(V + W)_{z^*} &= (V - W)\gamma_{z^*}. \end{aligned} \right\} \quad (2.19)$$

Such a transformation is known as a Bäcklund transformation. Suppose we define two functions ϕ and ψ by

$$\phi = V + W ,$$

$$\psi = V - W ,$$

then equations (2.19) can be written as

$$\gamma \phi_{z^*} = \psi \gamma_{z^*} ,$$

$$\gamma \psi_z = \phi \gamma_z .$$

Integrating these gives

$$\phi(z, z^*) = \hat{f}(z) + \int_{\delta^*}^{z^*} \psi(z, \tau) \gamma_\tau(z, \tau) \gamma^{-1}(z, \tau) d\tau , \quad (2.20)$$

$$\psi(z, z^*) = \hat{g}(z^*) + \int_{\delta}^z \phi(t, z^*) \gamma_t(t, z^*) \gamma^{-1}(t, z^*) dt , \quad (2.21)$$

which are formally reminiscent of Vekua's approach. Imposing the condition that $V(\delta, z^*) = V(z, \delta^*) = 1$ and that $W(z, \delta^*) = W(\delta, z^*) = -1$ gives

$$\phi(z, z^*) = \int_{\delta^*}^{z^*} \psi(z, \tau) \gamma_\tau(z, \tau) \gamma(z, \tau) d\tau , \quad (2.22)$$

$$\psi(z, z^*) = 2 + \int_{\delta}^z \phi(t, z^*) \gamma_t(t, z^*) \gamma(t, z^*) dt . \quad (2.23)$$

Consider solving these equations by an iterative method. Choose

$\phi^{(0)} = 0$, then $\psi^{(0)} = 2$ and

$$\phi^{(1)} = 2 \int_{\delta^*}^{z^*} \gamma^{-1}(z, \tau) \gamma_\tau(z, \tau) d\tau = 2 \log \gamma(z, z^*) - 2 \log \gamma(z, \delta^*) ,$$

$$\psi^{(1)} = .2 + 2 \int_{\delta}^z \gamma^{-1}(t, z^*) \gamma_t(t, z^*) [\log \gamma(t, z^*) - \log \gamma(t, \delta^*)] dt .$$

The last equation cannot be integrated exactly due to the presence of $\gamma(t, \delta^*)$, but a second approximation to $V = \frac{1}{2}(\phi + \psi)$ is

$$V = 1 + \log \gamma(z, z^*) - \log \gamma(z, \delta^*) .$$

Identifying V as the Riemann function of (2.16) gives the following which need to be satisfied

$$1) \quad \gamma(\delta, z^*) = \gamma(\delta, \delta^*) ,$$

$$2) \quad \gamma_z(z, z^*) \gamma_{z^*}(z, z^*) + \gamma(z, z^*) \gamma_{zz^*}(z, z^*) \log\{\gamma(z, z^*)/\gamma(z, \delta^*)\} = 0 .$$

If these conditions were transformed back to real variables then their meaning would not be obvious due to the complex form of $\gamma(z, \delta^*)$.

2.5 A Fundamental Solution Via An Expansion

It is evident from the methods considered so far that $\rho^{-\frac{1}{2}} \log R$ plays an important part in the fundamental solution. In this section the possibility is therefore considered of obtaining a fundamental solution by means of a sequence of functions of which $\rho^{-\frac{1}{2}} \log R$ is the leading term.

Let the fundamental solution of $L[\phi] \equiv \nabla \cdot (\rho \nabla \phi) = 0$ be

$$\begin{aligned} S(x, y) &= - [\rho^{-\frac{1}{2}} + R^2 B_2(x, y) + O(R^4)] \log R , \\ &= - \hat{S}(x, y) \log R , \quad (\text{say}) . \end{aligned}$$

This function will satisfy all the conditions needed to remove the singularities of $L[S]$ if $L[\hat{S}] = 0$. Then substituting \hat{S} into the

complex form of $L[\hat{S}]$ and examining the coefficients of R leads to the requirement that

$$\begin{aligned} & (-\rho^{\frac{1}{2}})_{zz^*} + (z - \delta)\{\rho(B_2)_z + \frac{1}{2} \rho_z B_2\} \\ & + (z^* - \delta^*)\{\rho(B_2)_{z^*} + \frac{1}{2} \rho_{z^*} B_2\} + \rho B_2 = 0. \end{aligned} \quad (2.24)$$

If B_2 is taken to be $\rho^{-\frac{1}{2}}$ then the above condition reduces to $(\rho^{\frac{1}{2}})_{zz^*} = \rho^{\frac{1}{2}}$, which has been shown to give a fundamental solution $\rho^{-\frac{1}{2}} K_0(R)$ (see section 2.3). If on the other hand we take $\rho B_2 = (\rho^{\frac{1}{2}})_{zz^*}$, then this imposes the condition $(\rho^{\frac{1}{2}})_{zz^*} = \lambda \rho^{\frac{1}{2}}$, where λ is a constant, which leads to a fundamental solution $\rho^{-\frac{1}{2}} K_0(\lambda R)$ (see section 2.3). Either of these two conditions may not be realistic. Also if a function of the form

$$S(z, z^*) = -\rho^{-\frac{1}{2}} \text{Exp}[-(z - \delta)(z^* - \delta^*)] \log R,$$

is considered, it leads to the condition

$$(\rho^{\frac{1}{2}})_{zz^*} = (R^2 - 1) \rho^{\frac{1}{2}},$$

again the validity of this condition is unknown.

2.6 Transformation To The Hodograph Plane

So far we have been considering the continuity equation as a linear equation in ϕ with ρ being regarded as some known function. Now consider the non-linear equation which is obtained when (1.1a) and (1.1b) are combined to give the full potential equation. Since this equation is non-linear, direct application of the boundary integral method is not possible, but it is possible to obtain a linear system of equations if we transform the problem into the hodograph plane. We express the

(u, v) coordinates of the hodograph plane in terms of the stream function $\psi(x, y)$ and the velocity potential $\phi(x, y)$ by

$$u = \phi_x = \rho^{-1}\psi_y, \quad v = \phi_y = -\rho^{-1}\psi_x.$$

Then provided $\phi_{xx}\phi_{yy} - 2\phi_{xy} \neq 0$ one can consider u, v as the independent variables. Introducing the polar coordinates (q, θ) by $u = q \cos \theta$, $v = q \sin \theta$, from the equation of continuity we obtain

$$d\phi + i\rho^{-1}d\psi = qe^{-i\theta}dz. \quad (2.25)$$

From Bernoulli's equation we know that, for an adiabatic gas, ρ is a function of the speed q only. Using this fact and

$$\begin{aligned} \rho q z_q &= (\rho \phi_q + i\psi_q)e^{i\theta}, \\ \rho q z_\theta &= (\rho \phi_\theta + i\psi_\theta)e^{i\theta}, \end{aligned} \quad (2.26)$$

together with the compatibility condition $z_{q\theta} = z_{\theta q}$ gives

$$\begin{aligned} \phi_q &= q[(\rho q)^{-1}]_q \psi_\theta, \\ \rho \phi_\theta &= q\psi_q. \end{aligned} \quad (2.27)$$

These last equations are known as the hodograph equations and they are linear whatever the form of the pressure-density relation. From Bernoulli's equation, with c denoting the local velocity of sound and M the local mach number, we have

$$q\rho dq = -c^2 d\rho, \quad (2.28)$$

i.e. $q^2 \rho [(\rho q)^{-1}]_q = M^2 - 1$

It is convenient to change from q to a new variable

$$\lambda(q) = - \int_q^c q^{-1} (1 - M^2)^{\frac{1}{2}} dq, \quad (2.29)$$

to introduce a new function

$$z(\lambda) = \rho^{-1} (1 - M^2)^{\frac{1}{2}}, \quad (2.30)$$

and to define

$$\phi(\lambda, \theta) = (z(\lambda))^{\frac{1}{2}} \phi^*(\lambda, \theta). \quad (2.31)$$

Then eliminating ψ from (2.27) gives

$$\phi_{\lambda\lambda}^* + \phi_{\theta\theta}^* = z^{\frac{1}{2}} (z^{-\frac{1}{2}})_{\lambda\lambda} \phi^*. \quad (2.32)$$

Serrin [1959] indicates that for a local mach number below 0.7 it would be acceptable to approximate $(z^{-\frac{1}{2}})_{\lambda\lambda}$ by zero, thus giving

$$\phi_{\lambda\lambda}^* + \phi_{\theta\theta}^* = 0. \quad (2.33)$$

Although this equation only has a range of $0 \leq M < 0.7$, this is not the prime difficulty; the main problem will be in transforming back to real space, since the mapping between the (x, y) plane and the (u, v) plane is not bijective.

2.7 Subtracting The Flow At Infinity

In section (1.2) the application of the boundary integral method to incompressible flow was based on the perturbation velocity potential, thus simplifying the far field boundary integral. For the treatment of slightly compressible flow with $\log R$ as the approximate fundamental solution, the use of perturbation potential will also reduce the effect of the residual field integral and hence give an advantage over the full

velocity potential which we have been considering so far in this section. So writing the full potential in the form $Ux + \bar{\phi}(x, y)$, where U is the free stream velocity and substituting into (1.1a) we obtain

$$\nabla \cdot (\rho \nabla \bar{\phi}) = U \rho_x, \quad (2.34)$$

which differs from (1.1a) due to the term on the right hand side. If this equation is to be solved by a boundary integral method, there are two main ways to set about its formulation. The first is to consider $U \rho_x$ as a forcing term in (2.34) and to linearize the left hand side as described in section (1.3). Then on the first iteration of (2.34) ρ is constant and so the system to be solved is similar to that for the full potential: but on subsequent iterations, even if the fundamental solution is obtained, a field integral will exist due to the forcing term. The second way to deal with (2.34) is to substitute ρ_x from Bernoulli's equation (for an adiabatic gas) to give

$$L(\bar{\phi}) \equiv \nabla \cdot (\rho \nabla \bar{\phi}) + \alpha \bar{\phi}_{xx} + \beta \bar{\phi}_{xy} = 0, \quad (2.35)$$

with

$$\alpha = \rho^{2-\gamma} M_\infty^2 (\bar{\phi}_x - 1), \quad \beta = \rho^{2-\gamma} M_\infty^2 \bar{\phi}_y,$$

where γ is the ratio of the specific heats and M_∞ is the mach number at infinity; then we linearize (2.35) such that, after application of the divergence theorem with an exact fundamental solution, no field integral remains. Thus we solve (2.35) and (1.1b) by iteration, in such a way that at every iteration of (2.35) ρ , α , β take on known values.

When comparing these two approaches one has to decide whether it is more desirable to find a fundamental solution of a linearized (1.1a)

and, after application of the divergence theorem to the linearized (2.34), to always have a field integral, or, to seek a fundamental solution of a more complex equation, i.e. (2.35), with a view to removing all field integrals. In this second case, we combine the linearized (2.35) with its adjoint equation for a function ψ

$$M(\psi) \equiv \nabla \cdot (\rho \nabla \psi) + (\alpha \psi)_{xx} + (\beta \psi)_{xy} \quad , \quad (2.36)$$

and from the divergence theorem obtain

$$\int_D \{\psi L(\bar{\phi}) - \bar{\phi} M(\psi)\} d\Omega = \int_C \left\{ \rho \left[\psi \frac{\partial \bar{\phi}}{\partial n} - \bar{\phi} \frac{\partial \psi}{\partial n} \right] + [\alpha (\psi \bar{\phi}_x - \bar{\phi} \psi_x) - \psi \alpha_x \bar{\phi} + \beta \psi \bar{\phi}_x] \frac{\partial x}{\partial n} - [\bar{\phi} \beta \psi_x + \bar{\phi} \beta_x \psi] \frac{\partial y}{\partial n} \right\} ds \quad , \quad (2.37)$$

where C is the integral around the body only; the contribution from the outer boundary is zero because $\bar{\phi}$ tends to zero at infinity. Let a fundamental solution of $M(\psi) = 0$ be $S(x, y) = -A(x, y) \log R + B(x, y)$. Then for a point $\underline{\xi}$ on the body the singular contribution to (2.37) is

$$\pi \bar{\phi}(\underline{\xi}) \left\{ \rho A + \frac{1}{2} \alpha A \right\} \Big|_{\underline{\xi}}$$

To enable application of either Vekua's or Bergman's method to (2.36) we need to transform it into normal form. Now the characteristics of (2.36) are given by

$$\frac{dy}{dx} = \frac{\beta \pm \{4\rho(\rho + \alpha) - \beta^2\}^{\frac{1}{2}}}{2(\rho + \alpha)} \quad , \quad (2.38)$$

but as the analytical forms of ρ , α , β are not known at each iteration (2.38) cannot be integrated exactly, and it is difficult to see how to

apply these methods. There is an alternative approach by Hadamard [1923], which is the real variable version of the Vekua-Bergman methods, but this requires the transformation of $M(\psi) = 0$ into an equation of the form $\nabla^2\psi + \bar{a}(x, y)\psi_x + \bar{b}(x, y)\psi_y + \bar{c}(x, y)\psi = 0$ which again proves difficult. With the apparent impossibility of obtaining a fundamental solution to (2.36) the second approach loses its attraction. It may therefore be better to consider the first case, which uses a forcing term and to try approximating the resulting field integral.

In any case, obtaining a fundamental solution is not the end of the matter, for if a boundary integral method is to be applied to a problem, both the fundamental solution and its normal derivative are required on the boundary. Thus it may actually be simpler to use a simple singular function, with a known normal derivative and then approximate the residual field integral, rather than to work with a very complicated accurate fundamental solution. We explore this possibility in the next section.

3 Possible Techniques Of Solution Using A Field Integral

3.1 Introduction

From the previous section it is clear that calculating a fundamental solution is not a simple task, and even if one could be obtained it might well take a complicated form, which would make evaluation of the boundary integral far from easy. So the question arises as to what can be done if an exact form of the fundamental solution cannot be obtained. The main point is that a field integral is then present, following the use of the divergence theorem. If the singular function used is sufficiently 'close' to the exact fundamental solution, then it might be possible to neglect the field integral, but in general this will not be so. In this section some ideas are put forward for approximating the field integral and their relative merits considered.

3.2 Artificial Boundary Curves

One possibility is to solve the problem inwards from infinity by constructing annular regions in the fluid domain and applying (1.7) in each region.

Consider a nested series of closed curves C^j , $j = 0, 1, \dots, J$ placed at regular intervals away from the body C^0 , with N nodes on each curve. Then on C^j values of the unknowns ϕ^j and ϕ_n^j need to be calculated. Outside C^j the governing equation is taken as Laplace's equation, i.e. incompressible flow, which on use of (1.4), relates ϕ^j and ϕ_n^j by N equations in $2N$ unknowns. The divergence theorem is applied between C^j and C^{j+1} and the resulting field integral approximated by a quadrature rule which uses the values of ϕ^j and ϕ^{j+1} only. Thus

we obtain $2N$ equations linking $4N$ unknowns. Since ϕ_n^0 is known, we have in all $(2J + 1)N$ equations in $(2J + 1)N$ unknowns.

Unless J is very small this method will compare unfavourably with a field method as there are twice as many unknowns at each point and the main equations link $2N$ points. So this will not be followed up any further for the moment.

3.3 Use Of An Asymptotic Form

We can reduce the number of unknowns in section 3.2 by $2N$ if instead of applying Laplace's equation outside C^J we assume some asymptotic relation between the unknowns on C^J and C^{J-1} .

Alternatively the number of artificial boundary curves in section 3.2 might be reduced to just one by using an asymptotic form for the velocity potential outside it, for example the Rayleigh-Janzen expansion (Shiffer [1960]) which has the form $\phi = \phi_0 + M_\infty^2 \phi_1 + M_\infty^4 \phi_2 + \dots$. It is known that this breaks down near the body but it could be used to give the boundary condition on a sufficiently far removed curve.

Whilst this method may be acceptable for simple objects, it is not so easy to implement in the case of arbitrary shaped bodies, since obtaining even the first few terms of the Rayleigh-Janzen expansion becomes a major task.

3.4 Approximation By Finite Differences Or Finite Elements

If the boundary integral formula is used for the whole domain between the body C^0 and an outer artificial boundary C^J , then an approximation

to the field integral using field point values of the potential is needed. So consider the field integral

$$\int_D \phi L[S] d\Omega \quad (3.1)$$

where S is a singular function, D is the whole fluid domain and L is an elliptic operator. As in section 3.2 this can be approximated by a two-dimensional quadrature rule

$$\sum_{i=1}^N \sum_{j=0}^J \phi_{ij} L[S_{ij}] w_i w_j \quad (3.2)$$

where the field points lie on J closed curves extending away from the body and each curve has N elements, in such a way that a regular grid is formed around the body. The coefficients w_i, w_j are weight functions appropriate to the quadrature rule used.

The field point values ϕ_{ij} may then be calculated in terms of boundary values by use of finite difference or finite element approximations to the field equation $L[\phi] = 0$: the outer boundary values will need to be given by one of the asymptotic forms described in section 3.3.

To make this method as efficient as possible we consider only one or two curves around the body i.e. $J = 1, 2$. Then since the field integral is to be approximated only near the body, a singular function which causes it to be dominant there is needed. Once the field integral has been approximated in terms of the unknown values of ϕ on the body, the system can be solved iteratively in such a way that the approximation to the field integral uses the values of ϕ from the previous iteration.

This gives the iterative process in the form

$$K\phi^{[m]} = \{\underline{b} + F(\phi^{[m-1]})\}, \quad (3.3)$$

where K is the boundary matrix, \underline{b} the boundary vector and F the approximation to the field integral. By this means a matrix decomposition at every iteration is avoided. An example of the method is considered in section 4.

4 Preliminary Numerical Experiment

4.1 Introduction

In this section we describe two numerical experiments to check the feasibility of the ideas put forward in the previous section. First we consider solving an equation which has a known fundamental solution, by means of an integral equation in which we use a singular function significantly different from it and then approximate the field integral as described in section 3.4. Results can then be compared to those obtained using the known fundamental solution. In the second experiment various approximations to the fundamental solution are compared, with the field integral completely ignored.

4.2 Numerical Experiment Using A Field Integral

In this part we consider a moving ellipse in a stationary incompressible fluid. Let ϕ be the perturbation velocity potential which satisfies Laplace's equation. Apply (1.2) to ϕ and a function S , where S has a logarithmic singularity at $R = 0$ but which is not equal to $-\log R$, i.e. it is distinct from the fundamental solution. Then it is desirable to choose S such that the value of

$$\int_D \phi \nabla^2 S d\Omega ,$$

is only significant near the body. We take $S = K_0(R)$, the modified Bessel function of the second kind which has exponential decay as R tends to infinity. Let the ellipse have a ratio of semi-major axis to semi-minor axis of 10 : 1 with a non-dimensionalized semi-major axis of length 2. So for a fixed point ξ on the body, with

$R = |\underline{x} - \underline{\xi}|$ and \underline{x} traversing around the ellipse the values of $K_0(R)$ and $-\log R$ will be significantly different over much of the boundary. We then have two distinct integral equations, one which uses a fundamental solution and one which uses the modified Bessel function

i.e.

$$\pi\phi = \int_C \left\{ \log R \frac{\partial\phi}{\partial n} - \phi \frac{\partial}{\partial n} \log R \right\} ds, \quad (4.1)$$

$$\pi\phi = \int_C \left\{ \phi \frac{\partial}{\partial n} K_0(R) - K_0(R) \frac{\partial\phi}{\partial n} \right\} ds + \int_D \phi K_0(R) d\Omega, \quad (4.2)$$

the boundary condition on the body is given by the values $\nabla\phi \cdot \underline{n} = \nabla(Ux) \cdot \underline{n}$, where U is the free stream velocity that would be imposed if the ellipse were at rest in a moving fluid - we take its value as unity. The solution of (4.1) has already been considered in section 1.2 and we have only to consider (4.2). The body is discretised as for (4.1), to give

$$\pi\phi_i = \sum_{j=1}^N \phi_j \int_{C_j} \frac{\partial}{\partial n} K_0(R) ds - \sum_{j=1}^N \phi_j^{(n)} \int_{C_j} K_0(R) ds + \int_D \phi K_0(R) d\Omega, \quad i = 1, \dots, N, \quad (4.3)$$

The sums can be evaluated by the methods used in section 1.2, except for the singular element. There we re-write $K_0(R)$ as

$$K_0(R) = [K_0(R) + \log R] - \log R,$$

whose first term is then regular and equal to zero at $R = 0$, and the logarithmic singularity term can be evaluated as before.

Following the method outlined in section 3.4 we solve Laplace's equation in the field by finite differences. Consider the problem in

elliptic coordinates (ξ, η) , with the body lying on $\xi = a =$
 constant $= 0.1$. Let the finite difference grid be given by the curves
 $\xi = a + hj$, $j = 0, 1, \dots, J$ and $\eta = 2\pi i/N = ik$, $i = 1, 2, \dots, N$,
 where h and k are the step sizes. Then the finite difference equation
 is

$$\frac{1}{h^2} [\phi_{j-1}^i - 2\phi_j^i + \phi_{j+1}^i] + \frac{1}{k^2} [\phi_j^{i-1} - 2\phi_j^i + \phi_j^{i+1}] = 0, \quad (4.4)$$

where ϕ_j^i is the value of ϕ at $(a+jh, ik)$. For the boundary condition
 on $\xi = a + hJ$ we use an asymptotic relation connecting ϕ on
 $\xi = a + hJ$ and $\xi = a + h(J-1)$. To obtain this relation we note
 that the perturbation velocity potential past a circle behaves like $1/r$
 away from the body, thus

$$\phi \Big|_{r=\alpha h} \sim \frac{1}{\alpha} \phi \Big|_{r=h}. \quad (4.5)$$

So for bodies which have a conformal mapping onto a circle a similar relation
 exists and in the case of our ellipse

$$\phi \Big|_{\xi=a+hJ} \sim \left(\frac{J-1}{J} \right) \phi \Big|_{\xi=a+h(J-1)}. \quad (4.6)$$

This leads to a system for which the values of ϕ between $\xi = a + h$
 and $\xi = a + h(J-1)$ can be obtained in terms of the values on $\xi = a$.

Finally we approximate the field integral by a two-dimensional
 trapezium rule giving

$$w_i w_j = \frac{1}{4} V_{ij}, \quad (4.7)$$

where V_{ij} is the area of field element ij (see figure 2). Since the
 two-dimensional trapezium rule uses values of $\phi K_0(R)$ at the four corners

of the element, when evaluating it over a field element which has $R = 0$ at one of its corners the field element is combined with the adjacent element which shares the point $R = 0$ (see figure 3). As a first approximation the two-dimensional trapezium rule was used to approximate the integral over the combined field element.

If J and N are taken too large then the finite difference mesh will have a considerable number of points and solution of the difference equation will then become a major part of the work. Thus the advantage of the boundary integral method will be eroded. So an exercise in economy was carried out in which we took $J = 2$, $N = 8, 16, 32$ and calculated the approximations to the field integral in each case. Then we also considered the possibility of using the values of the field integral in the $N = 8$ case to generate solutions in the $N = 16, 32$ case without having to re-work the field integral. To do this we took the functions obtained for the field integral, in terms of the unknown Φ 's, and interpolated them around the body by assuming an angular variation in their value. Thus we obtained the results given in tables 1, 2 and 3.

By the symmetry of the problem only the quadrant $0 \leq \eta \leq \frac{1}{2}$ is considered. Table 1 shows the percentage error from the true potential obtained from the boundary integral method applied to (4.1). Table 2 shows the percentage error obtained with (4.2), where we have approximated the field integral for each N . Lastly table 3 shows the results when we use $N = 8$ and interpolate the value of the field integral around the body. The oscillatory errors in Table 2 are due to the approximation of the field integral. Since the results were obtained by subdividing the field elements in the η coordinate direction only, we would not expect any increase in accuracy of the finite difference approximation to ϕ in the strip. Also the trapezium rule procedure for approximating the field integral over the combined element is too crude when the mesh is refined. On the other hand, by using interpolation with the ^{refined} mesh (table 3), the oscillations are considerably reduced.

4.3 Numerical Experiment Neglecting The Field Integral

Here we consider applying the boundary integral method to compressible flow past a circle at a free stream mach number of 0.39, where the variables are scaled such that both the free stream velocity and the density are unity at infinity. The problem will be solved by iterating between (1.1a) and (1.1b) in such a way that the function ρ in (1.1a) will take the values from the previous iteration of the velocity potential, thus linearizing (1.1a) such that use can be made of (1.7). The density is initially taken as one everywhere. Since a fundamental solution of (1.1a) has not been found, when the full velocity potential $\hat{\phi}$ and a singular function S are applied to (1.7) a field integral remains. In this section we look at the consequences of neglecting this field integral.

Let the singular function be of the form

$$S(x, y) = -A(x, y) \log R,$$

where A is a regular function. Then after applying (1.7) we get

$$\begin{aligned} \pi(\rho A \hat{\phi}) \Big|_{\underline{\xi}} &= \int_C A \log R \frac{\partial \hat{\phi}}{\partial n} - \hat{\phi} \frac{\partial}{\partial n} (A \log R) \rho dS \\ &- \int_V \{ \hat{\phi} \nabla \cdot (\rho \nabla A \log R) - A \log R \nabla \cdot (\rho \nabla \hat{\phi}) \} d\Omega. \end{aligned}$$

$$\underline{\xi} \in C, \quad (4.8)$$

Differentiating Bernoulli's equation with respect to the body normal gives

$$\rho^{\gamma-2} \frac{\partial \rho}{\partial n} = -M_\infty^2 \nabla \hat{\phi} \cdot \nabla \frac{\partial \hat{\phi}}{\partial n}, \quad (4.9)$$

and since we have no flow through the body the normal derivative of ϕ is zero so that

$$\frac{\partial \phi}{\partial n} = 0 . \quad (4.10)$$

This also holds when using the perturbation potential since in either case we are using the same density distribution.

As suggested by Vekua's approach the choice of $A = \rho^{-\frac{1}{2}}$ was taken and compared with $A = 1$. These forms of A are of particular interest because if the true fundamental solution had the form $-A \log R$ with A a function of ρ only, then for the full potential all the boundary integrals would be evaluated exactly. This is due to the special geometry of the circle and the fact that $\log R$ has as a complex conjugate function an angle. Also with $A \equiv A(\rho)$ there will be no difficulty in applying the boundary conditions. Thus by using this form of function in (4.3) we are sure that any difference which occurs in the results of the full potential compared with the true ones are from the neglect of the field integral.

The following three cases have been considered

- 1) the full potential and $A = \rho^{-\frac{1}{2}}$,
- 2) the perturbation potential and $A = 1$,
- 3) the perturbation potential and $A = \rho^{-\frac{1}{2}}$.

For the full potential with $A = 1$ we did not obtain convergence of the solution, indicating the importance of the field integral in that case.

Taking η to be the angle measured from the forward stagnation point, figure 4 shows the local mach numbers obtained in the above three

cases, plus the exact incompressible solution and the compressible solution obtained by Williams [1979] using a finite element method, for which the maximum error is at $\eta = \frac{1}{2}\pi$ and is around 0.5%. The figure shows that for mach numbers up to 0.5 case 3 gives the best result, whereas nearer $\frac{1}{2}\pi$ case 2 approximates the solution better.

5 Conclusion

When considering the solution of compressible fluid flow problems by a boundary integral method it has been seen that obtaining an accurate form for the fundamental solution is not a simple task. Whilst it has been observed in section 4.2 that by using a singular function distinct from the fundamental solution reasonable accuracy may be achieved by approximation of the field integral, it is seen from section 4.3 that for the boundary integral method to reach its full potential the fundamental solution needs to be obtained. The most promising means for getting an accurate representation of a fundamental solution of the linearized equation (1.1a) comes from Vekua's work, which can give a series solution for the function for any analytically assumed form for ρ . Transformation to the hodograph plane on the other hand produces equations for which simple fundamental solutions can be found over a large range, but transformation back to real space is always a problem.

This is a preliminary study mainly aimed at setting out the alternative approaches that might be pursued. Thus any conclusions must be regarded as very tentative at this stage.

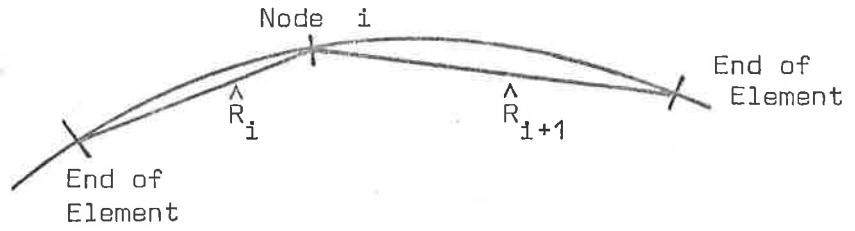


FIGURE 1.

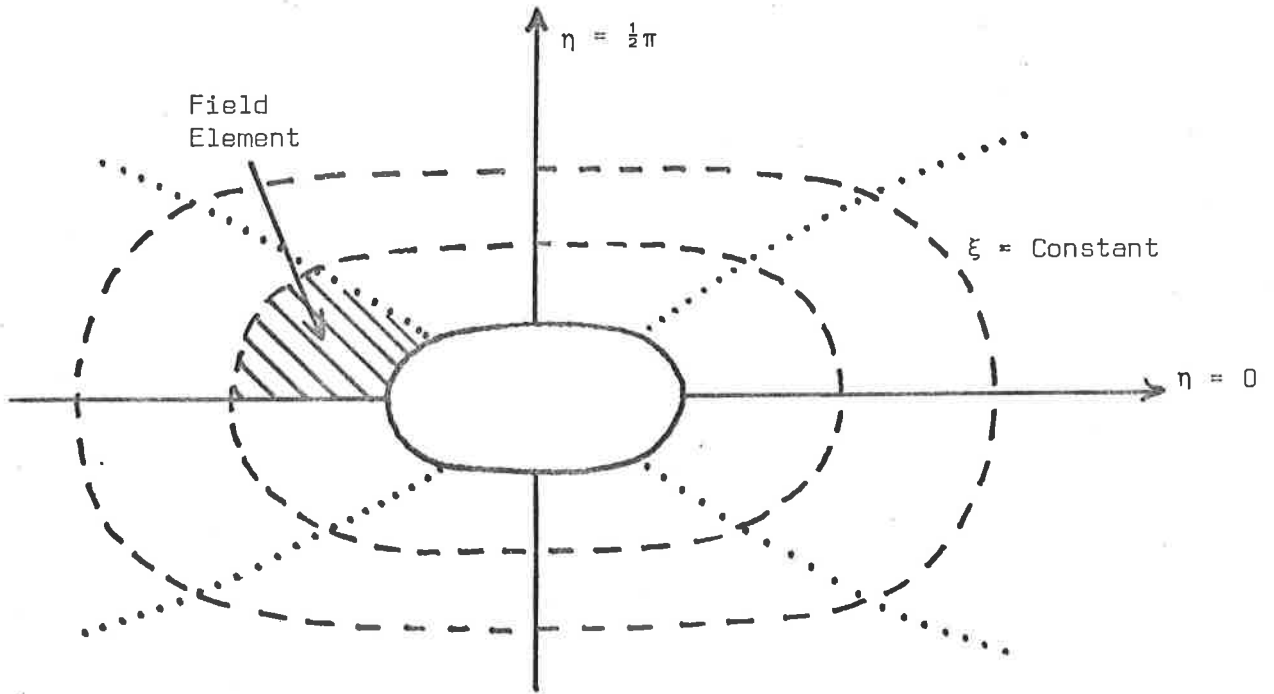


FIGURE 2.

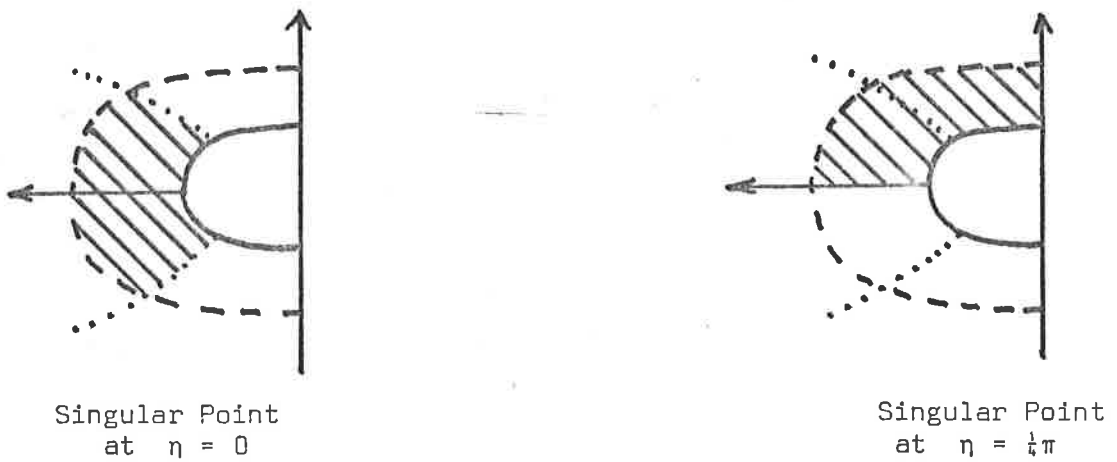


FIGURE 3.

TABLE 1.

η	N = 8	N = 16	N = 32
0	+40.56	+11.21	+2.33
$\pi/8$	/	- 1.73	-0.19
$\pi/4$	- 4.09	-0.53	-0.10
$3\pi/8$	/	-0.40	-0.08

Percentage Error In Potential Obtained From Using
The Exact Fundamental Solution Compared With
True Solution For N Elements.

TABLE 2.

η	N = 8	N = 16	N = 32
0	+34.50	+27.23	+25.96
$\pi/8$	/	+ 3.18	+ 9.98
$\pi/4$	- 2.14	- 1.92	+11.93
$3\pi/8$	/	+ 0.93	-12.30

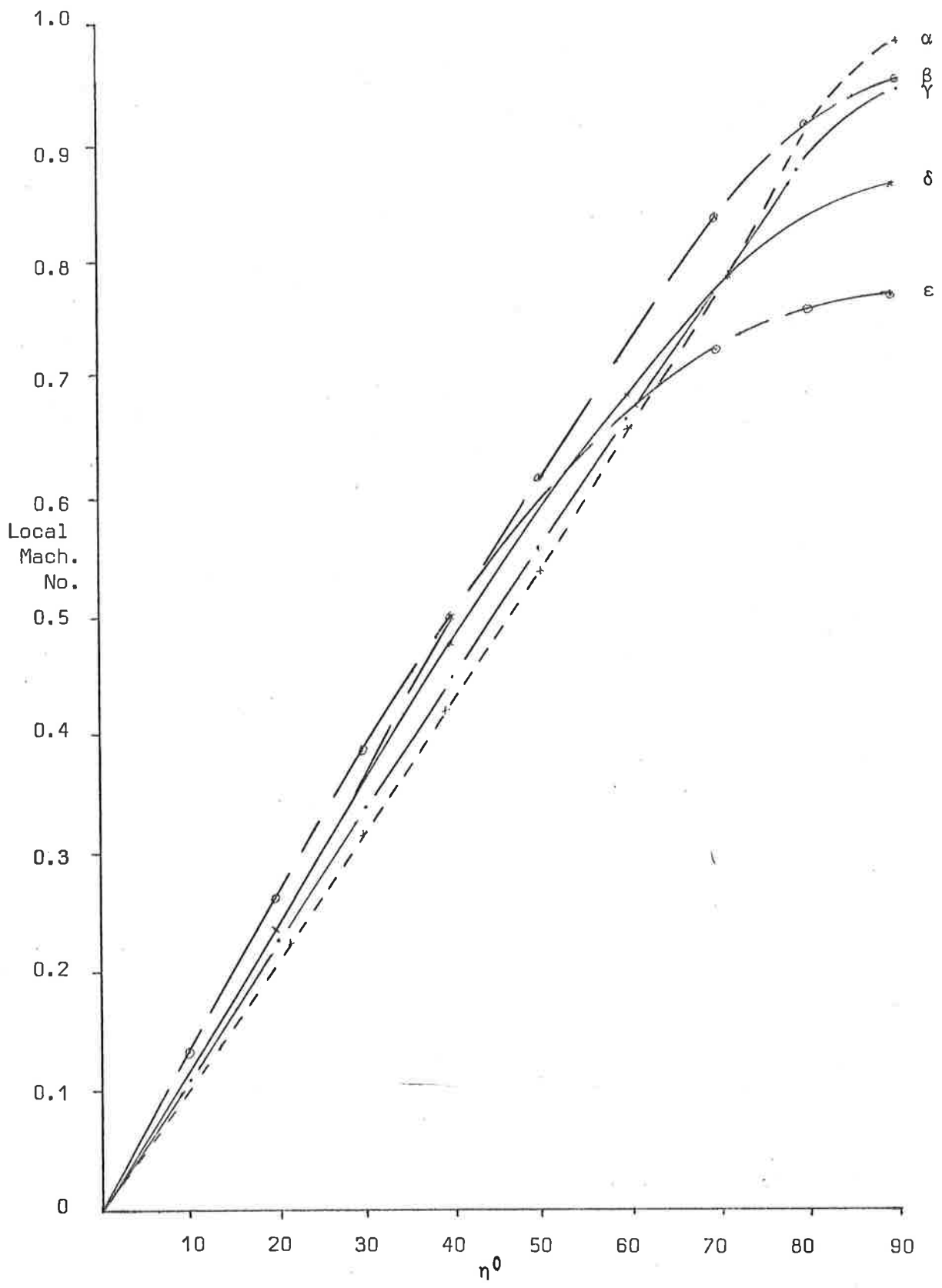
Percentage Error In Potential Obtained From Using
The Bessel Function As The Singular Function
And The Calculated Values Of The Field Integral.

TABLE 3.

η	N = 8	N = 16	N = 32
0	+25.54	+10.08	+6.51
$\pi/8$	/	- 4.52	+2.38
$\pi/4$	-24.51	- 7.09	-1.16
$3\pi/8$	/	-10.39	-4.43

Percentage Error In Potential Obtained From Using
The Bessel Function As The Singular Function And
The Interpolated Values Of The Field Integral.

α : Case 1
 β : Williams Compressible Solution
 γ : Case 2
 δ : Case 3
 ϵ : Exact Incompressible



FLOW AROUND A CIRCLE AT $M_\infty = 0.39$

FIGURE 4.

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