NUMERICAL SIMULATION OF HOT FLUID
INJECTION INTO POROUS MEDIA.

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ABSTRACT

This report develops a mathematical model for the injection of a hot fluid into a porous media, with applications to heavy oil recovery. The resulting differential equations are then solved numerically by a finite element method.
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§1 INTRODUCTION

The first stage in the extraction of light oils (oils with a low viscosity) is by Primary Recovery, a method which simply involves sinking a well into the reservoir and letting the fluid flow out under the natural pressure of its surrounds (see[4] & [5]). However this method usually removes less than half of the initial oil in place, and therefore secondary recovery techniques are used to remove the remaining oil, for example water flooding (see [4]). There also exist heavy oil fields, in which the oil is at such a high viscosity that recovery by primary or secondary methods is unfeasible. A number of different techniques exist for lowering the viscosity of the oil and thus allowing extracting, for example in-situ combustion (see[15]), injection of carbon dioxide, or steam injection (see [10] & [15]), there being an increased interest in the numerical simulation of these methods (see[1],[2],[3],[6] & [7]).

Steam injection, as its name suggests, involves the injection of steam under high pressure into the reservoir where it acts as a heating agent. After a specified period of time the steam is switched off and the melted oil flows out due to the pressures in the reservoir. It is known that the steam forms a front which propagates through the reservoir and the period for which steam is injected is dependent on the position of the front. Therefore accurate tracking of the front is essential, but existing numerical methods tend to give unsatisfactory oscillatory behaviour in the solution.

In this report we consider both modelling and numerical techniques for the simpler problem of injecting hot water into a reservoir of cold water. We have analysed this problem analytically and used a numerical method which appears to give good resolution without oscillations.

A mathematical model for the simpler problem is developed in section two of the report, with the resulting algebraic and differential equations being non-dimensionalised in section three. Analytic solutions are sought
for the non-dimensional equations in section four. In section five a
Finite Element technique (with fixed nodes) is used to obtain numerical
solutions for the equations, the results of which can be seen in section
six, with the conclusions being contained in section seven.
$2$ MATHEMATICAL MODEL

As a first step we shall consider the injection of hot water into a porous medium containing cold water, the temperature difference being of the order of $50^\circ$C.

We consider a cylindrical reservoir of radius $L$ and height $h$, with a well bore of radius $R$ dropping vertically through the centre of the reservoir (see fig. 1). The reservoir is considered homogeneous in all rock properties and isotropic with respect to permeability.

Fluid is injected from the well bore wall into the reservoir. The boundary condition at this point assumes the pressure is radially symmetric with a given pressure distribution vertically within the well bore. For the purpose of this report we shall assume a uniform pressure distribution within the well bore.

The strata lies horizontally with impermeable boundaries, the permeability throughout the rest of the well being homogeneous and isotropic. Therefore if we ignore the effects due to gravity the resulting flow may be assumed to be purely horizontal. As the reservoir is assumed to be radially symmetric with respect to all the rock properties, the flow of a fluid through
the reservoir will be radially symmetric.

§ 2.1 INITIAL AND BOUNDARY CONDITIONS

Initially the reservoir is at constant pressure, density and temperature, $P_{\text{int}}$, $\rho_{\text{int}}$ and $T_{\text{int}}$, respectively. At the outer boundary of the reservoir ($r = L$) there exists an insulated no-flow boundary, i.e., no heat or fluid flows across it. Also a no-flow condition is imposed on the top and bottom of the reservoir, but this is automatically satisfied, as we are considering only horizontal flow. For time greater than zero, water is injected at a constant mass rate $q$ at temperature $T_{\text{inj}}$ at the well bore ($r = R$) (see fig.2).

![Flow Diagram](image)

Flow out of the well bore is only in the radial direction. Therefore, as the surface area of the well bore is $2\pi Rh$,

$$\text{flow} = \frac{q}{2\pi Rh} \frac{Kg m^{-2} S^{-1}}{(mass/unit \ area/\ unit \ time)}$$

§ 2.2 GOVERNING EQUATIONS

As we are dealing with the flow of a heated liquid (the heat not being sufficient enough to change the phase of the fluid) through a porous medium, the unknowns in this problem are (I) the temperature of the liquid, (II) the pressure of the liquid, (III) the density of the liquid and (IV) the velocity of the liquid. The other properties, e.g., viscosity of the liquid, density of the rock etc., are taken as constants (see § 2.3.1). Therefore we require four equations for the four unknowns. These are (a) Darcy's law, which relates
the velocity of a fluid in a porous medium to its pressure gradient, (b) conservation of mass, (c) conservation of energy and (d) an equation of state: for this the definition of isothermal compressibility was chosen.

\section*{§ 2.2.1 Darcy's Law}

In 1856 Darcy (see [4]) derived an empirical law for the single phase flow of a fluid through a porous medium. Later King Hubbert (see [5]) obtained the same law from the Navier-Stokes equation of motion for a viscous fluid. The law may be stated in the form.

\[ V = \frac{K}{\mu} (\nabla p - \rho g d) \]  

(1)

where

\( V \) - superficial velocity (volume/unit area/unit time)
\( \mu \) - viscosity
\( p \) - pressure
\( \rho \) - density
\( g \) - gravity
\( d \) - depth
\( K \) - absolute permeability tensor (a function of the rock).

In this report we are ignoring the effects due to gravity and assuming a radially symmetric flow, so (1) simplifies to

\[ V = -\frac{K}{\mu} \nabla p \]  

(2)

where \( \nabla \) now takes the form \( \frac{\partial}{\partial r} \hat{r} \)

\( \hat{r} \) being the unit vector in the radial direction.

\section*{§ 2.2.2 Conservation of Mass}

If we consider an arbitrary volume \( R \) with boundary \( \partial R \) contained
in the porous region, then conservation of mass yields

\[
\begin{aligned}
\text{rate of change of mass in } R &= \text{flow of mass out of } R + \text{mass created in } R \\
&\quad \text{per unit volume}
\end{aligned}
\]

i.e.

\[
\frac{\partial}{\partial t} \left( \phi \rho_w \right) - \nabla \cdot \left( \rho_w \mathbf{V}_w \right) + \int_{\partial R} \rho_w \hat{n} \cdot \mathbf{d}T \quad \text{for } \int_{R} \mathbf{d}T > 0
\]

where

\[
\phi = \frac{V_{\text{pores}}}{V_{\text{tot}}} = \text{Volume occupied by pores} / \text{total volume}
\]

\[
\hat{n} = \text{outward unit normal to } R
\]

\[
q = \text{mass injected per volume, which is zero in the present problem}
\]

as fluid is injected or extracted only at the edge of the region. Therefore mass injected enters only as a boundary condition.

From the divergence theorem

\[
\int_{\partial R} \rho_w \mathbf{V}_w \cdot \hat{n} \mathbf{d}T = \int_{R} \nabla \cdot \left( \rho_w \mathbf{V}_w \right) \mathbf{d}T
\]

or

\[
\int_{R} \frac{\partial}{\partial t} (\phi \rho_w) + \nabla \cdot (\rho_w \mathbf{V}_w) \mathbf{d}T = 0
\]

and as we are considering an arbitrary region \( R \)

\[
\frac{\partial}{\partial t} (\phi \rho_w) + \nabla \cdot (\rho_w \mathbf{V}_w) = 0
\]

where the operator \( \nabla \cdot E \) is taken as

\[
\nabla \cdot E = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} E \right)
\]
§ 2.2.3 CONSERVATION OF ENERGY

Again considering an arbitrary volume $R$, conservation of energy gives:

$$\frac{\partial}{\partial t} \left( \phi \rho_w U_w + (1 - \phi) \rho_r U_r \right) dT = \int_R q_h \left( \frac{\partial H_w}{\partial T} \right) dV - \int_{\partial R} \left( \rho_w H_w \frac{V_w}{\partial V} \right) \cdot \frac{dn}{\partial N} - \int_{\partial R} k_\theta \nabla \cdot \frac{\nabla T}{\partial N} dS,$$

(ignoring effects due to radiation, because of their small magnitude), i.e.

$$\frac{\partial}{\partial t} \left( \phi \rho_w U_w + (1 - \phi) \rho_r U_r \right) dT = \int_R q_h \left( \frac{\partial H_w}{\partial T} \right) dV - \int_{\partial R} \left( \rho_w H_w \frac{V_w}{\partial V} \right) \cdot \frac{dn}{\partial N} - \int_{\partial R} k_\theta \nabla \cdot \frac{\nabla T}{\partial N} dS,$$

where

$U_w$ - specific internal energy

$H_w$ - specific enthalpy

$k_\theta$ - coefficient of thermal conductivity

$q$ - heat injected into $R$, taken as zero in the present problem as heat is injected at the edge of the reservoir: it is therefore again treated as a boundary condition.

After applying the divergence theorem,

$$\int_R \left\{ \frac{\partial}{\partial t} \left( \phi \rho_w U_w + (1 - \phi) \rho_r U_r \right) + \nabla \cdot \left( \rho_w H_w \frac{V_w}{\partial V} \right) - \nabla \cdot k_\theta \nabla T \right\} dV = 0,$$

but because $R$ is an arbitrary volume, we obtain

$$\frac{\partial}{\partial t} \left( \phi \rho_w U_w + (1 - \phi) \rho_r U_r \right) + \nabla \cdot \left( \rho_w H_w \frac{V_w}{\partial V} \right) - \nabla \cdot k_\theta \nabla T = 0 \quad (4)$$

§ 2.2.4 ISOTHERMAL COMPLEXIBILITY

As we are only dealing with a temperature range of $\sim 50^\circ C$, whereas
the pressure range should be of the order 500psi, we can assume that the
density of water is a function of the pressure only. Therefore, from the
definition of isothermal compressibility

$$C_w = - \frac{1}{V} \frac{\partial V}{\partial P}$$

where

$$C_w$$ - coefficient of isothermal compressibility for water.

But

$$\rho = \frac{m}{V}$$

so

$$C_w = - \frac{\rho}{m} \frac{\partial m}{\partial P} = - \frac{1}{\rho} \frac{\partial \rho}{\partial P} .$$

Integration gives

$$C_w \rho = \log \rho + \text{const} ,$$

but initially $$\rho = \rho_{\text{int}}$$ when $$p = p_{\text{int}}$$:

therefore

$$C_w \rho_{\text{int}} - \log \rho_{\text{int}} = \text{const}$$

which gives

$$C_w \rho - C_w \rho_{\text{int}} = \log \rho - \log \rho_{\text{int}}$$

$$C_w (\rho - \rho_{\text{int}}) = \log \left( \frac{\rho}{\rho_{\text{int}}} \right)$$

$$\rho = \rho_{\text{int}} \exp \left( C_w (\rho - \rho_{\text{int}}) \right)$$  \hspace{1cm} (6)

A consequence of using the definition of isothermal compressibility as the
equation of state is that it decouples the equation for the conservation
of mass (3) from the equation for the conservation of energy (4).

§ 2.3 PHYSICAL ASSUMPTIONS

2.3.1 ROCK PROPERTIES

Due to the homogeneity and isotropy of the rock, $$\phi$$, $$K$$, $$k_0$$ and $$\rho_r$$
can be assumed to be constant throughout the reservoir, for all time.

2.3.2 THERMODYNAMIC PROPERTIES

Defining

\[ U = \int_C v \, d\theta \]

\[ H = \int_C p \, d\theta \]

where

\( C_v \) - specific heat at constant volume,

\( C_p \) - specific heat at constant pressure,

it can be shown that (see [9])

\[ U = H - \frac{p}{\rho} . \]  \hspace{1cm} (7)

If we assume \( C_p^W \) and \( C_p^R \) are constants and that \( H_W = H_R = 0 \) when

\( T = T_{base} \) (a base temperature)

then

\[ H_W = C_p^W (T - T_{base}) \]  \hspace{1cm} (8)

\[ H_R = C_p^R (T - T_{base}) . \]  \hspace{1cm} (9)

Also if the pressure in the rock is taken to be the same as in the water, then by (7)

\[ U_W = H_W - \frac{p}{\rho_W} \]  \hspace{1cm} (10)

and

\[ U_R = H_R - \frac{p}{\rho_R} . \]  \hspace{1cm} (11)
§ 2.3.3 VISCOSITY

From [10] we take the viscosity of water $\mu$ to be given by the approximate formula

$$\frac{1}{\mu} = 0.14 + \theta/30 + 9 \times 10^{-8} \cdot \theta^2 \cdot \text{cp}^{-1} \cdot (\theta - 0^\circ C)$$

(see fig 3).

Fig 3

For the temperature range we are interested in it can be seen that $\mu$ varies by approximately 0.4 cp, but for the purpose of this report we will assume $\mu$ to be constant.
§ 2.4 SIMPLIFICATION OF EQUATIONS

Substituting (10) and (11) into (4) gives

\[ \frac{\partial}{\partial t} \left( \phi p_w (H_w - \frac{P}{\rho_w}) + (1 + \phi) \frac{\partial}{\partial t} \left( \phi \rho_w (H_w - \frac{\rho_w}{\rho_r}) \right) \right) + \nabla \cdot \left( \rho_w (H_w \nabla V_w) - \nabla \right) \cdot k_{\theta_\perp} \nabla T = 0 \]

\[ \frac{\partial}{\partial t} \left( \phi p_w H_w \right) + \frac{\partial}{\partial t} \left( (1 - \phi) \rho_r H_r \right) - \frac{\partial p}{\partial t} + \nabla \cdot \left( \rho_w (H_w \nabla V_w) - \nabla \right) \cdot k_{\theta_\perp} \nabla T = 0 \]

\[ H_w \frac{\partial}{\partial t} \left( \phi p_w \right) + \frac{\partial}{\partial t} \left( \phi p_w \frac{\partial H_w}{\partial t} + (1 - \phi) \frac{\partial}{\partial t} \left( \rho_r \frac{\partial H_r}{\partial t} \right) - \frac{\partial p}{\partial t} \right) \]

\[ + H_w \nabla \cdot \left( \rho_w \nabla V_w \right) + \rho_w \nabla \cdot V_w \cdot V_h - \nabla \cdot k_{\theta_\perp} \nabla T = 0 \]

\[ H_w \frac{\partial}{\partial t} \left( \phi p_w \right) + \frac{\partial}{\partial t} \left( \phi p_w \frac{\partial H_w}{\partial t} \right) + \phi \frac{\partial}{\partial t} \left( \rho_w \frac{\partial H_w}{\partial t} \right) + (1 - \phi) \frac{\partial}{\partial t} \left( \rho_r \frac{\partial H_r}{\partial t} \right) - \frac{\partial p}{\partial t} \]

\[ + \rho_w \nabla \cdot V_w \cdot V_h - \nabla \cdot k_{\theta_\perp} \nabla T = 0. \]

But from (3)

\[ \frac{\partial}{\partial t} \left( \phi p_w \right) + \nabla \cdot \left( \rho_w V_w \right) = 0 \]

so that equation (4) simplifies to

\[ \phi p_w \frac{\partial H_w}{\partial t} + (1 - \phi) \frac{\partial}{\partial t} \left( \rho_r \frac{\partial H_r}{\partial t} \right) - \frac{\partial p}{\partial t} + \rho_w \nabla \cdot V_w \cdot V_h - \nabla \cdot k_{\theta_\perp} \nabla T = 0. \]

Then substituting (8) and (9) in the above gives

\[ (\phi p_w \frac{C_w}{\rho} + (1 - \phi) \frac{C_r}{\rho}) \frac{\partial T}{\partial t} - \frac{\partial p}{\partial t} + \rho_w \nabla \cdot V_w \cdot C_w \frac{\partial T}{\partial T} - \nabla \cdot k_{\theta_\perp} \nabla T = 0 \]

and, finally substituting (2), namely

\[ V_w = - \frac{K}{\mu} \frac{\partial \rho}{\partial \rho_r} \]

into equation (3), and the above equation we obtain
\[
\frac{\partial}{\partial t} \left( \phi \rho_w \right) - \frac{1}{r} \frac{\partial}{\partial r} \left( \rho_w \phi \frac{K}{\mu} \frac{\partial p}{\partial r} \right) = 0
\]

(12)

and

\[
(\phi \rho_w c_p^w + (1-\phi) \rho_r c_p^r) \frac{\partial T}{\partial t} - \frac{\partial p}{\partial t} - \rho_w c_p^w \frac{K}{\mu} \frac{\partial p}{\partial r} \frac{\partial T}{\partial r} - \frac{1}{r} \frac{\partial}{\partial r} \left( r k \frac{\partial T}{\partial r} \right) = 0
\]

(13)

along with (6)

\[
\rho = \rho_{\text{int}} e^{c_w(p-p_{\text{int}})}
\]

which gives us three equations for the unknowns \( p, T \) and \( \rho_w \).

§ 2.5 Boundary Conditions

At \( r = R \) (well bore radius) water is injected at a constant mass rate \( q \) at temperature \( T_{\text{inj}} \).

The flow into the reservoir per unit area is

\[
\rho_w V_w = -\frac{q}{2\pi rh} \frac{\hat{r}}{r}
\]

and, substituting for \( V_w \), we obtain

\[-\frac{K_p}{\mu} \frac{\partial p}{\partial r} = \frac{q}{2\pi Rh}\]

which implies that

\[
\begin{align*}
\frac{\partial p}{\partial r} &= -\frac{q}{2\pi RhK_p} \\
\text{at} \quad r &= R
\end{align*}
\]

and \( T = T_{\text{inj}} \).

At \( r = L \) (outer boundary) no heat or fluid flows across the boundary, therefore

\[
\rho_w V_w = 0 \Rightarrow \frac{\partial p}{\partial r} = 0
\]

and

\[
\frac{\partial T}{\partial r} = 0
\]

at \( r = L \)

(15)
Summarising, (12), (13) and (6) form our set of three equations, with boundary conditions (14) and (15) and initial conditions

\[ p = \rho_{\text{int}}, \quad T = T_{\text{int}} \quad \text{and} \quad p = p_{\text{int}}. \]

§ 3 NON-DIMENSIONAL FORM

We now non-dimensionalise the equations of §2 so as to try and make the variation in the variables lie in the range 0(1), giving us a better idea of the size of the coefficients of the differential equations and thus leading to simplification of the system of equations. Also equation (6) will be assumed to be a linear function of pressure (see table 1).

Non-dimensional variables are represented by an overbar.

Let \( p_{\text{max}} \), \( \rho_{\text{max}} \), and \( t_{\text{max}} \) be the maximum pressure, density, and time obtained in this problem.

### TABLE 1

<table>
<thead>
<tr>
<th>Relationship between variable and non-dimensionalised variable</th>
<th>Range of non-dimensionalised variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r = L \bar{r} )</td>
<td>( \bar{r} \in [\bar{R}/L, 1] )</td>
</tr>
<tr>
<td>( T = T_{\text{int}} + (T_{\text{inj}} - T_{\text{int}}) \bar{T} )</td>
<td>( \bar{T} \in [0, 1] )</td>
</tr>
<tr>
<td>( p = \rho_{\text{int}} + \rho \bar{p} )</td>
<td>( \bar{p} \in [0, (p_{\text{max}} - p_{\text{int}})/\rho] )</td>
</tr>
<tr>
<td>( t = \bar{t} )</td>
<td>( \bar{t} \in [0, t_{\text{max}}/\tau] )</td>
</tr>
<tr>
<td>( \rho_{\bar{w}} = \rho_{\text{int}} + b \bar{p} )</td>
<td>( \rho_{\bar{w}} \in [0, (p_{\text{max}} - \rho_{\text{int}})/b \rho_{\text{max}}] )</td>
</tr>
</tbody>
</table>

Let

\[ \delta = R/L \]

\[ T_{\text{Diff}} = T_{\text{inj}} - T_{\text{int}}, \]
where \( \hat{p} \) is chosen such that \( \frac{3\hat{p}}{\partial r} = -1 \) at \( \hat{r} = \delta \). Then

\[
\frac{3\hat{p}}{\partial r} = \frac{\hat{p}}{\partial r} \left( \frac{L}{\hat{r}} \right)^2 = \frac{q}{\mu} \quad \text{at} \quad \hat{r} = R, \quad (r=\delta) \quad \text{by (14)},
\]

and hence

\[
\hat{p} = \frac{q}{\mu} \frac{L}{2\pi R h \rho_w K}.
\]

\( \tau \) is chosen so as to match the most dominant terms in the energy equation.

If we place \( p = p_{\text{int}} + \hat{p} \tilde{p} \) into (6),

\[
p = p_{\text{int}} e^{\hat{q} \tilde{p} p}
\]

Then expanding \( e^{\hat{q} \tilde{p} p} \) in the form

\[
p = p_{\text{int}} \left( 1 + c_{\hat{w} \tilde{p} p} + (c_{\hat{w} \tilde{p} p})^2 + \ldots \right)
\]

where

\[
c_{\hat{w} \tilde{p} p} \sim 10^{-3}
\]

\( p_{\text{int}} \sim 10^3 \)

we have

\[
p \sim p_{\text{int}} + b \hat{p} \tilde{p}
\]

where

\[
b = p_{\text{int}} c_{\hat{w} \tilde{p}} \sim 2
\]

Substituting the expressions in table 1 into equations (12) and (13) and the boundary conditions (14) and (15) gives

\[
\frac{1}{\tau} \frac{\partial}{\partial t} \left( \phi (p_{\text{int}} + b \hat{p}) \right) - \frac{1}{L^2} \frac{\partial}{\partial r} \left( (r k \rho_{\text{int}} + b \hat{p}) \frac{\partial}{\partial r} \left( p_{\text{int}} + \hat{p} \tilde{p} \right) \right) = 0
\]
and
\[
\left( \phi \rho_{\text{int}} + b \hat{p} \right) \frac{C_p}{\rho} + \left( 1 - \phi \right) \rho_r \frac{C_r}{\rho_r} \left( \frac{1}{\tau} \frac{\partial}{\partial t} \left( T_{\text{int}} + T_{\text{Diff}} \right) \frac{\partial}{\partial t} \right) - \frac{1}{\mu L^2} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} \left( p_{\text{int}} + \hat{p} \hat{p} \right) \right) \frac{\partial}{\partial r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} \left( r \rho_\theta \left( T_{\text{int}} + T_{\text{Diff}} \right) \right) = 0
\]

with
\[
\frac{\partial \bar{p}}{\partial r} = -1 \text{ at } \bar{r} = \delta , \quad \frac{\partial \bar{p}}{\partial r} = 0 \text{ at } \bar{r} = 1 , \quad \bar{p} = 0 \text{ at } \bar{t} = 0
\]

and
\[
\bar{T} = 1 \text{ at } \bar{r} = \delta , \bar{t} > 0 , \quad \frac{\partial \bar{T}}{\partial r} = 0 \text{ at } \bar{r} = 1 , \quad \bar{T} = 0 \text{ at } \bar{t} = 0
\]

This simplifies to
\[
\frac{\partial \bar{p}}{\partial t} - \frac{1}{\mu \phi b L^2} \left( \frac{\partial}{\partial r} \left( K \rho_{\text{int}} \frac{T_{\text{int}}}{\rho} + K b p \frac{T_{\text{int}}}{\rho} \right) \right) \frac{\partial \bar{p}}{\partial r} = 0 \tag{17}
\]

and
\[
\left( \phi \rho_{\text{int}} C_p^{\text{W}} + \left( 1 - \phi \right) \rho_r C_r^{\text{R}} \right) \left( \frac{\partial}{\partial t} \left( T_{\text{Diff}} \right) \frac{\partial}{\partial t} \right) + \rho b C_p^{\text{W}} \frac{T_{\text{Diff}}}{\rho} \left( \frac{\partial}{\partial t} \left( T_{\text{Diff}} \right) \frac{\partial}{\partial t} \right) + \rho \frac{\partial \bar{T}}{\partial r} \left( \frac{\partial}{\partial r} \left( r \rho_\theta \right) \right) = 0
\]

Gives
\[
\left( \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} \right) \right) \left( \frac{\partial}{\partial t} \right) + \rho b C_p^{\text{W}} \frac{T_{\text{Diff}}}{\rho} \left( \frac{\partial}{\partial t} \left( T_{\text{Diff}} \right) \frac{\partial}{\partial t} \right) + \rho \frac{\partial \bar{T}}{\partial r} \left( \frac{\partial}{\partial r} \left( r \rho_\theta \right) \right) = 0
\]

(18)
From the data set (see [11] given in appendix A, the two dominant terms of the energy equation are

\[
(\phi \rho_{\text{int}} C_P^w + (1-\phi) \rho_r C_P^r) \frac{T_{\text{Diff}}}{\tau} = f_1, \text{ say}
\]

and

\[
\frac{\rho_{\text{int}} C_P^w K \rho T_{\text{Diff}}}{\mu L^2} = h_1, \text{ say}.
\]

Thus \( \tau \) is chosen such that \( h_1 = f_1 \)

\[
\tau = \frac{\tau f_1}{h_1} = \frac{(\phi \rho_{\text{int}} C_P^w + (1-\phi) \rho_r C_P^r) T_{\text{Diff}}}{\rho_{\text{int}} C_P^w K \rho T_{\text{Diff}}} \frac{\mu L^2}{h_1}.
\]

Now substitute \( \tau \) into (17) and (18) and then divide equation (18) by the coefficient of \( \frac{\partial \tilde{T}}{\partial t} = f_1 \)

\[
\frac{\partial \tilde{T}}{\partial t} = f_1
\]

to give the two non-dimensional differential equations which follow.

We have

\[
\frac{\partial \tilde{p}}{\partial \tilde{t}} \frac{1}{r} \frac{\partial}{\partial \tilde{r}} \left( \tilde{r} (\alpha + \beta \tilde{p}) \frac{\partial \tilde{p}}{\partial \tilde{r}} \right) = 0 \tag{19}
\]

with

\[
\frac{\partial \tilde{p}}{\partial \tilde{r}} = -1 \text{ at } \tilde{r} = \delta
\]

\[
\frac{\partial \tilde{p}}{\partial \tilde{r}} = 0 \text{ at } \tilde{r} = 1
\]
\[ \tilde{p} = 0 \quad \text{at} \quad \tilde{t} = 0 \]

and

\[ (1 + \lambda \beta) \frac{\partial \tilde{T}}{\partial \tilde{t}} - (1 + \lambda \beta) \frac{\partial \tilde{p}}{\partial \tilde{r}} \tilde{r} \frac{\partial \tilde{T}}{\partial \tilde{r}} - \nu \tilde{\alpha} (\tilde{r} \frac{\partial \tilde{T}}{\partial \tilde{r}}) = \eta \frac{\partial \tilde{p}}{\partial \tilde{r}} \]

with

\[ \tilde{T} = \tilde{T}_{\text{inj}} \quad \text{at} \quad \tilde{r} = \delta, \tilde{t} > 0, \]

\[ \frac{\partial \tilde{T}}{\partial \tilde{r}} = 0 \quad \text{at} \quad \tilde{r} = 1, \]

\[ \tilde{T} = 0 \quad \text{at} \quad \tilde{t} = 0. \]

Again from the data set of appendix A the coefficients may be calculated as

\[ \alpha = 1687.80 \]

\[ \beta = 3.643 \]

\[ \lambda = 5.92488 \times 10^{-4} \]

\[ \omega = 2.15843 \times 10^{-3} \]

\[ \nu = 1.38088 \times 10^{-4} \]

\[ \eta = 3.0395 \times 10^{-2}. \]

§ 4 ANALYTIC SOLUTIONS

In this section we shall simplify the equations of §3 and seek some analytic solutions.

As \( \beta \) is small compared to \( \alpha \) in equation (19), we can as a first approximation place \( \beta = 0 \) in (19) giving

\[ \frac{\partial \tilde{p}}{\partial \tilde{t}} - \alpha \frac{\partial}{\partial \tilde{r}} (\tilde{r} \frac{\partial \tilde{p}}{\partial \tilde{r}}) = 0 \]
with
\[ \frac{\partial \tilde{p}}{\partial r} = 0 \quad \text{at} \quad r = 1, \]
\[ \frac{\partial \tilde{p}}{\partial r} = 1 \quad \text{at} \quad r = \delta. \]

**Fig 4**

\[ \frac{\partial \tilde{p}}{\partial r} = 0 \]
\[ \frac{\partial \tilde{p}}{\partial r} = -1 \]

- effect of injection felt at outer boundary \( r = 1 \)

After the injection has been taken place for a sufficient period of time such that the effect of the outer boundary \( r = 1 \) has been felt (see fig 4) - a semi-steady state condition occurs, that is to say if the injection rate is constant, then the reservoir pressure will increase in such a way that

\[ \frac{\partial \tilde{p}}{\partial t} \]
constant for all \( r \) and \( t \) (see [4]).

From (5) we have

\[ C_w = -1 \ \frac{\partial \tilde{V}}{\partial t} \quad \text{and} \quad C_w \ V \ \frac{\partial \tilde{p}}{\partial t} = - \frac{\partial \tilde{v}}{\partial t} = \text{Volume rate of injected fluid.} \]
But the volume rate of injected fluid = \( \frac{q}{p_w} \), so that

\[
\frac{dp}{dt} = \frac{q}{C_w \sqrt{p_w}} \quad \Rightarrow \quad V = \pi (1-R)^2 \phi \ .
\]

Hence

\[
\frac{dp}{dt} = \frac{q}{C_w \pi (1-R)^2 \ h \phi_T} = \frac{\hat{p}}{T} \frac{\delta \hat{p}}{\delta t} .
\]

\[
\frac{\delta \hat{p}}{\delta t} = \frac{q}{p \ C_w \pi (1-R)^2 \ h \phi} \cdot \psi (\phi_0, 0.901) \ .
\]

(22)

Then (19) becomes

\[
\frac{\delta}{\delta r} \left( r \frac{\delta \hat{p}}{\delta r} \right) = \frac{\hat{r} \hat{p}}{\alpha} = r \sigma \quad \text{where} \quad \sigma = \frac{\psi}{\alpha} 2 \delta
\]

\[
\frac{\delta \hat{r}}{\delta r} = r^2 \sigma + C \quad (C=C(t)) ;
\]

when \( \hat{r} = 1 \ \frac{\delta \hat{r}}{\delta r} = 0 \quad \forall \hat{r} \)

which implies that

\[
C = - \frac{\alpha}{\psi} \ 2 \delta .
\]

Letting \( \sigma = 2 \delta \)

\[
\frac{\delta \hat{p}}{\delta r} = \delta \hat{r}^2 - \delta
\]

\[
\{ \text{note that when} \ \hat{r} = \delta, \ \frac{\delta \hat{r}}{\delta r} = -1 + \delta^2 \hat{r} - 1 \}
\]

\[
\hat{p} = \frac{\delta}{2} \hat{r}^2 - \delta \log r + C_o \quad C_o = C_o(t) .
\]
Then differentiating with respect to $t$,

$$\frac{\partial \tilde{p}}{\partial t} = \frac{3C_0}{\partial t} = \psi \quad C_0 = \psi \tilde{t} + \text{const}$$

so that

$$\tilde{p} = \frac{\delta}{2} \tilde{r}^2 - \delta \log \tilde{r} + \psi \tilde{t} + \text{const}.$$  \hspace{1cm} (23)

which gives an equation for the non-dimensional pressure in the reservoir.

To solve equation (20) we require $\frac{\partial \tilde{p}}{\partial r}$, which from (23) is

$$\frac{\partial \tilde{p}}{\partial r} = \frac{\delta}{\tilde{r}} - \frac{\delta}{r}.$$  

---

**Fig. 5**

From Fig. 5 it can be seen that for $\tilde{r} < 0.05$, $\delta \tilde{r} \approx 0$, therefore

$$\frac{\partial \tilde{p}}{\partial r} \approx \frac{\delta}{r}.$$  \hspace{1cm} (24)
Substituting into equation (20) gives

\[
(1+\lambda p) \frac{\partial \tilde{T}}{\partial t} + (1+wp) \frac{\partial}{\partial r} \frac{\partial \tilde{T}}{\partial r} - \nu \frac{\partial}{\partial r} (r \frac{\partial \tilde{T}}{\partial r}) = \eta \frac{\partial \tilde{p}}{\partial r}.
\]

Next if we assume that \( \lambda, w \) and \( \eta \) are zero we have

\[
\frac{\partial \tilde{T}}{\partial t} + \frac{\partial}{\partial r} \frac{\partial \tilde{T}}{\partial r} - \nu \frac{\partial}{\partial r} (r \frac{\partial \tilde{T}}{\partial r}) = 0
\]

with boundary conditions

\[
\tilde{T} = 0 \text{ at } \tilde{r} = 0 \quad \forall \tilde{r}
\]

\[
\tilde{T} = 1 \text{ at } \tilde{r} = \delta \quad \tilde{t} > 0
\]

\[
\frac{\partial \tilde{T}}{\partial \tilde{r}} = 0 \text{ at } \tilde{r} = 1 \quad \forall \tilde{t}.
\]

If we assume that \( \tilde{T} = 1 \) at \( \tilde{r} = 0 \) for \( \tilde{t} > 0 \)

and instead of \( \frac{\partial \tilde{T}}{\partial \tilde{r}} = 0 \) at \( \tilde{r} = 1 \) we use \( \tilde{T} = 0 \) at \( \tilde{r} = 0 \) \( \forall \tilde{t} \)

we can find a similarity solution to (25).

Let \( \tilde{T}(\tilde{r}, \tilde{t}) = \tilde{t}^\alpha y(x) \), \( x = \frac{\tilde{r}}{\tilde{t}^\beta} \)

\[
\frac{\partial \tilde{T}}{\partial \tilde{t}} = \alpha \tilde{t}^{\alpha-1} y + \tilde{t}^\alpha y_x X_t
\]

\[
X_t = -\beta \tilde{t}^{\beta-1} = \frac{-\beta \tilde{t}^{\beta-1}}{\tilde{t}^\beta} = -\beta x t^{-1}
\]

so

\[
\frac{\partial \tilde{T}}{\partial \tilde{t}} = \tilde{t}^{\alpha-1} (\alpha y - \beta y x_x)
\]
\[ \frac{\partial \tilde{T}}{\partial \tilde{r}} = \tilde{t}^{\alpha} y_{\tilde{x}} x_{\tilde{r}} \]
\[ \tilde{x}_{\tilde{r}} = \tilde{t}^{-\beta} \]
\[ \frac{\partial \tilde{T}}{\partial \tilde{r}} = \tilde{t}^{\alpha-\beta} y_{\tilde{x}} \]
\[ \frac{\partial (r \tilde{T})}{\partial \tilde{r}} = \tilde{t}^{\alpha-\beta} \frac{\partial (r y_{\tilde{x}})}{\partial \tilde{r}} = \tilde{t}^{\alpha-\beta} (y_{\tilde{x}} + r \frac{\partial y_{\tilde{x}}}{\partial \tilde{r}} x_{\tilde{r}}) \]
\[ = \tilde{t}^{\alpha-\beta} (y_{\tilde{x}} + r \tilde{y}_{xx} \tilde{t}^{-\beta}) \]
\[ = \tilde{t}^{\alpha-\beta} y_{\tilde{x}} + \tilde{t}^{\alpha-2\beta} r \tilde{y}_{xx} \tilde{t}^{-\beta} \]

Substituting (27), (28) and (29) into (25) gives
\[ \tilde{t}^{\alpha-1} (a y - \beta x y_{\tilde{x}}) + \delta \frac{\partial}{\partial \tilde{r}} \tilde{t}^{\alpha-\beta} y_{\tilde{x}} - \nu \frac{\partial}{\partial \tilde{r}} \tilde{t}^{\alpha-\beta} y_{\tilde{x}} - \nu t^{\alpha-2\beta} y_{xx} = 0 \]

Next substituting (26) \[ x = \frac{\tilde{r}}{t^{\beta}} \] into the above gives
\[ \tilde{t}^{\alpha-1} (a y - \beta x y_{\tilde{x}}) + \delta \frac{\partial}{\partial x} \tilde{t}^{\alpha-2\beta} y_{\tilde{x}} - \nu \frac{\partial}{\partial x} \tilde{t}^{\alpha-2\beta} y_{\tilde{x}} - \nu t^{\alpha-2\beta} y_{xx} = 0 \]
or
\[ \tilde{t}^{\alpha-1} (a y - \beta x y_{\tilde{x}}) + \tilde{t}^{\alpha-2\beta} (\delta - \nu) \frac{\partial}{\partial x} y_{\tilde{x}} - \nu y_{xx} = 0 \]

For a solution to exist
\[ \tilde{t}^{\alpha-1} = \tilde{t}^{\alpha-2\beta} \] which implies that
\[ \alpha - 1 = \alpha - 2\beta \quad \text{or} \quad \beta = \frac{1}{2} \]
which gives
\[
(\alpha - \nu) y_x - \nu y_{xx} = 0.
\] (30)

The new boundary conditions are

(a) \( \tilde{T} = 1 \) at \( \tilde{r} = 0, \tilde{t} > 0 \) \( \Rightarrow \tilde{T} = \tilde{t}^\alpha y(0) \forall \tilde{t} > 0 \)

(b) \( \tilde{T} = 0 \) at \( \tilde{t} = 0 \forall \tilde{r} \Rightarrow 0 = \lim_{\tilde{t} \to 0} \tilde{t}^\alpha y(x) \forall \tilde{r} \)

(c) \( \tilde{T} = 0 \) as \( \tilde{r} \to \infty \forall \tilde{t} \Rightarrow 0 = \tilde{t}^\alpha y(x) \to \infty \)

From (a) \( \alpha = 0 \) and \( y(0) = 1 \)

and from (c) \( y(\infty) = 0 \)

which gives from (26)
\[ \tilde{T} = y(x) \text{ where } x = \tilde{r} \frac{\tilde{t}^{1/2}}{\tilde{t}^{1/3}} \]

Substituting \( \alpha \) into (30) gives us an ODE for \( y \),

namely
\[
-\frac{1}{2} x y_x + (\delta - \nu) y_x - \nu y_{xx} = 0.
\]

\[
y_{xx} + \frac{1}{2\nu} xy_x - \frac{(\delta - \nu)}{\nu} y_x = 0.
\]

Letting \( \nu = \frac{1}{2\nu} \)
\( \kappa = \frac{(\delta - \nu)}{\nu} \)

where from the data set [11]
\( \nu = 3620.8794 \)
\( \kappa = 0.931136 \)

we obtain
\[
y'' + (\nu - \kappa) y' = 0 \quad x
\]

\[
y'' + (\nu - \kappa) y' = 0
\] (31)
Then letting \( z = y', \quad z' = y'' \),

\[
\frac{z'}{z} = \left( k - \frac{1}{k} \right) x
\]

\[
\log z = k \log x - \frac{1}{2} x^2 + \log c \quad (c\text{-const.})
\]

or

\[
\log y' = \log c \cdot x^k - \frac{1}{2} x^2
\]

\[
y' = c \cdot x^k \cdot e^{-\frac{1}{2} x^2}
\]

giving

\[
y = \int_{0}^{x} c \cdot x^k \cdot e^{-\frac{1}{2} x^2} \, dx + d \quad (d\text{-const}).
\]

When \( x = 0 \), \( y = 1 \) we find that \( d = 1 \)

and when \( x = \infty \), \( y = 0 \),

\[
y = 1 + c \int_{0}^{\infty} x^k \cdot e^{-\frac{1}{2} x^2} \, dx = 0
\]

so that

\[
c = -\frac{1}{\int_{0}^{\infty} x^k \cdot e^{-\frac{1}{2} x^2} \, dx}
\]

Therefore

\[
T(r, t) = y(x) = 1 - \int_{0}^{\infty} x^k \cdot e^{-\frac{1}{2} x^2} \, dx
\]

\[
\int_{0}^{\infty} x^k \cdot e^{-\frac{1}{2} x^2} \, dx
\]
where \( x = \frac{r}{t^{1/2}} \).

As \( k = 0.9311 \), the approximation \( k = 1.00 \) will lead to a simplification of the above equations, namely,

\[
\int_{0}^{x} x e^{-\frac{1}{2}x^2} \, dx = \left[ -\frac{1}{2} e^{-\frac{1}{2}x^2} \right]_{0}^{x} = \frac{1}{1} \left( 1 - e^{-\frac{1}{2}} \right)
\]

Then \( \lim_{x \to \infty} \int_{0}^{x} xe^{-\frac{1}{2}x^2} \, dx = \lim_{x \to \infty} \frac{1}{1} \left( 1 - e^{-\frac{1}{2}} \right) = \frac{1}{1} \)

which gives

\[
y = 1 - \frac{1}{1} \left( 1 - e^{-\frac{1}{2}} \right) = e^{-\frac{1}{2}}
\]

therefore

\[
\tilde{T}(\tilde{r}, \tilde{t}) = y = e^{-\frac{1}{2}} \quad \text{with} \quad x = \frac{\tilde{r}}{\tilde{t}^{1/2}}
\]

\[
\tilde{T}(\tilde{r}, \tilde{t}) = e^{-\frac{1}{2\tilde{t}}} \quad \text{for small} \quad \tilde{r}
\]

which along with (23) gives us two analytic solutions for the pressure and temperature in the reservoir.

§ 5 NUMERICAL METHODS

We require a numerical method capable of solving the system of non-dimensional differential equations of § 3. Although the majority of oil reservoir simulators use finite difference techniques on a fixed rectangular grid (see [2], [5], and [6]), we shall use a finite element method (see [13]) with fixed nodes on an irregular grid. This method was chosen for two reasons: firstly further simplification of equation...
(19) yields a linear parabolic equation and it is known the finite element method gives good results for this type of problem (see [13]) and secondly it is anticipated that it may be necessary to use finite element techniques with moving nodes in future work.

§ 5.1 THE FINITE ELEMENT METHOD

As in this section of the report we shall always be dealing with the non-dimensional form of the equations, we shall ignore the overbars and assume each variable is in its non-dimensional form.

We seek semi-discrete approximations for the unknowns $p$ and $T$, in the form

$$p(r, t) = \sum_{i=1}^{n+1} p_i(t) \phi_i(r)$$

$$T(r, t) = \sum_{i=1}^{n+1} T_i(t) \phi_i(r)$$

(33)

(34)

where $p_i$ and $T_i$ are coefficients and $\phi_i$ are linear basis functions (see fig (B)).

The region of solution is the line $r = \delta$ to $r = 1$ (using the symmetry of the problem), divided into $n$ intervals (elements).
From Fig 6 we note that
\[ \phi_1 = \phi^-_1 \]
\[ \phi_{n+1} = \phi^+_n \]
and \[ \phi^+_i \phi^-_i \quad i = 2, \ldots, n. \]

If we write equations (19) and (20) of §4 as

\[ P_t - L(p) = 0 \]  \hspace{1cm} (35)

and

\[ (1 + \lambda p) T_t + \eta P_t - L'(T, p) = 0, \]  \hspace{1cm} (36)

where \[ L(p) = \frac{1}{r} \frac{\partial}{\partial r} \left( r(\alpha + \beta p) \frac{\partial p}{\partial r} \right) \]

and

\[ L'(T, P) = (1 + \omega) \frac{\partial P}{\partial T} - \eta \frac{\partial}{\partial T} \left( r \frac{\partial T}{\partial r} \right) \]

then ODE's for the 2(n+1) unknowns \[ P_1, P_2, \ldots, P_{n+1} \] and \[ T_1, T_2, \ldots, T_{n+1} \] can be obtained by taking a Galerkin weak form of (35) and (36)

using test functions \[ \phi_i \quad i = 1, \ldots, n+1, \] to give

\[ \langle p_t + L(p), \phi_i \rangle = 0 \]  \hspace{1cm} (37)

\[ \langle (1 + \lambda p) T_t + \eta P_t + L'(T, p), \phi_i \rangle = 0 \]  \hspace{1cm} (38)  \quad i = 1, \ldots, n+1

where

\[ \langle a, b \rangle = \int_a^b ab d\tau. \]

In this case \[ \tau \] represents the volume of the reservoir.
with
\[\delta < r < 1\]
\[0 < z < h/L\]
\[0 < \theta < 2\pi\]
and \[d\tau = rd\theta dz\]
so that
\[\langle a, b \rangle = \int_{0}^{h/L} \int_{0}^{2\pi} \int_{0}^{1} ab rd\theta dz\]
\[\langle a, b \rangle = \frac{2mh}{L} \int_{0}^{1} ab rdr\]

(39)

Due to the lower triangular coupling of the equations equation (37) may be solved first, and the values obtained for the \(p_i\)'s used in the solution of (38).

By considering the term \(\alpha p\) in equation (19) and using the values obtained for \(\alpha\) and \(\beta\) in §3 it can be seen that \((\alpha + \beta) = 1887.60\ (1 + 0.0022p)\).

As it is thought that \(p \in [0, 1]\), it may be assumed for the purpose of this report that \(\beta = 0\), therefore simplifying equation (19) to

\[\begin{align*}
\frac{\partial p}{\partial t} - \frac{\alpha}{\partial r} (r \frac{\partial p}{\partial r}) &= 0, \\
\frac{\partial p}{\partial r} &= 0 \text{ at } t = 0, \\
\frac{\partial p}{\partial r} &= -1 \text{ at } r = \delta, t > 0, \\
\frac{\partial p}{\partial r} &= 0 \text{ at } r = 1, \forall t.
\end{align*}\]

From the analytic solution of §5 it can be seen that
\[p = -\delta \log r + O(r^2), \delta < r < 1,\]
therefore as \( p = -\delta \log r \) satisfies \((40)\), it was decided to transform the variable \( p \) to \( \tilde{p} \), where

\[
p = \tilde{p} - \delta \log r .
\]

This has the effect of changing the initial and boundary conditions of equation \((40)\) to

\[
\tilde{p} = \delta \log r \text{ at } t = 0 ,
\]

\[
\frac{\partial \tilde{p}}{\partial r} = 0 \text{ at } r = \delta \forall t ,
\]

\[
\frac{\partial \tilde{p}}{\partial r} = \delta \text{ at } r = 1 \text{ for } t > 0 .
\]

Having non-zero initial data allows us to equi-distribute the nodes (see \([14]\)) with respect to \( f = \delta \log r \). This has the effect of placing the majority of the nodes near \( r = \delta \), where it is expected that the greatest disturbance will occur. The new boundary conditions have the effect of introducing a small flux into the region at \( r = 1 \) with a no-flow condition at \( r = \delta \), rather than a large flux at \( r = \delta \) and a no-flow condition at \( r = 1 \).

Taking the weak form of \((40)\) with test functions \( \psi \) leads to a set of \( n+1 \) linear ODE's

\[
M \tilde{P} = \alpha K \tilde{P} + L \tag{41}
\]

where \( \tilde{p} = (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{n+1})^T \),

\( M \) being a mass matrix, \( K \) a stiffness matrix and \( L \) a load vector obtained from the Neuman boundary conditions. From the nature of the inner product \((39)\), both \( K \) and \( M \) are symmetric.

A simple one step 0 time-stepping scheme is employed to reduce \((41)\) to \( n+1 \) linear equations.
\[ M \left( \frac{\tilde{p}^{k+1} - \tilde{p}^k}{\Delta t} \right) = \alpha \lambda \tilde{p}^{k+1} + (1-\theta) \alpha \lambda \tilde{p}^k + L. \]  

(42)

\( \tilde{p}^{k+1} \) can then be obtained by applying an LU solver to the above

\[ \tilde{p}^{k+1}_i = \tilde{p}^k_i + \delta \log r_i. \]

With \( \tilde{p} \) known, (20) can be solved. This was done first with \( \lambda = w = \eta = 0 \) and later with \( \eta \neq 0 \).

After taking the weak form of (20) with test functions \( \phi \), we obtain the set of \( n \) linear ODE's (as \( T_1 = 1 \) for \( t > 0 \))

\[ M \dot{T} = \tilde{K} T + vK \dot{T} + L, \]

(43)

where \( M \) and \( K \) are the same mass and stiffness matrixes, as derived in equation (41), but \( \tilde{K} \) is a different stiffness matrix obtained from

\[ \left< \frac{\partial \tilde{p}}{\partial r} , \phi, \frac{\partial T}{\partial r} , \psi \right> \]

so that \( \tilde{K} = \tilde{K}(p) \) and \( L \) is a load vector, obtained from the Neuman boundary conditions which in the present case is always zero.

Note that \( \tilde{K} \) is unsymmetric.

Again using a one step \( \theta \) time stepping scheme we can reduce (43) to \( n \) linear equations.

\[ M \left( \frac{\tilde{T}^{k+1} - \tilde{T}^k}{\Delta t} \right) = \theta_1 \tilde{K} \tilde{p}^{k+1} + (1-\theta_1) \tilde{K} \tilde{p}^k + \theta_2 vK \tilde{T}^{k+1} + (1-\theta_2) vK \tilde{T}^k \]

(44)

and \( \tilde{T}^{k+1} \) may be obtained by applying an LU solver to the above.

If we take \( \eta \neq 0 \), then we have an extra term on the right hand side, namely
\(< \eta \frac{\partial p}{\partial t}, \phi_i \)\),

but if we only consider times after which \( p \) has reached semi-steady state, then by the analysis of \( \S 5 \),

\[ \frac{\partial p}{\partial t} \sim \text{constant} \]

and this can be calculated from the definition of isothermal compressibility as

\[ \frac{\partial p}{\partial t} = \Psi \] .

Therefore taking \( \eta \neq 0 \) changes (43) into

\[ M \frac{\tau}{T} = k \frac{\tau}{T} + \nu \frac{\tau}{T} + \frac{\tau}{L} + \frac{\tau}{L} \]

where \( \frac{\tau}{L} = \langle \eta \Psi, \phi_i \rangle, \ i = 1, \ldots, n+1 \)

Again a \( \theta \) time stepping scheme was used to reduce (45) to \( n \) linear equations

\[ M \left( \frac{\tau}{T}^{k+1} - \frac{\tau}{T}^k \right) = \theta \left( \frac{\tau}{T}^{k+1} - \frac{\tau}{T}^k \right) + (1-\theta) \frac{\tau}{T}^k + \frac{\tau}{T}^{k+1} + \frac{\tau}{T} \frac{\tau}{T} + \frac{\tau}{L} \]

\[ \Delta t \]

\[ \left( \frac{\tau}{L} = 0 \right) \]

and a \( LU \) solver was used to calculate \( \frac{\tau}{T}^{k+1} \)

\( \S \) 5.2 RELATIVE NORMS

If we have two piecewise linear functions \( f \) and \( g \) defined on the interval \([a, b]\), we often require a measure of the relative difference between them. If \( f \) and \( g \) are defined on the same grid, then a relative pointwise norm \( L_{r\infty} \), may be defined as
\[ L_r^2 = \sum_{i=1}^{n+1} \frac{(f_i - g_i)^2}{\left( \frac{f_i + g_i}{2} \right)^2} \]

taking \( n \) intervals and \( f_i \) and \( g_i \) being the values of \( f \) and \( g \) at node \( i \), respectively. However if \( f \) and \( g \) are defined on different grids with possibly different numbers of nodes a more general relative \( L_r^2 \) norm \( L_r \) must be used to measure the relative difference between \( f \) and \( g \), derived from

\[ L_r^2 = \frac{\int_a^b (f-g)^2 \, dr}{\left( \int_a^b \left( \frac{f+g}{2} \right)^2 \, dr \right)^{1/2}} \]

with appropriate quadrature.

§ 6 NUMERICAL RESULTS

§ 6.1 Pressure

Up to a time of \( t \approx 0.0001 \) transient flow occurs. This is due to the fact that the effects of the outer boundary of the reservoir have not yet affected the flow and can be seen in fig (7) where a fully implicit scheme (\( \theta = 1 \)) is used to solve equation (42) (as for the rest of the results).

Figures (8) and (9) show the pressure distribution for times up to \( t = 0.01 \), the result for fig (9) having twice as many elements as the result for fig (8). A relative \( L_r^2 \) norm for the difference between them gave \( L_r = 9.527 \times 10^{-4} \), while a relative pointwise norm between fig (9) and the analytic solution at
this time fig (10) gave \( L_\tau = 3.2642 \times 10^{-3} \). Figures (11) and (12) show the numerical and analytic solution up to \( t = 1.00 \), the relative pointwise norm between these results being \( L_\tau = 9.5792 \times 10^{-4} \).

§ 6.2 Temperature (n=0)

Placing \( \eta = 0 \) has the physical effect of putting the internal energy equal to the enthalpy.

The results shown in figures (13) and (14) show the temperature up to time \( t = 1.00 \) (using implicit time stepping) with the results in figure (4) having twice the number of nodes as the results in figure (13). A relative \( L_2 \) norm for the difference between the two solutions gave \( L_\tau = 2.4318 \times 10^{-3} \), while a relative pointwise error between fig (14) and the approximate analytic solution in fig (15) gave \( L_\tau = 1.2328 \times 10^{-2} \). Figure (16) shows the temperature up to \( t = 1.00 \) but with \( \theta_1 = 0 \) (the convection term calculated using explicit time stepping), a pointwise norm between this and the analytic solution giving \( L_\tau = 1.2652 \times 10^{-2} \).

§ 6.3 Temperature(n ≠ 0)

Having \( \eta ≠ 0 \) should allow for heating by compressibility, thus increasing the temperature through out the whole reservoir, not only near the well bore as in the case \( \eta = 0 \). This phenomenon can clearly be seen in figures (17) and (18), the results of figure (17) being calculated using a fully implicit time stepping scheme \( (\theta_1 = \theta_2 = 1) \), whereas the result in figure (18) uses a time stepping scheme with \( \theta = 1 \) (explicit on the convection term), a relative \( L_2 \) norm for the difference between the two results giving \( L_\tau = 3.7417 \times 10^{-3} \).
§ 6.4 Convective flow

It was next decided to check the robustness of the numerical method by letting $v \to 0$, the flow being purely convective when $v = 0$. Physically, this is equivalent to lowering the thermal conductivity of the rock until the heat transfer is only by convection in the water.

Figures (19) and (20) show that $v$ can be reduced by factors of 10 and 20 respectively, and the method will still reproduce the solution with 100 elements. However if $v$ is reduced by a factor of 100, figure (21) shows that oscillations occur using 100 elements, but if 200 elements are used a smooth solution is again obtained (fig (22)). Similarly, if $v$ is reduced by a factor of 200, a solution using 100 elements leads to oscillations (fig (23)), but using 200 elements gives an oscillation-free solution. Finally taking $v = 0$ and using 100 nodes, we obtained very bad oscillations (fig (25)), but again using 200 elements they are almost totally eliminated.
§ 7 CONCLUSIONS

In this report we have discussed the modelling of hot fluid injection into porous media, with analytic and numerical solutions being obtained to the resulting differential equations. In section two we showed that the problem could be written as a system of three equations, one algebraic and two differential. In section three, the three equations were non-dimensionalised with the algebraic equation being simplified to a linear function of pressure, which when substituted into the two differential equations resulted in two differential equations for the unknowns pressure and temperature. After simplifying the equations it was possible to obtain analytic solutions for the pressure and temperature, this being done in section 4, with numerical methods being described in section 5. The results were displayed in section 6.

The numerical method coped very well with the simplified pressure equation (40), as expected, with good comparisons between the numerical and analytic solutions. The method also gave good results for the temperature equation (since most of the activity takes place near the well bore), the nodal distribution used for the pressure equation proving very suitable for the temperature problem.

When the flow became purely advective \((v \to 0)\) the method still coped well, but at the cost of a large number of elements, mainly distributed near the well bore. Therefore if the problem was run to a greater time, such that the temperature front reached further into the well, it is expected the results would break down due to the lack of nodes in the region.
Also if we considered a 2-D problem the method would become very expensive in terms of computer time. In order to have enough nodes to resolve the solution some kind of adaptive gridding would therefore appear to be necessary.
APPENDIX

For completeness, we include the data set [see [14]] used for this report.

Reservoir parameters

Outer radius \( L = 300.0 \text{m} \)

Height \( h = 30.0 \text{m} \)

Well bore radius \( R = 0.08 \text{m} \)

Rock Properties

Porosity \( \phi = 0.2 \)

Permeability \( K = 1.97385 \times 10^{-3} \text{ m}^2 \)

Density \( \rho_r = 2.643 \times 10^3 \text{ Kg m}^{-3} \)

Thermal Conductivity \( K_\theta = 1.8028 \text{ Jm}^{-1} \text{s}^{-1} \text{K}^{-1} \)

Specific heat at constant volume \( C^r_v = 1.0 \times 10^3 \text{ J Kg}^{-1} \text{K}^{-1} \)

Water Properties

Thermal compressibility \( C_w = 4.3511 \times 10^{-10} \text{ pascals}^{-1} \)

Specific heat at constant volume \( C^w_v = 4.0 \times 10^3 \text{ J Kg}^{-1} \text{K}^{-1} \)

Viscosity \( \mu = 3.0 \times 10^{-4} \text{ Kg m}^{-1} \text{s}^{-1} \)

Stock Tank Conditions

Temperature \( T_{\text{stc}} = 288.0 \text{K} \)

Density of Water \( \rho_{\text{stc}} = 1.0 \times 10^3 \text{ Kg m}^{-3} \)

Atmospheric Pressure \( p_{\text{atm}} = 1.0135 \times 10^5 \text{ pascals} \)
Initial Conditions in Reservoir

Temperature  \( T_{\text{int}} = 310.0 \text{ K} \)
Density of Water  \( \rho_{\text{int}} = 1.0 \times 10^3 \text{ kg/m}^3 \)
Pressure  \( P = 1.0341 \times 10^7 \text{ pascals} \)

Injection Conditions

Mass injection rate  \( q = 0.16406 \text{ Kgs}^{-1} \)
Heat injection rate  \( q_h = 7.4040 \times 10^5 \text{ J/s}^{-1} \)
Temperature  \( T_{\text{inj}} = 386.0 \text{ K} \)
REFERENCES


