D E P A R T M E N T   O F   M A T H E M A T I C S

ON BEST PIECEWISE CONSTANT L_2 FITS WITH ADJUSTABLE NODES

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Numerical Analysis Report 12/90

U N I V E R S I T Y   O F   R E A D I N G

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Abstract

In this report a simple procedure is used to determine the best piecewise constant L_2 fit to a given function of a single variable with adjustable nodes.

§1. Theory

This report is an extension (or in some ways a truncation) of [1], in which best piecewise linear L_2 fits were considered.

Let $f(x)$ be a given continuously differentiable function of x and denote by u_k the best constant L_2 fit to $f(x)$ in the interval (x_{k-1}, x_k).

Then

$$\delta \int_{x_{k-1}}^{x_k} \left\{f(x)-u_k\right\}^2 dx = 0 \qquad u_k \in \Pi_k \tag{1}$$

or

$$\int_{x_{k-1}}^{x_k} \left\{f(x)-u_k\right\} \delta u_k \, dx = 0 \qquad \delta u_k \in \Pi_k \tag{2}$$

where Π_k is the family of constant functions on the interval (x_{k-1}, x_k). For an interval (x_o, x_{N+1}) which is the union of intervals (x_{k-1}, x_k), $(k=1,\ldots,n+1)$, the best L_2 fit to $f(x)$ amongst piecewise constant functions discontinuous at x_k, $(k=1,\ldots,n)$, is also given by (1) and (2), $(k=1,\ldots,n+1)$, since the problems decouple.

Now consider the problem of determining the best L_2 fit $u(x)$ to $f(x)$ amongst all discontinous piecewise constant functions on the fixed interval (x_o, x_{n+1}) on a variable partition $(x_1, x_2,\ldots, x_k,\ldots,x_n)$ of the interval. Then

$$\delta \int_{x_o}^{x_{n+1}} \left\{f(x)-u(x)\right\}^2 dx = \delta \sum_{k=1}^{n+1} \int_{x_{k-1}}^{x_k} \left\{f(x)-u_k\right\}^2 dx = 0 \tag{3}$$

where $u(x) = U\{u_k\}$ and the $x_k, (k=1,\ldots,n)$, are also varied. It is convenient to introduce here a new independent variable ξ which remains fixed, while x joins u as a dependent variable, both now depending on ξ and denoted by $\hat{x}$ and $\hat{u}$, respectively. Then (3) becomes

$$\delta \sum_{k=1}^{n+1} \int_{x_{k-1}}^{x_k} \left\{f(\hat{x}(\xi))-\hat{u}_k\right\}^2 \frac{d\hat{x}}{d\xi}\, dx = 0 \tag{4}$$

with $\hat{u}(\xi) = U\{\hat{u}_k\}$.

Taking the variations of the integral in (4) gives

$$\int\left\{2\left\{f(\hat{x}(\xi))-\hat{u}_k\right\}\left\{f'(\hat{x}(\xi))\ \delta\hat{x}-\delta\hat{u}_k\right\}\frac{d\hat{x}}{d\xi}+\left\{f(\hat{x}(\xi))-\hat{u}_k\right\}^2\frac{d}{d\xi}(\delta\hat{x})\right\}d\xi. \tag{5}$$

Integrating the last term by parts leads to

$$-\int 2\left\{f(\hat{x}(\xi)) - \hat{u}_k\right\} f'(\hat{x}(\xi)) \frac{d\hat{x}}{d\xi}\,\delta\hat{x}\, d\xi$$

$$+\sum_{k=1}^{n+1}\left\{(f(\hat{x}(\xi)) - \hat{u}_k)^2_{k-1}\ \delta\hat{x}_{k-1} + f(\hat{x}(\xi)) - \hat{u}_k)^2_k\ \delta\hat{x}_k\right\}. \tag{6}$$

Collecting terms and returning to the x,u notation, (4) yields

$$\sum_{k=1}^{n+1}\int_{x_{k-1}}^{x_k} 2\left\{f(x) - u(x)\right\}\delta u_k dx + \sum_{j=1}^{n}\left[(f(x) - u_k)^2\right]_j \delta x_j=0 \tag{7}$$

where the summation is over nodes j and the square bracket notation $[\,]_j$ denotes the jump in the quantity at the node j.

The conditions at an extremum are

$$\int_{x_{k-1}}^{x_k} \left\{f(x)-u_k\right\}\delta u_k dx = 0 \qquad (8)$$

$$\left[(f(x_k) - u_k)^2\right]_j \delta x_j = 0 \quad \forall k . \qquad (9)$$

With δu in the space of piecewise constant functions the orthogonality condition (8) is equivalent to

$$\int_{x_{k-1}}^{x_k} \left\{f(x)-u(x)\right\} \pi_k(x)\, dx = 0 \qquad (10)$$

where $\pi_k(x)$ is the characteristic function in the element k (see fig. 1). On the other hand, we may set $\delta u_k = 0$, $\delta x_j \neq 0$ to obtain from (9)

$$\left[\left[f(x_k)-u_k\right]^2\right]_j = 0 . \qquad (11)$$

Using L,R for values to the left and right of the (variable) node j, it follows from (11) that either

$$(f - u_L) = f - u_R \Rightarrow u_L = u_R \qquad (12)$$

or

$$-(f - u_L) = f - u_R \Rightarrow u_L + u_R = 2f \qquad (13)$$

It is easy to verify that the latter corresponds to monotonic behaviour of f while the former may exceptionally occure at maxima or minima (see fig. 2).

The solution of the problem (10),(11) is then the set of best constant fits in separate elements which have the continuity property (12) or the averaging property (13).

§2. The Algorithm

The algorithm used here to find the best piecewise constant L_2 fit with variable nodes is in two stages (carried out repeatedly until convergence), corresponding to the choices of variations referred to in §1 above.

$$\text{Stage (i)} \quad \delta x_j = 0 \ , \quad \delta u_j = \pi_k \quad (k=1,2,\ldots,\ n+1) \tag{14}$$

This stage of the algorithm corresponds to the best L_2 fit amongst constant functions discontinuous at prescribed nodes, as in (1),(2).

$$\text{Stage (ii)} \quad \delta x_j \neq 0 \ (j=1,2,\ldots,n), \quad \delta u_k = 0 \ (k=1,2,\ldots,\ n) \tag{15}$$

This stage corresponds to finding x_j such that (11) holds, with u restricted to points lying on the piecewise constant approximation (possibly linearly extrapolated) in element k .

As remarked in [1], the algorithm is analogous to minimising a quadratic function $f(x,y)$ using two search directions v1 and v2 spanning the plane. Starting from some initial guess we may alternately minimise f in the directions v1 and v2 . Similarly, to find the best L_2 fit we may begin with an initial guess $\{x_j\},\{u_j\}_L,\{u_j\}_R$. Stage (i) is to find the minimum in the linear manifold specified by the variations given in (14) and so solve (10) for new $\{u_j\}_L,\{u_j\}_R$ with the x_j fixed. Stage (ii) is to find the minimum in the linear manifold specified by the variations given in (15) and so solve (11) for new $\{x_j\}$ by the implementation of (13) as described below.

Note that the calculation of x_j from (13) is implicit since f depends on x_j and u_L,u_R are new values. Any standard algorithm may be used to extract x_j : here we use the elementary bisection method.

In the case of (12) there is no solution for x_j unless $u_L = u_R$. In this exceptional case any x_j in the element is a solution.

The L_2 error of the fit described here is never worse than the error of the interpolant u_I which is well known [2] to satisfy

$$||u_I - f||_2 \leq \frac{n^{-1}}{\pi}||f'||_2 \tag{17}$$

on (0,1) . This order of accuracy is borne out in practice (table 1), as is second order for the corresponding piecewise linear approximation [1] (table 2). (See also Appendix).

§3 Results

We show results for three examples,

(a)	$e^{-20(1-x)}$	$0 \leq x \leq 1$	11 interior nodes
(b)	$\tanh\{20(x-0.5)\}$	$0 \leq x \leq 1$	11 interior nodes
(c)	$\sin 2\pi x$	$0 \leq x \leq 1$	11 interior nodes

In each case the initial grid is equally spaced. In each example the trajectories of the nodes as they move towards their final positions are shown together with the function and the fit obtained. The process is said to have converged when the relative error in the L_2 norm of $f(x) - u(x)$ is less than 10^{-4}. (The number of iterations appears on the ordinate axis of the trajectories. The iteration is of Jacobi type and no attempt has been made to accelerate the convergence.)

Table 1a **exp(−20x)**

no. of internal nodes	8	16	32
relative error in $\|\|.\|\|_2$	0.1163	0.0843	0.0604
convergence rate	0.875	0.929	0.959

Table 1b **sin 2πx**

no. of internal nodes	8	16	32
relative error in $\|\|.\|\|_2$	0.3317	0.2448	0.1784
convergence rate	0.993	0.876	0.912

Table 1c **tanh 20(x−½)**

no. of internal nodes	8	16	32
relative error in $\|\|.\|\|_2$	0.1773	0.1274	0.0909
convergence rate	0.917	0.953	0.972

Table 2a exp(−20x)

no. of internal nodes	8	16	32
relative error in $\|\|.\|\|_2$	0.03256	0.01711	0.00888
convergence rate	1.747	1.856	1.892

Table 2b sin 2πx

no. of internal nodes	8	16	32
relative error in $\|\|.\|\|_2$	0.09727	0.05259	0.02742
convergence rate	1.6	1.774	1.88

Table 2c tanh 20(x−½)

no. of internal nodes	8	16	32
relative error in $\|\|.\|\|_2$	0.05611	0.02994	0.01555
convergence rate	1.672	1.812	1.892

§4. References

[1] Baines, M.J. and Carlson N.N. (1990). On Best Piecewise Linear L_2 Fits with Adjustable Nodes. Numerical Analysis Report 6/90, Department of Mathematics, University of Reading, U.K.

[2] See e.g. Porter D. & Stirling D.S.G. (1990). Integral Equations; A Practical Treatment from Spectral Theory to Applications. CUP.

[3] Carey G.F. & Dinh, H.T. (1985). Grading Functions and Mesh Redistribution. Siam J. Numer. An. 22, 1028.

Appendix A

In this appendix, following [1] and [3], we give an asymptotic equidistribution result for the convex case. From (10) it follows that u-f vanishes at at least one point in each element, r_k say. Then, since $u' = 0$,

$$\int_{r_k}^{x} f'(\xi)\, d\xi = \int_{r_k}^{x} (f'(\xi) - u'(\xi))d\xi = f(x) - u_k \tag{A1}$$

Hence

$$\int_{x_{k-1}}^{x_k} (f(x)-u_k)^2 dx = \int_{x_{k-1}}^{x_k} \left\{ \int_{r_k}^{x} f'(\xi)d\xi \right\}^2 dx \tag{A2}$$

$$\leq \int_{x_{k-1}}^{x_k} \left\{ (x_k - x_{k-1})\ f'_{max,k} \right\}^2 dx \tag{A3}$$

where $f'_{max,k}$ is the maximum norm of f' in element k .

Now, if $E'(x)$ is an equidistributing function,

$$(x_k - x_{k-1})\ E'(\theta_k) = \text{a constant, } C\ , \tag{A4}$$

where $x_{k-1} < \theta_k < x_k$, and we have from (A3)

$$\int_{x_{k-1}}^{x_k} (f-u_k)^2 dx \leq C^2 \int_{x_{k-1}}^{x_k} \left\{ E'(\theta_k) \right\}^{-2} \left\{ f'_{max,k} \right\}^2 dx \tag{A5}$$

so that

$$\int_{x_o}^{x_n} (f-u_k)^2 dx \leq C^2 \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \left\{E'(\theta_k)\right\}^{-2} \left\{f'_{max,k}\right\}^2 dx \quad \text{(A6)}$$

Finally, as in [3], we approximate the right hand side of (A6) by the integral

$$C^2 \int_{x_o}^{x_n} \left\{E'(x)\right\}^{-2} \left\{f'(x)\right\}^2 dx \quad \text{(A.7)}$$

and minimise over functions $E(x)$, yielding

$$\frac{d}{dx}\left[\left\{E'(x)\right\}^{-3} \left\{f'(x)\right\}^2\right] = 0 \quad \text{(A8)}$$

or

$$E'(x) \propto \left\{f'(x)\right\}^{2/3} \quad \text{(A.9)}$$

$$E(x) \propto \int^{x} \left\{f'(\xi)\right\}^{2/3} d\xi \quad \text{(A.10)}$$

which may be regarded as the asymptotically equidistributed function.

Appendix B

In this appendix we extend the result in the main body of the report to general extremals.

For the problem of finding the extremal of the integral

$$\int F(x,u)dx \tag{B1}$$

over piecewise linear discontinuous functions $u(x)$ with variable nodes, we folllow the same procedure as in §1, obtaining

$$\int_{x_{k-1}}^{x_k} F_u(x,u_k)\delta u_k \, dx = 0 \tag{B2}$$

$$\Big[F(x,u)\Big]_j \, \delta x_j = 0 \qquad \forall k \tag{B3}$$

in place of (8) and (9). Then (10) and (11) become

$$\int_{x_{k-1}}^{x_k} F_u(x,u) \; \pi_k(x) \; dx = 0 \tag{B4}$$

$$\Big[F(x,u_k)\Big]_j = 0 \; . \tag{B5}$$

The corresponding algorithm is to solve (B4) for u_k in each element with fixed x_j (stage (i)) and then to solve (B5) for the x_j with u restricted to the stage (i) solution, possibly extrapolated (stage (ii)). Both problems are nonlinear and may or may not have unique solutions.

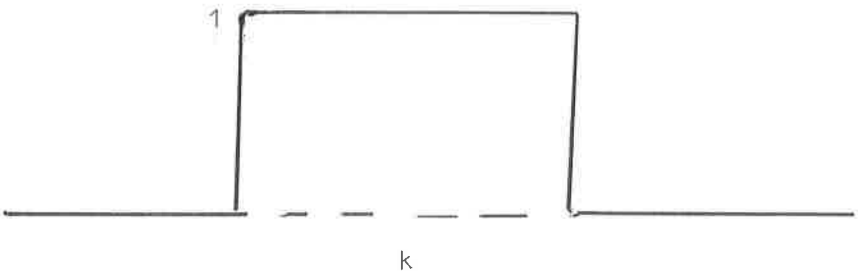
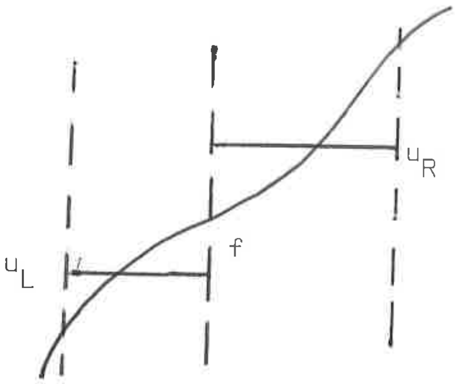
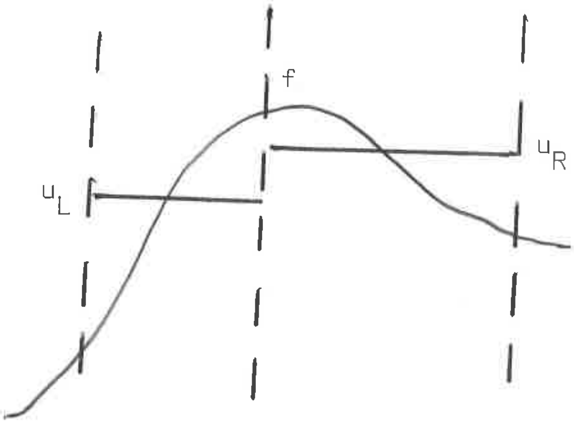


fig. 1 $\pi_k(x)$

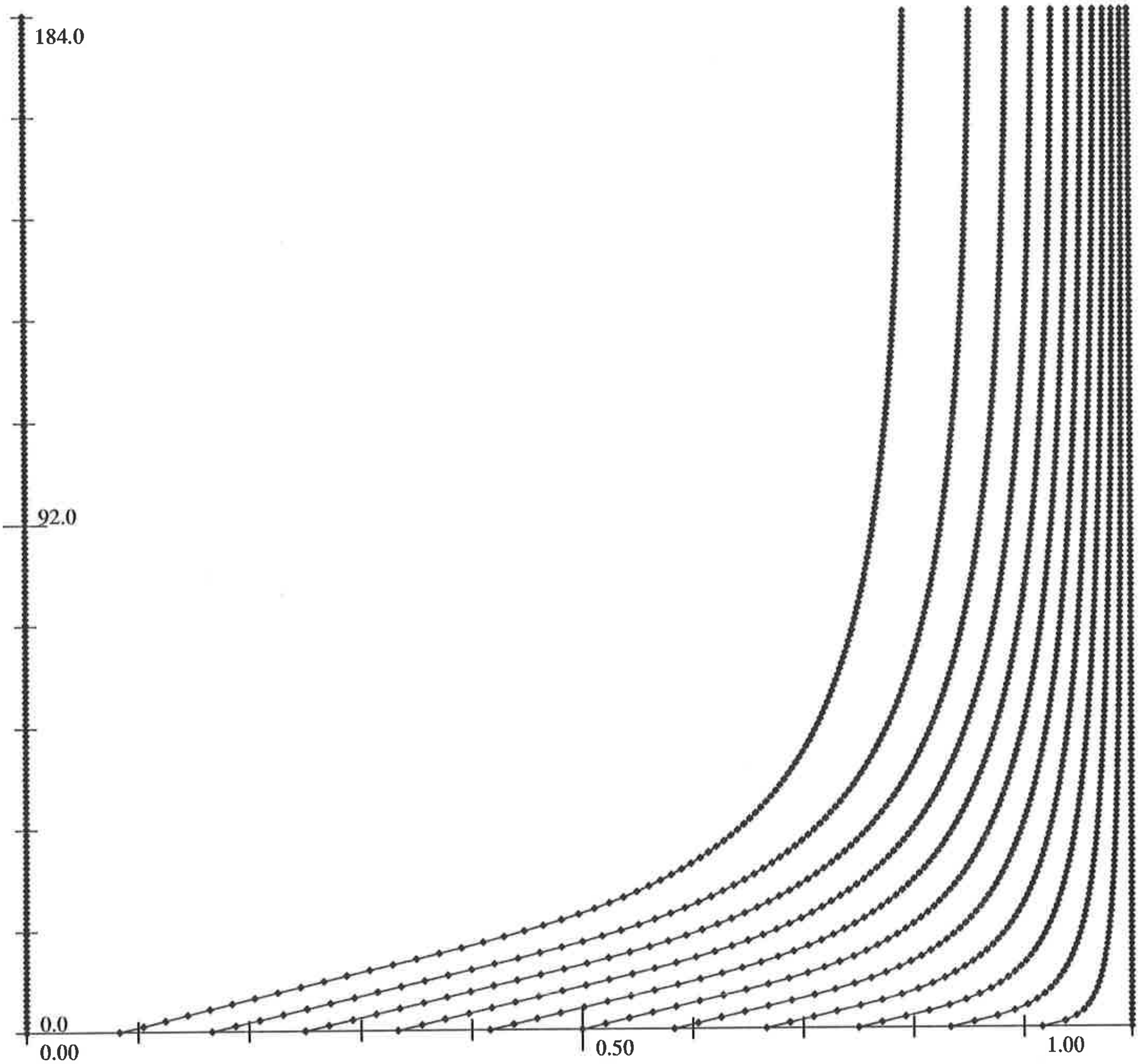


(13)



(12)

fig. 2



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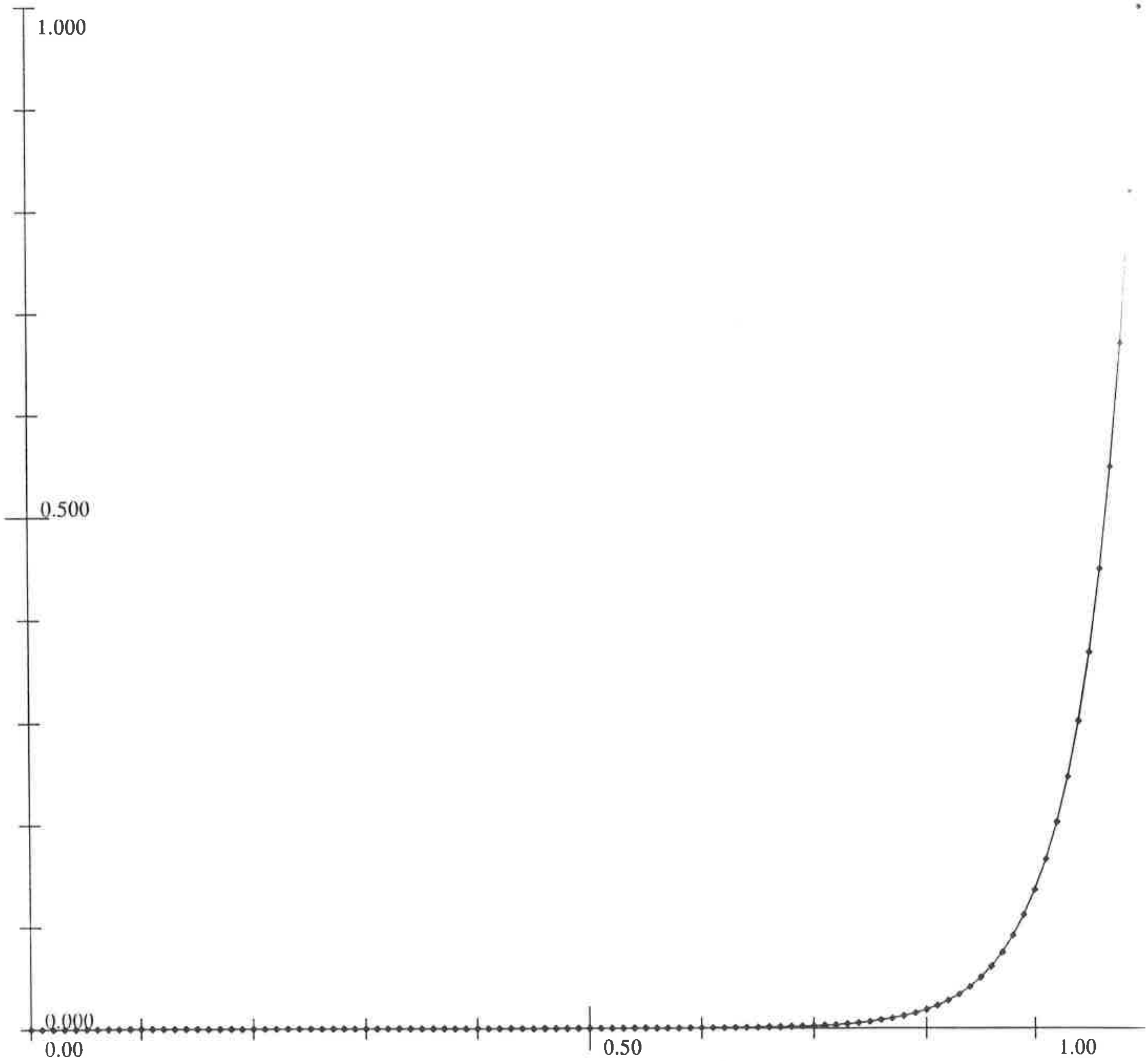

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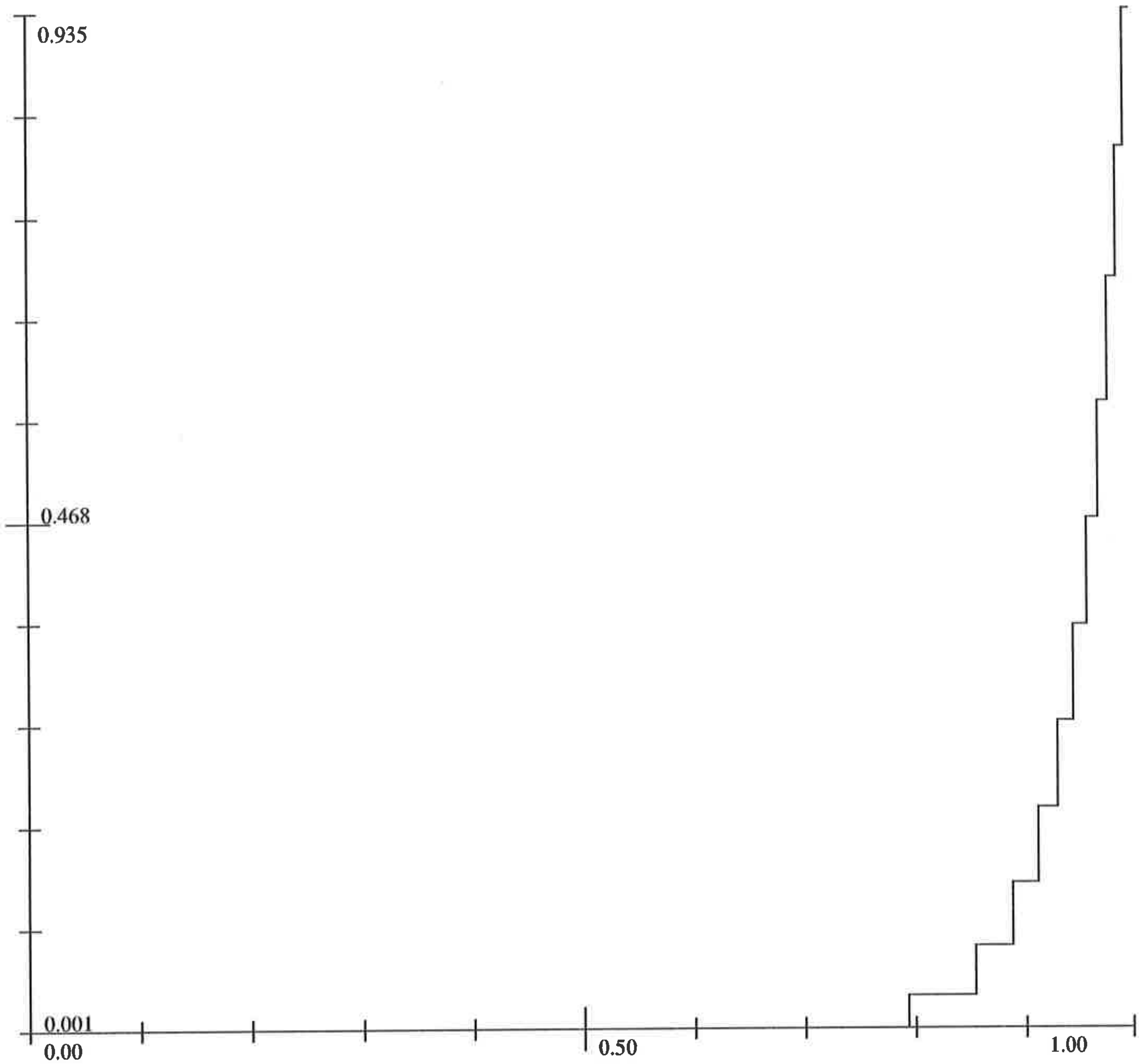

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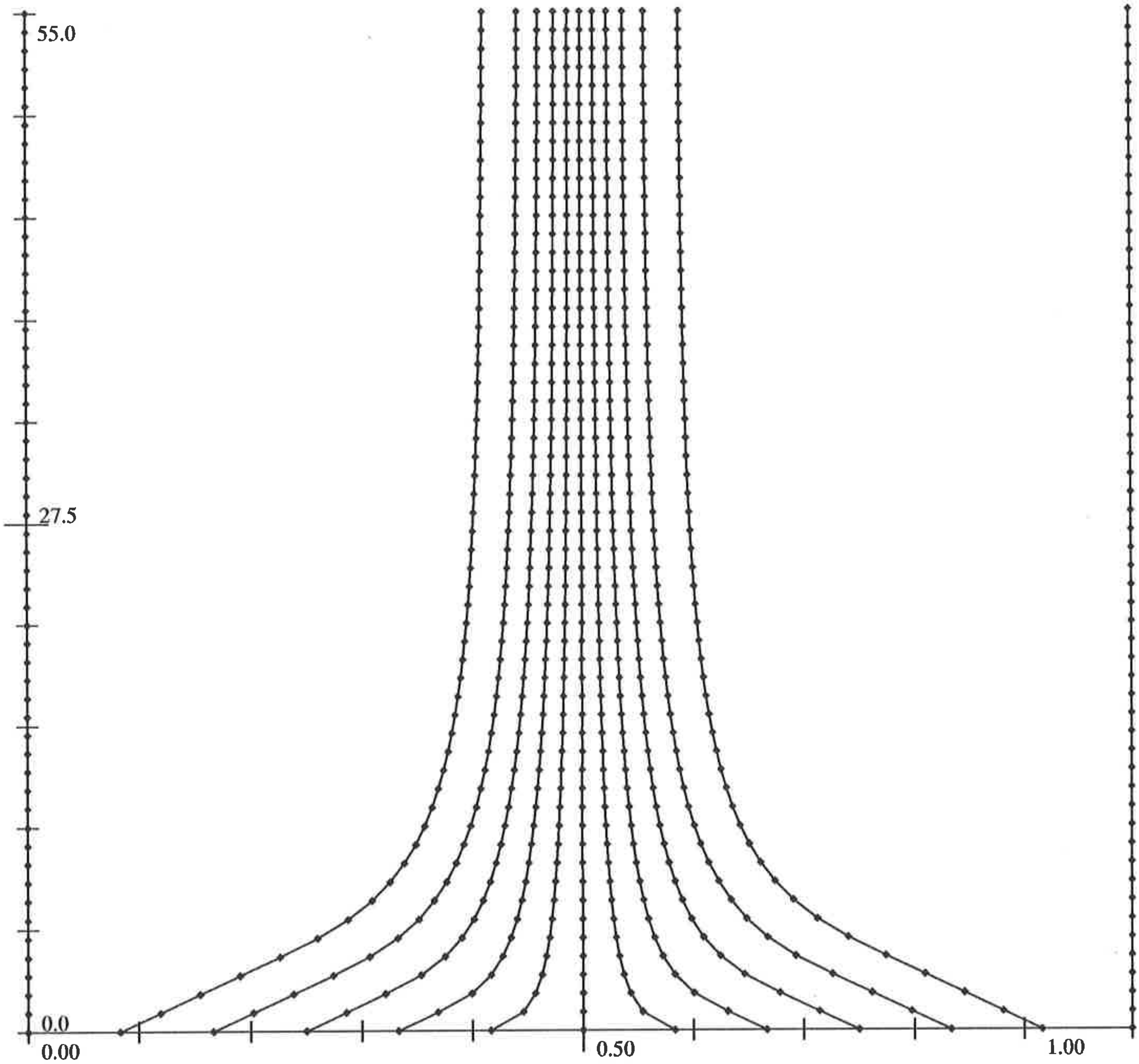

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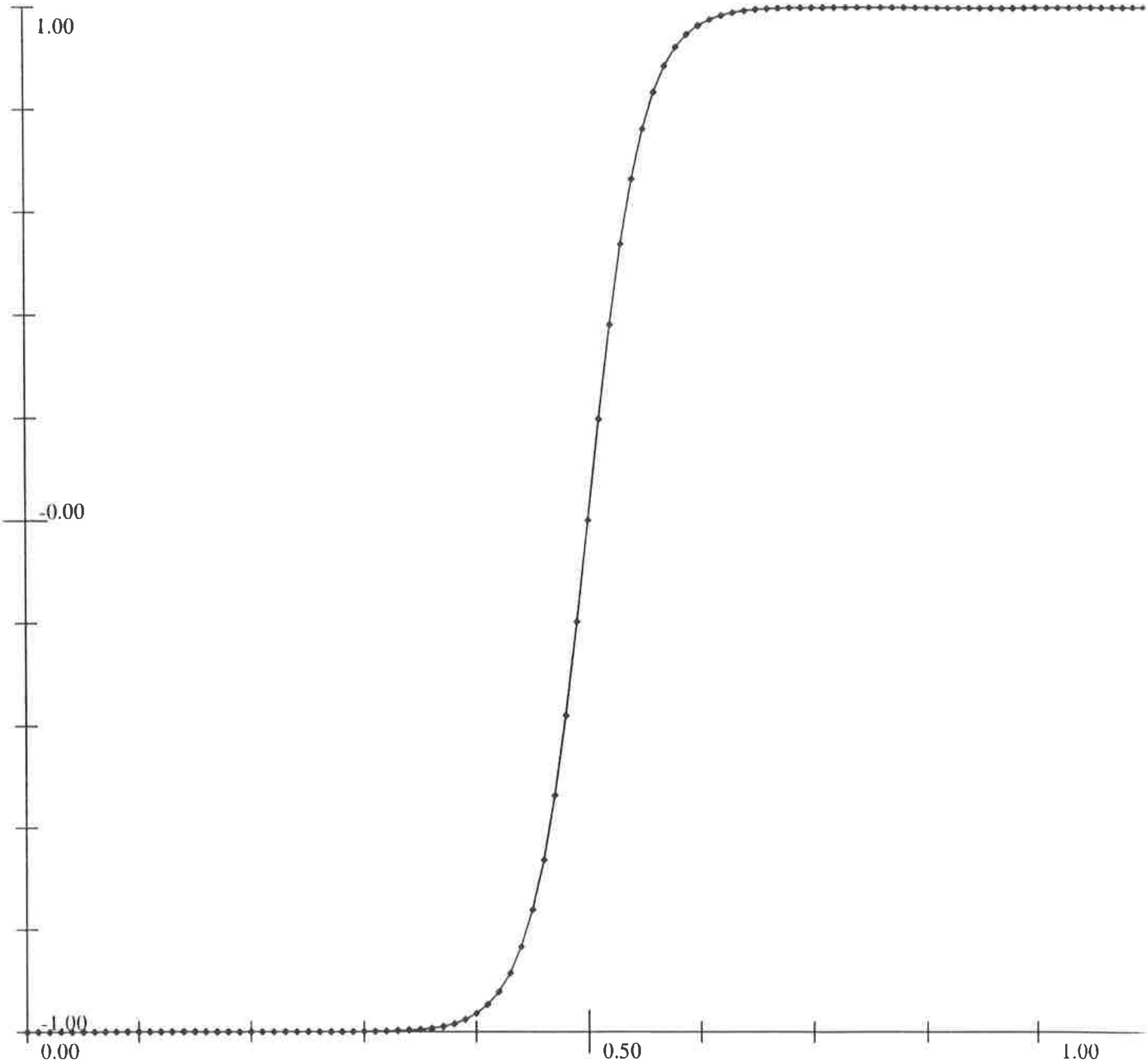

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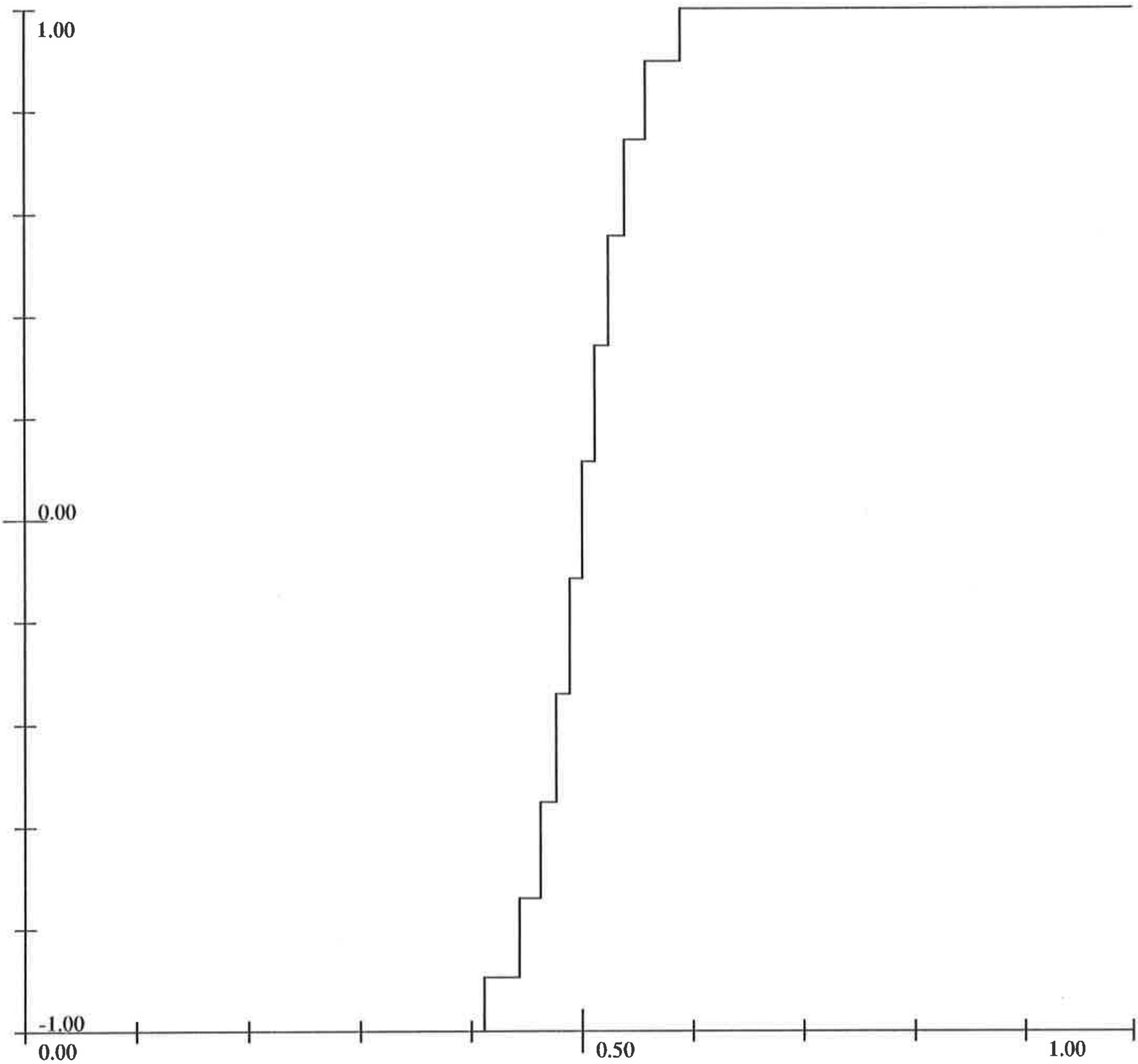
tanh 20 (x - 0.5) function		



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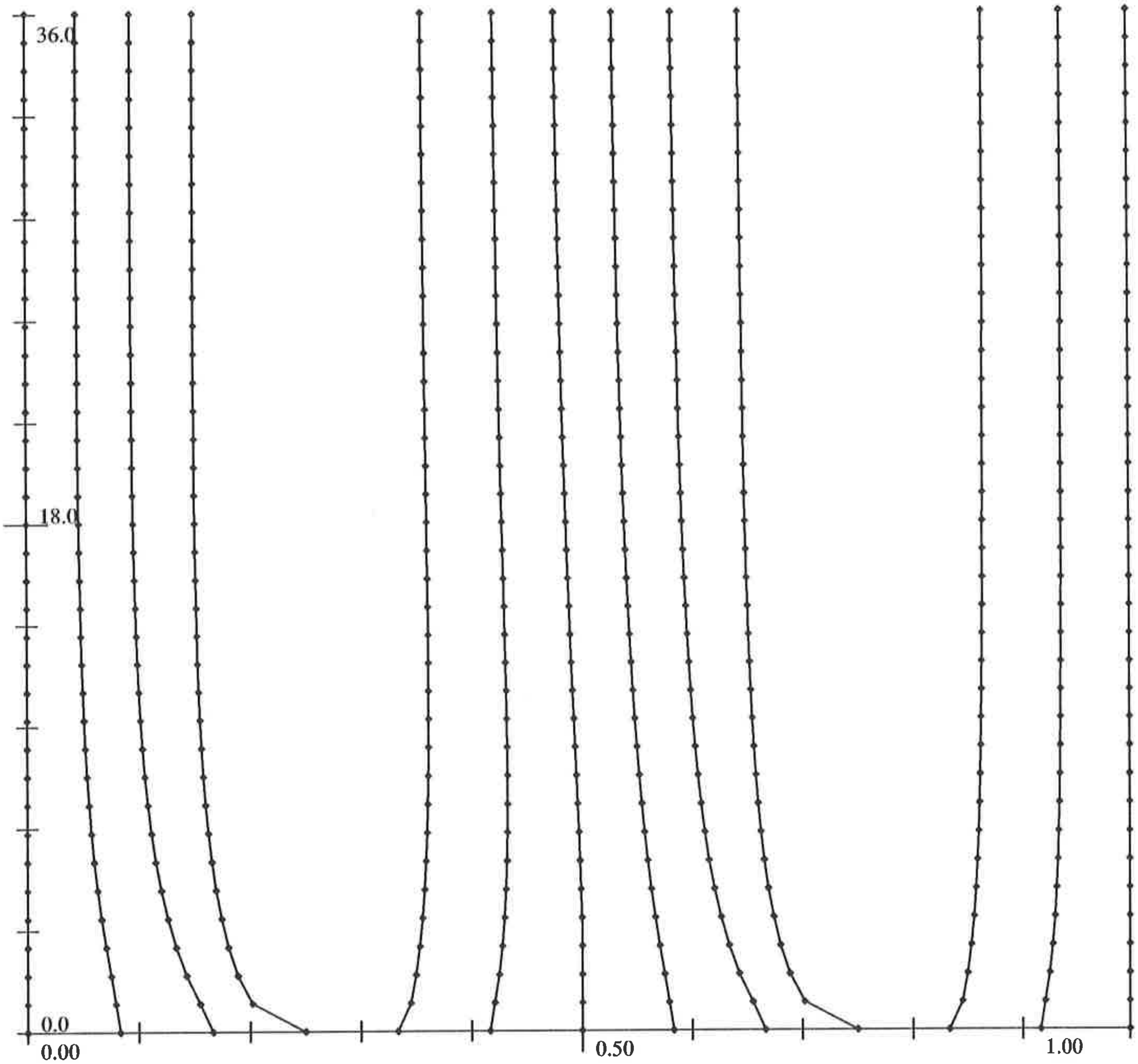

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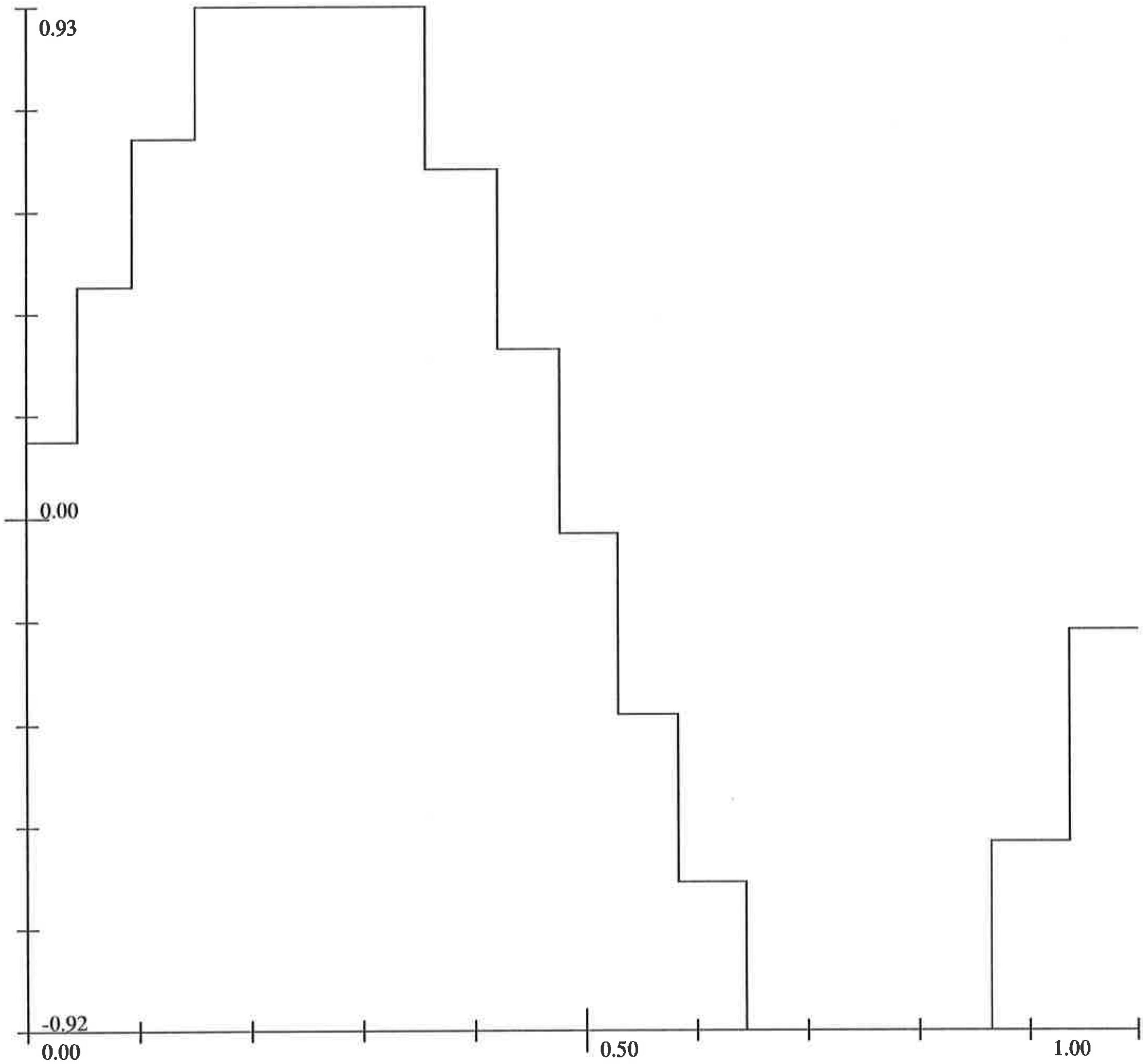

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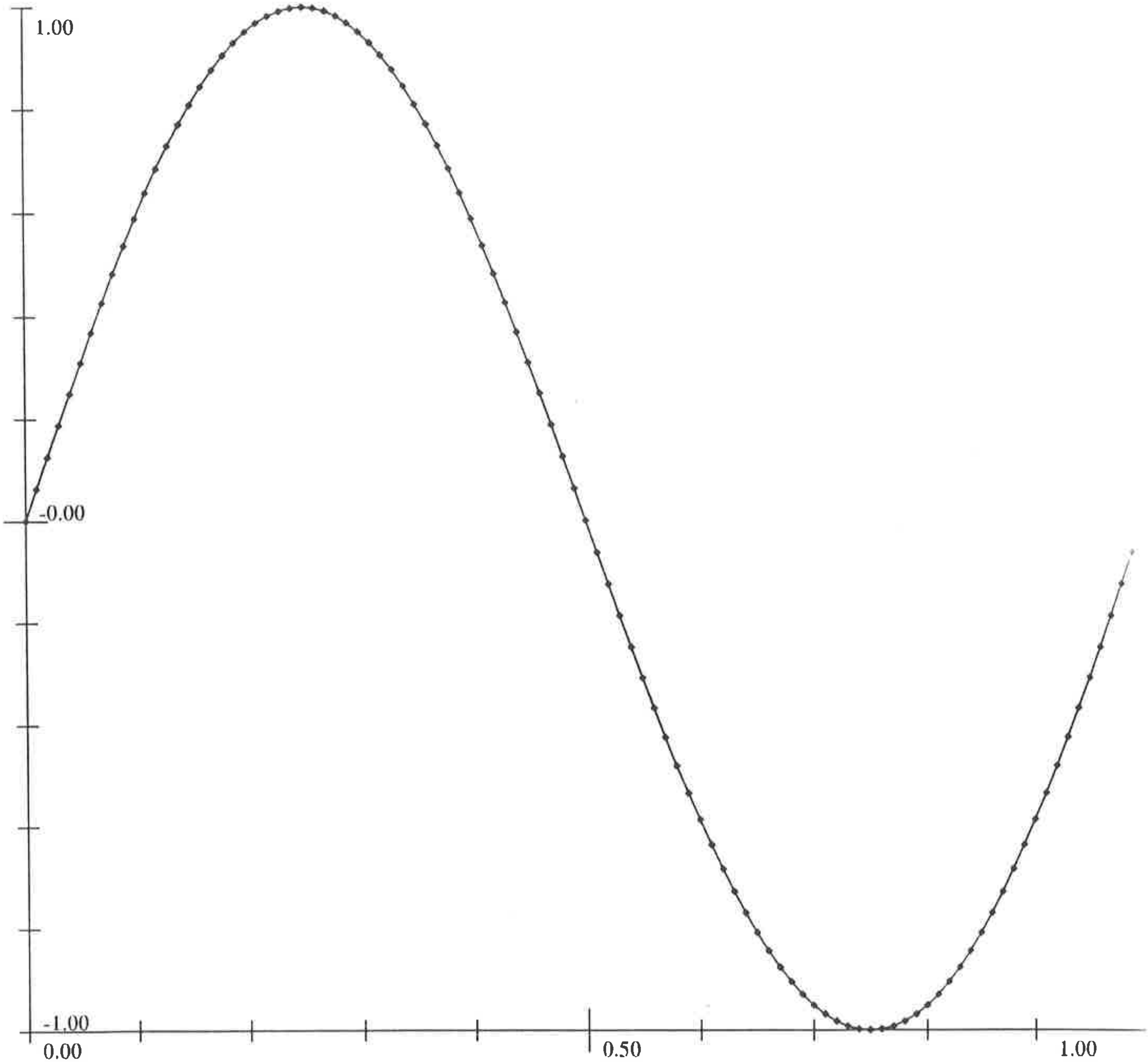

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