THE UNIVERSITY OF READING

Moving Finite Element, Least Squares and Finite Volume Approximations of Steady and Time-Dependent PDEs in Multidimensions.

by

M J Baines

Numerical Analysis Report 11/99

DEPARTMENT OF MATHEMATICS

Abstract

We review recent advances in Galerkin and Least Squares methods for approximating the solutions of first and second order PDEs with moving nodes in multidimensions. These methods use unstructured meshes and minimise the norm of the residual of the PDE over both solutions and nodal positions in a unified manner. Both finite element and finite volume schemes are considered, as are transient and steady problems.

For first order scalar time-dependent PDEs in any number of dimensions, residual minimisation always results in the methods moving the nodes with the (often inconvenient) approximate characteristic speeds. For second order equations, however, the Moving Finite Element (MFE) method moves the nodes usefully towards high curvature regions. In the steady limit, for PDEs derived from a variational principle, the MFE method generates a locally optimal mesh and solution: this also applies to Least Squares minimisation.

The corresponding Moving Finite Volume (MFV) method, based on the l_2 norm, does not have this property however, although there does exist a finite volume method which gives an optimal mesh, both for variational principles and Least Squares.

1 Introduction

In this paper we consider standard Galerkin and Least Squares methods on moving meshes in multidimensions. The capabilities of mesh movement in approximating the solution of PDEs are yielding their secrets slowly, largely because there have been significant difficulties in handling the complex nonlinearities inherent in the problem and in controlling the mesh to prevent tangling. In the recent past techniques employed have included various forms of equidistribution ([1]-[3]), usually based on solution shape criteria, and minimisation based on the residual of the PDE ([4]-[6]). Equidistribution is a highly effective technique for the distribution of nodes in one dimension. However, there have remained question marks over how to choose the equidistribution criteria and to what purpose (although see [7]). Minimisation techniques, on the other hand, allow immediate access to multidimensions but here node distribution is less well understood. In this paper we concentrate on residual minimisation as the criterion for moving the nodes.

We begin by recalling the basis of the Moving Finite Element (MFE) method of Miller ([8]-[12]) together with some of its properties. We then go on to discuss L_2 Least Squares methods on moving meshes, with examples, and the relationship between the two methods in the steady case.

In the second part of the paper we describe Moving Finite Volume and discrete l_2 Least Squares methods are proposed using the same approach. These methods have their own character and their properties differ from the L_2 case when the nodes are allowed to move.

Finally we discuss local optimisation methods for minimising these norms and conclude with a summary of the schemes and their properties.

2 Finite Elements

The Galerkin finite element method for the generic scalar PDE

$$u_t = Lu \tag{1}$$

where L is a second order space operator, e.g. $Lu = -a\partial_x u + \sigma\partial_x^2 u$, is a semi-discrete method based on a weak form of the PDE. One way of deriving the weak form is by constructing the unique minimiser of the L_2 residual of the PDE with respect to *the time derivative* U_t via

$$\min_{U_t} ||U_t - LU||_{L_2}^2 \tag{2}$$

where U is the finite-dimensional approximation to u. Differentiating (2) with respect to U_t and using an expansion of U in terms of basis functions $\phi_j(\underline{x})$ in the form

$$U = \sum_j U_j(t)\phi_j(\underline{x}) \tag{3}$$

yields the Galerkin equations

$$\langle \phi_j, U_t - LU\rangle = 0 \tag{4}$$

where the bracket notation denotes the L_2 inner product and ϕ_j is the j'th basis function for the finite-dimensional subspace which contains U and therefore U_t. We take the functions U and U_t to be piecewise continuous and the basis function ϕ_j to be of local finite element type. The resulting matrix system may be solved in time using a suitable ODE package in the style of the method of lines.

2.1 Steady State

In principle the Galerkin method may be used to solve the time-*independent* equation $Lu = 0$ by driving solutions of the time-dependent equation (1) to steady state.

To reach steady state the velocity U_t in (4) may be replaced by an explicit time discretisation with index n and time step τ and the discrete equation

$$\left\langle \phi_j, \frac{U^{n+1} - U^n}{\tau} - LU^n \right\rangle = 0 \tag{5}$$

used as an iteration to drive U^n to convergence. The steady state solution satisfies the weak form

$$\langle \phi_j, LU\rangle = 0 \tag{6}$$

Not only may the Galerkin equations (4) or (5) be used to iterate to steady state but the mass matrix may be replaced by any positive definite matrix.

2.2 An Optimal Property of the Steady Galerkin Equations

If the differential equation $Lu = 0$ can be derived from a variational principle, i.e. there exists a function $F(\underline{x}, u, \nabla u)$ such that

$$Lu = -\frac{\partial F}{\partial u} + \nabla . \frac{\partial F}{\partial \nabla u}, \tag{7}$$

then since

$$\frac{\partial}{\partial U_j} \int F(\underline{x}, U, \nabla U) d\underline{x} = \int \left(\frac{\partial F}{\partial U} \frac{\partial U}{\partial U_j} + \frac{\partial F}{\partial \nabla U} \frac{\partial \nabla U}{\partial U_j} \right) d\underline{x}$$

$$= \int \left(\frac{\partial F}{\partial U} - \nabla . \frac{\partial F}{\partial \nabla U} \right) \frac{\partial U}{\partial U_j} d\underline{x} = - \int LU \phi_j d\underline{x} \tag{8}$$

the Galerkin equations (6) provide an optimal U for variations of the functional

$$I(F) = \int F(\underline{x}, u, \nabla u) d\underline{x} \tag{9}$$

in the approximation space of U. The functional (9) is minimised by solutions of the weak form (6) with LU given by (7), i.e. solutions of

$$\left\langle \phi_j, \frac{\partial F}{\partial U} \right\rangle + \left\langle \nabla \phi_j, \frac{\partial F}{\partial \nabla U} \right\rangle = 0 \tag{10}$$

In (10) integration by parts has been used over a local patch of elements surrounding node j (see fig.1) with the assumption that the finite element basis functions ϕ_j vanish on the boundary of the patch. Not only may the Galerkin equations (4) or (5) be used to iterate to steady state but the mass matrix may be replaced by any positive definite matrix.

In particular, if $F(\underline{x}, u, \nabla u) = \frac{1}{2} \left(f(\underline{x}, u, \nabla u) \right)^2$, the functional

$$J(f) = \frac{1}{2} \int \left(f(\underline{x}, u, \nabla u) \right)^2 d\underline{x} \tag{11}$$

is minimised by solutions of the weak form

$$\left\langle \phi_j, \frac{\partial f^2}{\partial U} \right\rangle + \left\langle \nabla \phi_j, \frac{\partial f^2}{\partial \nabla U} \right\rangle = 0 \tag{12}$$

3 Moving Finite Elements

The Moving Finite Element (MFE) procedure ([8]-[12]) is a semi-discrete moving mesh finite element method based on the two coupled weak forms of the PDE

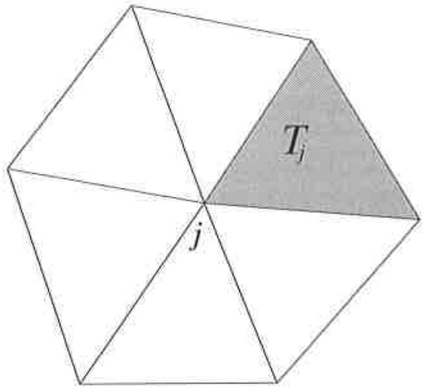

Figure 1: A local patch of elements surrounding node j.

arising from the minimisation in section 2 when the node locations are allowed to depend on time. Thus U becomes an explicit function of the $\underline{X}_j(t)$ (the nodal positions). Then, using the result

$$\frac{\partial U}{\partial \underline{X}_j} = (-\nabla U)\,\phi_j \tag{13}$$

which holds if the basis functions ϕ_j are of *linear* finite element type (see [8],[9] or [13]), the derivative of U with respect to t becomes

$$\begin{aligned}\frac{\partial U}{\partial t}|_{moving\underline{X}} &= \frac{\partial U}{\partial t}|_{fixed\underline{X}} + \sum_j \frac{\partial U}{\partial \underline{X}_j}.\frac{d\underline{X}_j}{dt} \\ &= \frac{\partial \hat{U}}{\partial t} + \sum_j (-\nabla U)\,\phi_j.\frac{\partial \underline{X}_j}{dt} \\ &= \dot{U} - \nabla U.\,\dot{\underline{X}} \end{aligned} \tag{14}$$

say, where U is given by (3) and $\hat{U}$ and $\underline{X}$ are independent functions of t whose time derivatives have expansions

$$\dot{U} = \frac{\partial \hat{U}}{\partial t} = \sum_j \frac{\partial U_j}{\partial t}\phi_j, \qquad \dot{\underline{X}} = \frac{d\underline{X}}{dt} = \sum_j \frac{d\underline{X}_j}{dt}\phi_j \tag{15}$$

(cf.(3)). These functions are taken to be continuous, corresponding to the evolution of a continuous linear approximation.

Using (14), minimisation of the residual in (2) over the coefficients $\dot{U}_j, \dot{\underline{X}}_j$ then takes the form

$$\min_{\dot{U}_j, \dot{\underline{X}}_j} \left\| \dot{U} - \nabla U . \dot{\underline{X}} - LU \right\|_{L_2}^2 \tag{16}$$

which, using (15), leads to the MFE or extended Galerkin equations

$$\left\langle \phi_j, \dot{U} - \nabla U . \dot{\underline{X}} - LU \right\rangle = 0 \tag{17}$$

$$\left\langle (-\nabla U)\, \phi_j, \dot{U} - \nabla U . \dot{\underline{X}} - LU \right\rangle = 0 \tag{18}$$

The resulting ODE system may be solved by a stiff ODE package as in the method of lines.

The method has been analysed in [10] and found to possess the following properties:

Property 1. For scalar first order time-dependent PDEs in any number of dimensions the method is an approximate method of characteristics.

Property 2. For scalar second order time-dependent PDEs in one dimension the method repels nodes from inflection points towards areas of high curvature. At steady state the nodes asymptotically equidistribute a power of the second derivative.

However the method also has intrinsic singularities.

- If ∇U has a component whose values are equal in adjacent elements (dubbed parallelism by Miller [8]), the system of equations (17)/(18) becomes singular and must be regularised in some way (see [8]-[12]).
- If the area of an element vanishes, the system again becomes singular and special action is required.

Each of these singularities also leads to poor conditioning of the corresponding matrix systems near to singularity. For these reasons the method is usually regularised by adding penalties the functional in (16).

3.1 Steady State

In the same way as for fixed meshes the MFE method may in principle be used to generate weak forms for approximately solving the *steady* PDE $Lu = 0$ by driving the MFE solutions to steady state (assuming convergence). From Property 2 of section 3.1 it may be expected that for scalar *second order* PDEs in one dimension the nodes will converge towards areas of high curvature. Property 1, however, indicates that for scalar *first order* PDEs the nodes continue to move with characteristic speeds and are not therefore expected to settle down to a steady state.

To reach a steady state we may replace the velocities $\dot{U}$ and $\dot{\underline{X}}$ by explicit time discretisations with index n and time steps τ, σ and use the resulting equations (17)/(18) to drive U^n, X^n to convergence, provided that is possible. Since we are only interested in the limit the mass matrix may be replaced by any positive definite matrix. The steady state solution satisfies the weak forms

$$\left\langle \begin{pmatrix} 1 \\ -\nabla U \end{pmatrix} \phi_j, LU \right\rangle = 0 \tag{19}$$

We note that from (16) the MFE method in the steady case implements the minimisation

$$\min_{\dot{U}, \dot{\underline{X}}} ||LU||^2_{L_2} \tag{20}$$

Although $\dot{U}$ and $\dot{\underline{X}}$ no longer appear in LU, the minimisation is over all functions lying in the space spanned by the $\{\phi_j, (-\nabla U)\phi_j\}$. In one dimension this space is also spanned by the discontinuous linear functions on the mesh (see [10]) (provided that U_x is not equal in adjacent elements).

3.2 The Optimal Property of the Steady MFE Equations

It has been shown in [13] that the optimal property of section 2.2 generalises to the steady MFE equations (19). If $Lu = 0$ can be derived from a variational principle then, as in section 2.2, solutions of the weak forms (19) provide a local optimum of (9) over the approximation space spanned by the set of functions $\{\phi_j, (-\nabla U)\phi_j\}$. We shall refer to this property as the Optimal Property. This result essentially follows from (14) modified to apply to variations, i.e.

$$\delta U|_{moving\underline{X}} = \delta U|_{fixed\underline{X}} - \nabla U.\delta\underline{X} \tag{21}$$

The MFE method may therefore be used as an iterative procedure to generate locally optimal meshes (see [13]). If desired, the MFE mass matrix may be replaced by any positive definite matrix (see [14]).

Substituting (7) into (19), the functional (9) is minimised by solutions of the two weak forms

$$\left\langle \phi_j, \frac{\partial F}{\partial U} \right\rangle + \left\langle \nabla\phi_j, \frac{\partial F}{\partial \nabla U} \right\rangle = 0 \tag{22}$$

$$\left\langle \phi_j, \frac{\partial F}{\partial \underline{x}} \right\rangle + \left\langle \nabla\phi_j, \left(F - \nabla U. \frac{\partial F}{\partial \nabla U} \right) \right\rangle = 0 \tag{23}$$

where the identity

$$\nabla. \left(F - \nabla U. \frac{\partial F}{\partial \nabla U} \right) = \frac{\partial F}{\partial \underline{x}} + \frac{\partial F}{\partial U} \nabla U - \left(\nabla. \frac{\partial F}{\partial \nabla U} \right) \nabla U \tag{24}$$

has been used to transform the second of (19) into the equivalent equation (23) which is formally suitable for piecewise linear approximation. In carrying out the integration by parts to arrive at (23) we have used the fact that the continuous piecewise linear finite element basis function ϕ_j vanishes on the boundary of the patch.

In particular, for the least squares functional (11) the weak forms are

$$\left\langle \phi_j, \frac{\partial f^2}{\partial U} \right\rangle + \left\langle \nabla \phi_j, \frac{\partial f^2}{\partial \nabla U} \right\rangle = 0 \tag{25}$$

$$\left\langle \phi_j, \frac{\partial f^2}{\partial \underline{x}} \right\rangle + \left\langle \nabla \phi_j, \left(f^2 - \nabla U . \frac{\partial f^2}{\partial \nabla U} \right) \right\rangle = 0 \tag{26}$$

4 Least Squares Finite Elements

Notwithstanding the use of the L_2 norm in the construction of the Galerkin and MFE methods in sections 2 and 3, from a fully discrete point of view the procedure used there is a restricted least squares minimisation because it is carried out only over the *velocities* $\dot{U}_j$ and $\dot{\underline{X}}_j$. The variables U_j and $\underline{X}_j$ are treated as constants, independent of $\dot{U}_j$ and $\dot{\underline{X}}_j$ and the coupling is ignored, as in the method of lines.

A fully discrete least squares approach is feasible, however, if u_t is discretised in time *before* the least squares minimisations are carried out. Minimisation is then over U_j and $\underline{X}_j$ rather than $\dot{U}_j$ and $\dot{\underline{X}}_j$ and the variational equations include additional terms that do not arise in the semi-discrete finite element formulations.

In what follows we shall restrict attention to *first order* space operators Lu depending on $\underline{x}, u$ and ∇u only.

4.1 Least Squares Finite Elements on a Fixed Grid

To describe the procedure in more detail consider a one-step (explicit or implicit) time discretisation of equation (1) of the form

$$\frac{u^{n+1} - u^n}{\Delta t} = Lu^* \tag{27}$$

where $*$ may denote n or $n+1$. The finite-dimensional approximation U^{n+1} at the next time step is then generated by least squares minimisation of the residual

$$R^* = \frac{U^{n+1} - U^n}{\Delta t} - LU^* \tag{28}$$

of (27) over the coefficients U_j^{n+1} via

$$\min_{U_j^{n+1}} ||R^*||_{L_2}^2 \tag{29}$$

In the explicit case ($* = n$) the gradient of (29) with respect to U_j^{n+1} gives rise to the weak form

$$\left\langle R^n, \frac{\phi_j^{n+1}}{\Delta t} \right\rangle = 0 \tag{30}$$

which is a simple time discretisation of (4).

However, in the implicit case ($* = n+1$) the gradient of (29) with respect to U_j^{n+1} leads to

$$\left\langle R^{n+1}, \frac{\phi_j^{n+1}}{\Delta t} - \frac{\partial LU^{n+1}}{\partial U_j^{n+1}} \right\rangle = 0 \tag{31}$$

Equation (31) is not simply an implicit time discretisation of (4) because of the additional terms in the test function.

4.2 Least Squares Moving Finite Elements (LSMFE)

Now consider minimisation of the L_2 norm in (29) over the nodal coordinates $\underline{X}_j^{n+1}$ as well as the coefficients U_j^{n+1}. This is the approach of the recent Least Squares Moving Finite Element (LSMFE) method [14] which was proposed partly in an attempt to overcome the difficulties which arise with first order PDEs when the nodes move with characteristic speeds.

By analogy with (16) consider the minimisation

$$\min_{U_j^{n+1}, \underline{X}_j^{n+1}} \|R^*\|_{L_2}^2 \tag{32}$$

where R^* is the residual

$$R^* = \frac{U^{n+1} - U^n}{\Delta t} - \nabla U^*. \frac{\underline{X}^{n+1} - \underline{X}^n}{\Delta t} - LU^* \tag{33}$$

(cf. (16)). In the explicit case ($* = n$) $\|R^n\|_{L_2}^2$ is quadratic in both sets of variables $U_j^{n+1}, \underline{X}_j^{n+1}$ and minimisation yields

$$\left\langle R^n, \begin{pmatrix} 1 \\ -\nabla U^n \end{pmatrix} \frac{\phi_j^{n+1}}{\Delta t} \right\rangle = 0 \tag{34}$$

which is a simple time discretisation of (17).

In the implicit case ($* = n+1$), on the other hand, the gradient of $\|R^{n+1}\|_{L_2}^2$ with respect to U_j^{n+1} gives

$$\left\langle R^{n+1}, \left(\frac{\phi_j^{n+1}}{\Delta t} - \frac{\underline{X}^{n+1} - \underline{X}^n}{\Delta t}. \nabla \phi_j^{n+1} - \frac{\partial}{\partial U_j^{n+1}} LU^{n+1} \right) \right\rangle = 0 \tag{35}$$

while that with respect to $\underline{X}_j^{n+1}$ gives (formally)

$$\left\langle R^{n+1}, \left(-\nabla U^{n+1}\right) \left(\frac{\phi_j^{n+1}}{\Delta t} - \frac{\underline{X}^{n+1} - \underline{X}^n}{\Delta t} . \nabla \phi_j^{n+1} - \frac{\partial}{\partial U_j^{n+1}} L U^{n+1} \right) \right\rangle$$

$$+ \oint \frac{1}{2} \left(R^{n+1}\right)^2 \phi_j^{n+1} \underline{\widehat{n}} ds = 0 \qquad (36)$$

using (13), where the boundary integral in (36) (which appears due to variations in the mesh) is taken over the boundaries of the elements in the patch containing node j (see fig.1). The unit normal $\underline{\widehat{n}}$ is measured inwards.

In deriving equations (35)/(36) the functions U^{n+1} and $\underline{X}^{n+1}$ appearing in the time-discretised terms in (33) are regarded as independent variables, but the U^{n+1} occurring in ∇U^{n+1} and LU^{n+1} are functions of $\underline{x}$ and $\underline{X}$. We have therefore used the chain rule

$$\frac{\partial LU}{\partial U_j} = \frac{\partial U}{\partial U_j} \frac{\partial LU}{\partial U} + \frac{\partial \nabla U}{\partial U_j} . \frac{\partial LU}{\partial \nabla U} = \frac{\partial LU}{\partial U} \phi_j + \frac{\partial LU}{\partial \nabla U} . \nabla \phi_j \qquad (37)$$

(see (3)) when differentiating ∇U^{n+1} and LU^{n+1} to obtain (36).

We shall refer to (35)/(36) as the transient LSMFE equations. These equations have been solved in [14] in one dimension and, in spite of hopes to the contrary, the method was found to still possess Property 1 of section 3, that for scalar first order time-dependent PDEs the method is an approximate method of characteristics. This property survives because, even though equations (35)/(36) differ from (17)/(18), approximate characteristic speeds still make the residual vanish. However,as we shall see, the method does generate the Optimal Property of MFE at the steady state.

4.3 Steady State Least Squares Moving Finite Elements

Consider now the steady state limit of (35)/(36) as $\Delta t \to \infty$, assuming that convergence takes place. Since $R^{n+1} \to LU$, equations (35) and (36) become

$$\left\langle LU, \frac{\partial}{\partial U_j} LU \right\rangle = \left\langle LU, (-\nabla U) \frac{\partial}{\partial U_j} (LU) \right\rangle + \oint \frac{1}{2} (LU)^2 \phi_j \underline{\widehat{n}} ds = 0 \qquad (38)$$

which may be written in the equivalent forms

$$\left\langle \frac{\partial (LU)^2}{\partial U}, \phi_j \right\rangle + \left\langle \frac{\partial (LU)^2}{\partial \nabla U}, \nabla \phi_j \right\rangle = 0 \qquad (39)$$

$$\left\langle \frac{\partial (LU)^2}{\partial \underline{x}}, \phi_j \right\rangle + \left\langle \left((LU)^2 - (\nabla U) . \frac{\partial (LU)^2}{\partial \nabla U} \right), \nabla \phi_j \right\rangle = 0, \qquad (40)$$

where we have used (37) and the identity (24) with $F = (LU)^2$. Equations (38) may be obtained by direct minimisation of $||LU||^2_{L_2}$ over U_j and $\underline{X}_j$.

Referring back to (25)/(26) we see that equations (39)/(40) are the *steady* MFE equations for the PDE

$$-\frac{\partial (Lu)^2}{\partial u} + \nabla . \frac{\partial (Lu)^2}{\partial \nabla u} = 0 \tag{41}$$

(see [13]) which corresponds to the Euler-Lagrange equation for the minimisation of the Least Squares functional $||Lu||^2_{L_2}$.

To solve the nonlinear equations (39)/(40) by iteration we may use the corresponding time-stepping method, (35)/(36), with n as the iteration parameter, or any other convenient iteration (see section 7).

4.4 Properties of the Steady LSMFE Equations

(i) As we have already seen in section 3.3, for steady problems the Least Squares functional $F(x, U, \nabla U) = (LU)^2$ leads to the weak forms (25)/(26) and therefore (39)/(40), and we have the Optimal Property, as expected.

(ii) In the LSMFE tests carried out in [14] on scalar first order *steady state* equations it is shown that the nodes no longer move with characteristic speeds, as in Property 1 of section 3.1, but instead move to regions of high curvature as in Property 2. This is a useful property and could have been expected because the least squares procedure in effect embeds the original first order equation in the *second order* equation (41) for which the MFE steady limit yields the asymptotic equidistribution of Property 2 of section 3.1.

(iii) In the particular case where LU takes the form of a divergence of a continuous function of U, we may apply an extension of the result in [15] which shows that, asymptotically, minimisation of $||LU||^2_{L_2}$ is equivalent to equidistribution of LU over all the elements in a certain sense. Thus for example in the case where

$$Lu = \nabla.(\underline{a}u) \tag{42}$$

with constant $\underline{a}$, the LSMFE method asymptotically equidistributes the piecewise constant residual $LU = \nabla.(\underline{a}U)$ in each element in the sense described in section 6.1 below.

We now give some illustrative examples of steady LSMFE.

4.5 Examples

(i) Take

$$Lu = u - f(x) \tag{43}$$

for which the steady LSMFE weak forms, from (39)/(40), are

$$\left\langle \begin{pmatrix} 1 \\ -\nabla U \end{pmatrix} \phi_j, (U - f(\underline{x})) \right\rangle = 0, \tag{44}$$

subject to boundary conditions, which provide a local minimum for the least squares variable node approximation problem

$$\min_{U_j,\underline{X}_j} \int (U - f(\underline{x}))^2 \, d\underline{x} \tag{45}$$

(Superior results for this problem can however be obtained by considreing piecewise linear discontinuous approximation - see section 4.6 below.)

(ii) Take

$$Lu = \underline{a}.\nabla u = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} \tag{46}$$

where $\underline{a} = (a, b)$ is constant. In this case equation (41) becomes the degenerate elliptic equation

$$\nabla.\left((\underline{a}.\nabla u)\,\underline{a}\right) = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\left(a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}\right) = 0 \tag{47}$$

The steady LSMFE weak forms, from (39)/(40), are

$$\langle \underline{a}.\nabla U, \underline{a}.\nabla \phi_j) \rangle = 0 \tag{48}$$

and

$$\left\langle \phi_j, (\underline{a}.\nabla U)\frac{\partial\,(\underline{a}.\nabla U)}{\partial \underline{x}} \right\rangle - \frac{1}{2}\left\langle \nabla\phi_j, (\underline{a}.\nabla U)^2 \right\rangle = 0, \tag{49}$$

subject to boundary conditions, which provide a local minimum for the least squares variable node approximation problem

$$\min_{U_j,X_j} \int (\underline{a}.\nabla U)^2 \, d\underline{x} \tag{50}$$

(see ([16],[19])). These are also the steady MFE equations for the second order degenerate elliptic equation

$$\nabla.\left((\underline{a}.\nabla u)\,\underline{a}\right) = 0 \tag{51}$$

If $\underline{a}$ depends on $\underline{x}$ but is continuous and divergence-free, then $LU = \underline{a}.\nabla U$ is the divergence of a continuous function and this example has the asymptotic equidistribution property referred to in section 4.4, in this case asymptotically equidistributing $\overline{\underline{a}}.\nabla U$ where $\overline{\underline{a}}$ consists of the element averages of $a(\underline{x})$.

(iii) Take

$$Lu = |\nabla u| = \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right)^{1/2} \tag{52}$$

for which equation (41) is Laplace's equation

$$\nabla^2 u = 0 \tag{53}$$

The steady LSMFE weak forms, from (39)/(40), are

$$\langle \nabla \phi_j, \nabla U \rangle = \left\langle \phi_j, \frac{\partial (\nabla U)^2}{\partial \underline{X}} \right\rangle - \left\langle \nabla \phi_j, (\nabla U)^2 \right\rangle = 0, \qquad (54)$$

subject to boundary conditions, which provide a minimum for the variable node Dirichlet problem (see e.g. [6])

$$\min_{U_j, \underline{X}_j} \int |\nabla U|^2 \, d\underline{x} \qquad (55)$$

4.6 The MBF Approach

If the functional F is independent of ∇U (the best approximation problem) we may take combinations of the variations δU_j and $\delta \underline{X}_j$ to design simpler sequential algorithms. The first variation of the square of the L_2 norm of $R = LU$, using (32),(35)/(36) and (38), gives

$$\delta \frac{1}{2} \|LU\|^2_{L_2} = \left\langle LU, \frac{\partial LU}{\partial U} \phi_j \right\rangle (\delta U_j - \nabla U . \delta \underline{X}_j) \qquad (56)$$

$$- \oint \frac{1}{2} (LU)^2 \phi_j \underline{\widehat{n}} . \delta \underline{X}_j ds \qquad (57)$$

Setting $\delta \underline{X}_j = 0$ gives the fixed mesh least squares equation

$$\left\langle LU \frac{\partial LU}{\partial U}, \phi_j \right\rangle = 0 \qquad (58)$$

while, setting

$$\delta U_j - \nabla U . \delta \underline{X}_j = 0$$

gives

$$\oint \frac{1}{2} (LU)^2 \phi_j \underline{\widehat{n}} ds = 0, \qquad (59)$$

an equation for updating the nodes locally which depends only on the integral over the boundaries of the patch containing node j. The constraint (4.6), which forces δU_j to vary with $\delta \underline{X}_j$, corresponds to linear interpolation/extrapolation on the current piecewise linear U function. This approach, which depends only on local problems, is called the Moving Best Fits (MBF) method in [10] and is the basis of best approximation algorithms in [19],[16].

5 Finite Volume Methods

We turn now to a discussion of the use of *discrete* l_2 norms with area weighting as a basis for generating finite volume schemes, to a large part the result of a simple quadrature applied to the L_2 norm used previously.

Define the discrete l_2 norm as the weighted sum over triangles of the average residual of the PDE, viz.

$$||R||_{l_2}^2 = \sum_T S_T \overline{R}_T^2 \tag{60}$$

(cf. (2), (16), (29) and (32)), where the suffix T runs over all the triangles of the region, S_T is the area of triangle T and $\overline{R}_T$ is the average value of the residual R over the vertices of T.

This norm coincides with the L_2 norm in the case where R is constant on each triangle. For then

$$||R||_{L_2}^2 = \int R^2 d\underline{x} = \sum_T \int_T R^2 d\underline{x} = \sum_T \overline{R}_T^2 \int_T d\underline{x} = \sum_T S_T \overline{R}_T^2 = ||R||_{l_2}^2 \tag{61}$$

as in example (ii) of section 4.5 where $R = LU = \underline{a}.\nabla U$, the advection speed $\underline{a}$ being constant and U piecewise linear. If the area weighting in (60) is omitted this link is lost. However, one objection to the use of Least Squares residuals is that when triangles become degenerate the norm of the derivative is unbounded. By redefining the norm in (60) with a squared weight S_T^2 instead of S_T the norm is always well-defined and still has an approximate equidistribution property (see [8]). Here we concentrate on (60), however: the modifications are straightforward.

The form (60) may be rewritten as a sum over nodes j, namely

$$||R||_{l_2}^2 = \frac{1}{3} \sum_j \sum_{\{T_j\}} S_{T_j} \overline{R}_{T_j}^2 \tag{62}$$

where $\{T_j\}$ runs over the patch of trangles abutting node j (see fig.1).

We may take $(\nabla U)_T$ to be the gradient associated with the linearly interpolated corner values of U in the triangle T, given by (see [13])

$$(\nabla U)_T = \left(\frac{-\sum^i U_i \Delta Y_i}{\sum^i X_i \Delta Y_i}, \frac{\sum^i U_i \Delta X_i}{-\sum^i Y_i \Delta X_i} \right) = \left(\frac{\sum^i Y_i \Delta U_i}{\sum^i X_i \Delta Y_i}, \frac{-\sum^i X_i \Delta U_i}{\sum^i Y_i \Delta X_i} \right) \tag{63}$$

where the sums run over the corners i of the triangle T and $\Delta X_i, \Delta Y_i, \Delta U_i$ denote the increments in the values of X, Y, U taken anticlockwise across the side of T *opposite* the corner concerned (see fig.1). This is of course identical to the finite element gradient with piecewise linear approximation. In the same notation the area of the triangle T is

$$S_T = \frac{1}{2} \sum_i X_i \Delta Y_i = -\frac{1}{2} \sum_i Y_i \Delta X_i \tag{64}$$

which incorporates a summation by parts.

5.1 Moving Finite Volumes

By analogy with the MFE method of section 3 a moving finite volume (MFV) method may be set up by minimising the residual

$$\left\| \dot{U} - \nabla U . \underline{\dot{X}} - LU \right\|_{l_2} \tag{65}$$

(see (60)) over $\dot{U}_j$ and $\underline{\dot{X}}_j$, which leads to

$$\sum_{\{T_j\}} S_{T_j} \left(\dot{U} - \nabla U . \underline{\dot{X}} - LU \right)_j = 0 \tag{66}$$

$$\sum_{\{T_j\}} S_{T_j} \left(\dot{U} - \nabla U . \underline{\dot{X}} - LU \right)_j (-\nabla U)_{T_j} = 0 \tag{67}$$

(cf. (17),(18)) where $\{T_j\}$ is the set of triangles abutting node j and the suffix j indicates that terms involving $\dot{U}$ and $\underline{\dot{X}}$ are evaluated at the node j while those involving ∇U and LU are evaluated over the triangle T_j.

Property 1 of section 3 still holds since the residual vanishes as before when $\underline{\dot{X}}$ approximates characteristic speeds. The method also has the same singularities as MFE, in particular when components of the gradients ∇U are the same in adjacent elements.

At the steady state we have the steady MFV equations

$$\sum_{\{T_j\}} S_{T_j} (LU)_j = 0 \tag{68}$$

$$\sum_{\{T_j\}} S_{T_j} (LU)_T (-\nabla U)_{T_j} = 0 \tag{69}$$

If Lu is derived from a variational principle, given by (7), these become

$$\sum_{\{T_j\}} S_{T_j} \left(-\frac{\partial F}{\partial U} + \nabla . \frac{\partial F}{\partial \nabla U} \right)_j = 0 \tag{70}$$

$$\sum_{\{T_j\}} S_{T_j} \left(-\frac{\partial F}{\partial U} + \nabla . \frac{\partial F}{\partial \nabla U} \right)_j (-\nabla U)_{T_j} = 0 \tag{71}$$

which, using the summation by parts implicit in (64), leads to

$$\sum_{\{T_j\}} S_{T_j} \left(-\frac{\partial F}{\partial U} - \frac{1}{2} \frac{\partial F}{\partial \nabla U} . \underline{n}_j \right)_j = 0 \tag{72}$$

$$\sum_{\{T_j\}} S_{T_j}\left(-\frac{\partial F}{\partial U}-\frac{1}{2}\frac{\partial F}{\partial \nabla U}\cdot\underline{n}_j\right)_j(-\nabla U)_{T_j}=0 \tag{73}$$

where $\underline{n}_j = (\Delta Y_j, -\Delta X_j)$ is the inward normal to the side opposite node j scaled to the length of that side (see [18] and fig.1).

5.2 A Discrete Optimisation

By analogy with (60) a discrete form of (9) is

$$I_d(F)=\sum_T S_T\overline{F(\underline{X},U,\nabla U)}_T=\sum_T S_T\frac{1}{3}\sum_{i=1}^{3}F\left(\underline{X}_i,U_i,(\nabla U)_T\right) \tag{74}$$

where the overbar denotes the average value over the vertices of T. (When Euler first derived his variational equation he used such a finite form, although on a fixed mesh of course.)

The sum in (74) may be rewritten as a sum over nodes j as

$$I_d(F)=\frac{1}{3}\sum_j\sum_{\{T_j\}} S_{T_j}\frac{1}{3}\sum_{i=1}^{3}F(\underline{X}_i,U_i,(\nabla U)_{T_j}) \tag{75}$$

where i runs over the corners of the triangle T_j.

A moving mesh method may be derived based on the two coupled equations which arise when (75) is optimised over both U_j and $\underline{X}_j$. Differentiating (75) with respect to U_j gives

$$\frac{\partial I_d}{\partial U_j}=\sum_{\{T_j\}} S_{T_j}\frac{1}{3}\left(\frac{\partial F}{\partial U}+\frac{\partial F}{\partial \nabla U}\cdot\frac{\partial \nabla U}{\partial U}\right)_j \tag{76}$$

leading to the equation

$$\sum_{\{T_j\}}\left(S_T\frac{\partial F}{\partial U}+\frac{1}{2}\frac{\partial F}{\partial \nabla U}\cdot\underline{n}\right)_j=0 \tag{77}$$

This is a finite volume weak form which corresponds to the finite element weak form (10). It is also identical to (72) showing that the optimal property of section 2.2 goes over to the steady state finite volume case when U_j varies.

Differentiating with respect to $\underline{X}_j$ gives

$$\frac{\partial I_d}{\partial \underline{X}_j}=\sum_{\{T_j\}} S_{T_j}\frac{1}{3}\left(\frac{\partial F}{\partial \underline{X}}+\frac{\partial \nabla U}{\partial \underline{X}}\cdot\frac{\partial F}{\partial \nabla U}\right)_j \tag{78}$$

which leads to

$$\sum_{\{T_j\}}\left(S_T\frac{\partial F}{\partial \underline{X}}-\left(\Delta U\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}-\frac{1}{2}S_T^{-1}\nabla U_T.\underline{n}\right)\frac{\partial F}{\partial \nabla U}\right)_j=0 \tag{79}$$

(cf.(73)). Equation (79) is the companion weak form to (73), corresponding to the second finite element weak form (23). However, this differs considerably from (23) showing that the optimal property of MFE does not go over to steady state MFV when $\underline{X}$ varies. This is because differentiation of a quadrature rule with respect to $\underline{X}$ is not the same as quadrature of the derivative. In fact, it is equations (77) and (79) which give the optimal property.

If $F = \frac{1}{2}(LU)^2$ equations (77) and (79) can be made the basis of a least squares method (LSMFV) for steady problems associated with the PDE (41) with $u_t = 0$. From (77) and (79) we have

$$\sum_{\{T_j\}} S_{T_j} \left(LU \frac{\partial LU}{\partial U} + \frac{1}{2} LU \frac{\partial LU}{\partial \nabla U} \underline{n} \right)_j = 0 \tag{80}$$

and

$$\sum_{\{T_j\}} \left[S_T LU - \left(\Delta U \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{1}{2} S_T^{-1} \nabla U \underline{n} \right) LU \frac{\partial U}{\partial \nabla U} \right]_j = 0 \tag{81}$$

6 Time-Dependent Least Squares Moving Finite Volumes

As in section 4, a fully discrete Least Squares Finite Volume method for time-dependent problems is obtained if u_t is discretised in time *before* the l_2 least squares minimisations are carried out. Minimisation is over U_j and $\underline{X}_j$ rather than $\dot{U}_j$ and $\dot{\underline{X}}_j$.

Consider again the one-step time discretisation of equation (1) in the form (27). Then on a moving mesh the solution at the next time step may be generated by the least squares minimisation of the implicit form of the residual over U_j^{n+1} and $\underline{X}_j^{n+1}$ via

$$\min_{U_j^{n+1}, \underline{X}_j^{n+1}} \left\| R^{n+1} \right\|_{l_2}^2 \tag{82}$$

where (cf. (60))

$$R^{n+1} = \frac{U^{n+1} - U^n}{\Delta t} - \nabla U^{n+1} . \frac{\underline{X}^{n+1} - \underline{X}^n}{\Delta t} - LU^{n+1} \tag{83}$$

(with ∇U^{n+1} defined as in (63)) and

$$\|R\|_{l_2}^2 = \langle R, R \rangle_{l_2}, \qquad \langle P, Q \rangle_{l_2} = \sum_{\{T_j\}} S_T \overline{P}_T \overline{Q}_T \tag{84}$$

Setting the gradients with respect to U_j^{n+1} and $\underline{X}_j^{n+1}$ to zero gives

$$\left\langle R^{n+1}, \frac{\partial \overline{R^{n+1}}}{\partial U_j^{n+1}} \right\rangle_{l_2} = 0 \tag{85}$$

and

$$\left\langle R^{n+1}, \frac{\partial \overline{R^{n+1}}}{\partial \underline{X}_j^{n+1}} \right\rangle_{l_2} + \frac{1}{2} \sum_{\{T_j\}} \left(\overline{R_{T_j}^{n+1}} \right)^2 \frac{\partial S_{T_j}^{n+1}}{\partial \underline{X}_j^{n+1}} = 0 \tag{86}$$

We shall refer to (85)/(86) as the transient LSMFV method. For scalar first order time-dependent PDEs the method is still an approximate method of characteristics since approximate characteristic speeds always make the residual R^{n+1} vanish. However, in the steady state it has other features.

6.1 Steady State Least Squares Finite Volumes

Consider now the steady limit. Then from (82) we are minimising $||LU||_{l_2}^2$ and (85)/(86) reduce to

$$\left\langle LU, \frac{\partial \left(\overline{LU}\right)}{\partial U_j} \right\rangle_{l_2} = 0 \tag{87}$$

and

$$\left\langle LU, \frac{\partial \overline{LU}}{\partial \underline{X}_j} \right\rangle_{l_2} + \frac{1}{2} \sum_{T_j} \overline{(LU)}_{T_j}^2 \frac{\partial S_{T_j}}{\partial \underline{X}_j} = 0 \tag{88}$$

which are identical to (80) and (81) (see [17],[18]).

It has been shown in [15] that if LU is the divergence of a continuous function, then the optimal values of $\overline{LU}$ are equidistributed in the sense that the double sum over elements

$$\sum_e \sum_{e'} \left(S_e \overline{LU}_e - S_{e'} \overline{LU}_{e'} \right)^2 \tag{89}$$

is minimised. Thus if $Lu = \underline{a}.\nabla u$ with constant $\underline{a}$, as in example (ii) of section 4.5 the piecewise constant $\overline{LU} = \underline{a}.\nabla U$ is equidistributed over the elements in this sense. The same result is only asymptotically true for the LSMFE method (see section 4.4).

6.2 Example

Consider again example (ii) of section 4.5, for which the steady state residual is

$$LU = \underline{a}.\nabla U \tag{90}$$

Then (85) and (86) reduce to

$$\left\langle \overline{LU}, \underline{a}.\frac{\partial (\nabla U)}{\partial U_j} \right\rangle_{l_2} = \left\langle \overline{LU}, \frac{\partial (\underline{a}.\nabla U)}{\partial \underline{X}_j} \right\rangle_{l_2} + \frac{1}{2} \sum_{T_j} \left(\overline{LU_T} \right)^2 \frac{\partial S_{T_j}}{\partial \underline{X}_j} = 0, \tag{91}$$

subject to boundary conditions, where from (63)

$$\frac{\partial (\underline{a}.\nabla U)_{T_j}}{\partial \underline{X}_j} = \Delta U_j \begin{pmatrix} -b \\ a \end{pmatrix} - \frac{1}{2} S_{T_j}^{-1} (\underline{a}.\nabla U)_{T_j} \underline{n}_j \tag{92}$$

Recall that $\underline{n}_j$ is the inward normal to the side of the triangle opposite j scaled by the length of that side and ΔU_j is the increment in U across that side, taken anticlockwise.

Equation (91) may be written

$$\sum_{T_j} (\underline{a}.\nabla U)_{T_j} \, (\underline{a}.\underline{n}_j) = 0 \tag{93}$$

and

$$\sum_{T_j} \left[(\underline{a}.\nabla U)_T \, \Delta U \begin{pmatrix} -b \\ a \end{pmatrix} - \frac{1}{2} (\underline{a}.\nabla U)_T^2 \, \underline{n} \right]_j = 0 \tag{94}$$

We observe that (93) is identical to (48)), noting that ∇U is constant and $\nabla \phi = S_T^{-1} \underline{n}$. However, (94) does not correspond to (49), even when $\underline{a}$ is constant, so the two methods are not identical under node movement.

7 Minimisation Techniques

The fully discrete least squares methods of sections 4 and 6, unlike the unsteady Galerkin methods of sections 2 and 3, provide a functional to reduce and monitor. It is therefore possible to take an optimisation approach to the generation of a local minimum. Note that the time stepping methods discussed earlier do not necessarily reduce the functional.

Descent methods are based upon the property that the first variation of a functional $\mathcal{F}$,

$$\delta \mathcal{F} = \frac{\partial \mathcal{F}}{\partial \underline{Y}} = \underline{g}^T \delta \underline{Y} \tag{95}$$

say, is negative when

$$\delta \underline{Y} = -\tau \underline{g} = -\tau \frac{\partial \mathcal{F}}{\partial \underline{Y}} \tag{96}$$

where τ is a sufficiently small positive relaxation parameter. For the present application the gradients $\underline{g}$ have already been evaluated in earlier sections. For example, in the LSMFE method the gradients $\underline{g}$ with respect to U_j and $\underline{X}_j$ appear on the left hand side of (35)/(36). Thus, writing $\underline{Y} = \{\underline{Y}_j\} = \{U_j, \underline{X}_j\}$ and $\underline{g} = \{\underline{g}_j\}$ the steepest descent method applied to (32) with $* = n+1$ is

$$\left(\underline{Y}_j^{n+1}\right)^{k+1} - \left(\underline{Y}_j^{n+1}\right)^k = -\tau_j^k \left(\underline{g}_j^{n+1}\right)^k \tag{97}$$

($k = 1, 2, \ldots$) where τ_j^k is the relaxation parameter. Choice of τ_j^k is normally governed by a line search or a local quadratic model.

The left hand side of (97) may be preconditioned by any positive definite matrix. The Hessian gives the full Newton method but may be approximated in various ways. In [14] a positive definite regularising matrix is used in place of the Hessian to generate the solution.

In the present application a local approach may be followed which consists of updating the unknowns one node at a time, using only local information. For a given j each step of the form (97) reduces the functional (32), even when the other $\underline{Y}^{n+1}$ values are kept constant. The updates may be carried out in a block (Jacobi iteration) or sequentially (Gauss-Seidel). A variation on the local approach is to update U_j and $\underline{X}_j$ sequentially, which gives greater control of the mesh. Descent methods of this type have been used by Tourigny and Baines [16] and Tourigny and Hulsemann [6] in the L_2 case and by Roe [17] and Baines and Leary [18] in the l_2 case.

8 Conclusions

We conclude with a summary of the main results.

- The MFE method is a Galerkin method extended to include node movement. Its main properties are

  (a) numerical imitation of the method of characteristics for first order equations in any number of dimensions

  (b) repulsion of nodes from inflection points for second order equations in one dimension

  (c) for
  $$Lu = -\frac{\partial F}{\partial u} + \nabla . \frac{\partial F}{\partial \nabla u} = 0 \tag{98}$$
  the steady MFE equations provide a local optimum for the variational problem
  $$\min_{U_j, \underline{X}_j} \int F(\underline{x}, U, \nabla U) d\underline{x} \tag{99}$$
  in a piecewise linear approximation space with moving nodes.

- The implicit semi-discrete in time Least Squares method (LSMFE) is a least squares method extended to include node movement. It differs from MFE through more complicated test functions and the extra term found in (36), although in the case of first order equations it shares with MFE the property of being a numerical method of characteristics. However, in the steady state the LSMFE equations for $Lu = 0$ are equivalent to the steady MFE weak forms for the PDE
  $$-\frac{\partial (Lu)^2}{\partial u} + \nabla . \left(\frac{\partial (Lu)^2}{\partial \nabla u} \right) = 0 \tag{100}$$

and therefore provide a local minimum for the variational problem

$$\min_{U_j,\underline{X}_j} \int (LU)^2 \, d\underline{x} \tag{101}$$

Moreover, it can be shown that, if LU is the divergence of a continuous flux function then the flux across element boundaries is *asymptotically* equidistributed over the elements.

- The LSMFV method is a moving mesh method based on minimisation of a weighted l_2 norm of the residual of the semi-discrete in time PDE over the solution and the mesh. It shares with LSMFE the property of generating approximate characteristic speeds. At steady state, however, it lacks the Optimal Property of LSMFE but it has the more precise property that, if LU is the divergence of a continuous flux function, then the flux across element boundaries is equidistributed discretely (not just asymptotically) over the elements in the sense of (89) [15].

- Solutions may be obtained by the minimisation procedures of optimisation theory applied to the appropriate norm. A local approach to optimisation is advantageous in preserving the integrity of the mesh.

The MFE, LSMFE and LSMFV methods have been shown to be effective in generating approximate solutions to scalar differential problems in multi-dimensions which exhibit shocks and contact discontinuities ([10]-[13]), ([17]-[20]). The MFE and LSMFV methods have also been effective in obtaining approximate solutions of systems of equations ([10]-[12], [18],[20]).

Finite Volume methods of the type discussed here do not give highly accurate solutions on coarse meshes. However high accuracy is not crucial as far as the mesh is concerned. Thus it may be argued that an LSMFV method is sufficiently accurate for the mesh locations but a more sophisticated method which is robust on distorted meshes, such as high order finite elements or multidimensional upwinding ([21],[22]), may be required for the solution on the optimal mesh.

An outstanding problem is how to avoid the generation of characteristic speeds by the MFE and MFV methods for first order equations. In the case of MFE a clue may be found in the LSMFE method, which embeds the original first order equation in a second order degenerate elliptic equation prior to moving the nodes. When solved by a relaxation method in an iterative manner, in effect applying a finite difference scheme to the associated parabolic equation, the resulting nodal speeds are not characteristic but instead move nodes from low curvature regions to high curvature regions, as required. Moreover the nodes tend to align themselves with characteristic curves, although they do not actually move along them. Although the resulting nodal speeds are effective in this sense, the LSMFE does not generate the correct solution to the first order equation since it has now been embedded a second order equation. Thus, if these speeds are to be used it is impossible for the discrete equations to be set up from a unified approach. Instead it is necessary to generate the speeds from the LSMFE method which must then be substituted into the Lagrangian form of the first order equation, to be solved separately using any convenient method.

9 References

[1] C.de Boor (1978) A Practical Guide to Splines. Springer-Verlag, New York.

[2] E.A. Dorfi, and L.O'C. Drury (1987). Simple Adaptive Grids for 1D Initial Value Problems. J.Comput.Phys., 69, 175-195.

[3] W.Huang et al (1994). Moving Mesh Partial Differential Equations. SIAM J Num An 31, 709-730 (1994).

[4] M.Delfour et al (1985). An Optimal Triangulation for Second Order Elliptic Problems. Comput.Meths.Applied Mech.Engrg., 50, 231-261.

[5] M.J.Baines (1998). Grid Adaptation via Node Movement. Applied Numerical Mathematics, 26, 77-96.

[6] Y.Tourigny and F.Hulsemann (1998). A New Moving Mesh Algorithm for the Finite Element Solution of Variational Problems. SIAM J Num Anal. 34, 1416-1438.

[7] C.Budd et al. (1999). Self-similar Numerical Solutions of the Porous -medium Equation using Moving Mesh Methods. Phil Trans Roy Soc Lond A, 357, pp1047-1077.

[8] K.Miller and R.N.Miller (1981). Moving Finite Elements I. SIAM J.Num.Anal. 18,1019-1032.

[9] K.Miller (1981). Moving Finite Elements II. SIAM J Num Anal. 18,1033-1057.

[10] M.J.Baines (1994). *Moving Finite Elements.* Oxford University Press.

[11] N.N.Carlson and K.Miller (1998). Design and Application of a Gradient Weighted Moving Finite Element Method I: in One Dimension. SIAM J Sci Comp, 19, 728-765.

[12] N.N.Carlson and K.Miller (1998). Design and Application of a Gradient Weighted Moving Finite Element Method II: in Two Dimensions. SIAM J Sci Comp, 19, 766-798.

[13] P.K.Jimack (1998). Local Minimisation of Errors and Residuals using the Moving Finite Element Method. University of Leeds Report 98.17, School of Computer Science

[14] K.Miller and M.J.Baines (1998). Least Squares Moving Finite Elements. OUCL report 98/06. Oxford University Computing Laboratory. Submitted to IMA J Num An.

[15] M.J.Baines (1999). Least Squares and Equidistribution in Multidimensions. Numerical Methods for Partial Differential Equations. 15, 605-615.

[16] Y.Tourigny and M.J.Baines (1997). Analysis of an Algorithm for Generating Locally Optimal Meshes for L_2 Approximation by Discontinuous Piecewise

Polynomials, Math. Comp., 66, 623-650.

[17] P.L.Roe (1996). Compounded of Many Simples. In Challenges in Computational Fluid Dynamics (Ventakrishnan, Salas and Chakravarthy (eds.)) 241-258, Kluwer (1997).

[18] M.J.Baines and S.J.Leary (1999). Fluctuations and Signals for Scalar Hyperbolic Equations on Adjustable Meshes. Numerical Analysis Report 6/98, Department of Mathematics, University of Reading. Communications in Numerical Methods in Engineering (to appear).

[19] M.J.Baines (1994). Algorithms for Optimal Discontinuous Piecewise Linear and Constant L_2 Fits to Continuous Functions with Adjustable Nodes in One and Two Dimensions. Math.Comp. 62, 645-669.

[20] M.J.Baines, S.J.Leary and M.E.Hubbard (1999). A Finite Volume Method for Steady Hyperbolic Equations with Mesh Movement. Proceedings of Conference on Finite Volumes for Complex Applications II (Vilsmeier, Benkhaldoun and Hanel (eds.)), 787-794, Hermes (1999).

[21] H.Deconinck, P.L.Roe and R. Struijs (1993). A Multidimensional Generalisation of Roe's Flux Difference Splitter for the Euler Equations. Computers and Fluids, 22,215.

[22] M.E.Hubbard and P.L.Roe (1999). Compact high-resolution Algorithms for Time-Dependent Advection on Unstructured Grids. International Journal for Numerical Methods in Fluids (to appear).