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An Analysis of some Higher-Order Triangular Elements and their Susceptibility to Hourglassing in Lagrangian Fluid Simulations.

by

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An Analysis of some Higher-Order Triangular Elements and their Susceptibility to Hourglassing in Lagrangian Fluid Simulations

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0 Abstract

In Lagrangian codes two of the main problems are hourglassing with bilinear quadrilateral elements and mesh locking with linear triangular elements. Higher order triangular elements, with their greater number of degrees of freedom, should not suffer from mesh locking. Hourglassing, though, is caused by the interaction of the element and the quadrature used to evaluate the integrals of the derivatives on it. Higher order triangular elements, as we shall see, unlike their linear counterparts, also suffer from this spurious mode. A higher order element/quadrature combination is found that does not suffer from hourglassing. The main fault of this combination is the extra cost involved over the more usual linear or bilinear element with centroid quadrature combination. A short discussion is given regarding this point.

1 Introduction

Lagrangian fluid codes typically require the solution of an equation

$$\int_V \rho \dot{\varepsilon} dV = -\int_V p \nabla . \mathbf{v} dV, \tag{1}$$

derived from the continuity equation, see Milne-Thomson[11] for example. In eq. (1) V is an arbitrary volume of fluid (usually an element for our purposes), ε is the internal energy, ρ is the fluid density, p is the pressure and $\mathbf{v}$ the fluid velocity. (We will normally only be concerned with the two-dimensional case, i.e. $\mathbf{v} = (u, v)^T$, but we will make no assumptions restricting ourselves to this situation). We have also used the common notation

$$\dot{\varepsilon} \equiv \frac{d\varepsilon}{dt} = \frac{\partial \varepsilon}{\partial t} + \mathbf{v}.\nabla \varepsilon.$$

Equation (1) tells us that in a divergence free flow there is no increase or decrease in the internal energy of a volume of fluid. It follows that, if the volume of fluid is distorted and hence the internal energy changes, then the divergence cannot be zero. Numerically, problems arise when the integral of the divergence (falsely) equals zero when the velocities deform the volume of fluid. Unchecked, they can swamp a numerical solution because no force is created to damp them down. This phenomenon of spurious velocity modes is referred to as hourglassing because of the characteristic patterns created in a mesh of regular bilinear elements. Another problem is caused when the nodes cannot move at all. This is called mesh locking and this question has largely been addressed, see Malkus & Hughes[9] for example, and we will not mention the problem further here.

There are several ways in which the spurious modes may be damped. Artificial viscosity can be used, see Maenchen & Sack[8], to produce nodal forces that damp the hourglassing forces. Artificial stiffness, Flanagan & Belytschko[5], is really a more sophisticated version of the artificial viscosity. See also Belytschko et al[2, 3]. Higher order quadrature, see Malkus & Hughes[9], can be used to overcome the problem of constant stresses and hence of hourglassing. Schulz[12] performed a Taylor series expansion of the stresses and retained terms other than the usual constant term to overcome this same problem. Schulz[12] also provides a short discussion on the merits of the above methods and provides additional references.

In this paper, though, we are not primarily interested in the treatment of the hourglassing mode. The purpose of this paper is to study higher order triangular elements and to determine to what degree the hourglassing mode is present.

We shall next look at the two well-known cases of the centroid integrated bilinear quadrilateral element and the linear triangular element to demonstrate the analysis. In the following section this analysis will be extended to higher order triangular element and quadrature combinations. In Section 3 we will discuss problems with implementing these higher order element/quadrature combinations. Finally we give a brief summary of what we have shown in this paper.

Following Margolin & Pyun[10] we consider a quadrilateral with u and v velocity components as shown in figure (1) and represent the 8 velocity components in the cell by an 8-dimensional vector,

$$\mathbf{v} = (u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4).$$

Margolin & Pyun[10] identify six of the eight degrees of freedom with six physical modes of motion and with six mathematical objects. The six physical modes are:

one pattern of horizontal translation

one pattern of vertical translation

one pattern of rotation

one pattern of horizontal strain

one pattern of vertical strain

one pattern of shear strain.

The six mathematical objects are:

$$\tilde{u}, \tilde{v}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}.$$

The idea is then to produce a basis for the 8-dimensional (velocity) space $\mathbf{l_1}, \mathbf{l_2}, \ldots, \mathbf{l_8}$ where the first six vectors correspond to the six mathematical quantities. $\mathbf{l_7}$ and $\mathbf{l_8}$ are then found by orthogonalization and must be spurious. Assuming the basis vectors to be orthonormal the spurious modes can then be damped as follows:

$$\mathbf{v}_{new} = \mathbf{v}_{old} - \alpha \sum_{j=7}^{8} (\mathbf{v}_{old}.\mathbf{l_j})\mathbf{l_j},$$

where α is a parameter to be chosen. A range of $0.01 < \alpha < 0.05$ is recommended by Margolin & Pyun[10].

In this paper we proceed slightly differently in that we will deliberately seek out the non-restoring modes. This is explained using the bilinear element as an example.

Using centroid quadrature, the integrals (subject to multiplication by the area of the element) of the two spatial gradients of interest are given by

$$\frac{\partial u}{\partial x} = \left((u_1 - u_3)(y_2 - y_4) + (u_2 - u_4)(y_3 - y_1)\right)/2A \tag{2}$$

$$\frac{\partial v}{\partial y} = \left((v_1 - v_3)(x_4 - x_2) + (v_2 - v_4)(x_1 - x_3)\right)/2A \tag{3}$$

where A is the area of the quadrilateral.

By observation we see that the only way for eqs. (2) and (3) to be zero is if

$$u_1 = u_3 \quad \text{and} \quad u_2 = u_4$$

and similarly in v. There are two independent ways of accomplishing this, either,

$$u_1 = u_3 = \quad 1 \quad = u_2 = u_4$$

or

$$u_1 = u_3 = 1 \quad \& \quad u_2 = u_4 = -1,$$

again similarly in v. The first produces no restoring forces but doesn't deform the element and so is quite allowable. This is just the uniform translation of an element. The second does deform the element and is therefore not allowable. This corresponds to the $\mathbf{l}_7$ vector of Margolin & Pyun[10] while the vector $\mathbf{l}_8$ is just the v version of this, i.e.,

$$\mathbf{l}_7 = \frac{1}{2}(1,-1,1,-1,0,0,0,0)$$

$$\mathbf{l}_8 = \frac{1}{2}(0,0,0,0,1,-1,1,-1).$$

Another simple example is given by considering the linear triangle. Since the derivatives are constants the choice of quadrature is immaterial; we then arrive at

$$\frac{\partial u}{\partial x} = \frac{1}{2A}\{(u_1-u_3)(y_2-y_3)-(u_2-u_3)(y_1-y_3)\}. \tag{4}$$

(Again, strictly speaking, to obtain the integral we need to multiply by the element area A.) It is easy to see from eq. (4) that for this expression to be zero it is necessary that

$$u_1 = u_3 \quad \text{and} \quad u_2 = u_3$$

$$\Rightarrow \qquad u_1 = u_2 = u_3.$$

This is the quite allowable non-deforming mode of uniform translation and so there are no spurious modes. The argument for $\partial v/\partial y$ is trivially the same.

2 Spurious Modes with other Elements and Quadratures

We will always transform our (irregular) triangle in (x,y) space onto the standard triangle in (p,q) space with nodes at (0,0), (1,0) and (0,1) as shown in figure (2). A transformation from (p,q) space to (x,y) space then needs to be calculated giving $x = x(p,q)$ and $y = y(p,q)$. See Zienkiewicz[16] for example. We note for future reference that this will also give us the form of the solution in (p,q) space, $u = u(p,q)$. From these functions we can then define the matrix

$$\begin{pmatrix} \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} \\ \frac{\partial y}{\partial p} & \frac{\partial y}{\partial q} \end{pmatrix}. \tag{5}$$

Inverting (5) gives the matrix

$$\begin{pmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \end{pmatrix},$$

from which we can then calculate the derivatives of interest, namely

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} \qquad \text{etc..}$$

2.1 Six-Noded Quadratic Triangle

Without loss of generality, to make the algebra simpler, we will take $(x_1, y_1) = (0,0)$. The six-noded quadratic triangular element, see Strang & Fix[13] for example, is transformed onto the standard triangle in (p,q) space, see figure 3, by the transformation

$$\begin{aligned} t &= (2p+2q-1)(p+q-1)t_1 - 4p(p+q-1)t_2 + p(2p-1)t_3 \\ &\quad + 4pqt_4 + q(2q-1)t_5 - 4q(p+q-1)t_6, \end{aligned}$$

where t can represent x, y or u.

2.1.1 Quadratic Triangle and Centroid Quadrature

Putting

$$\begin{aligned} D &= (y_4 - y_6)(4x_5 - 16x_2) + y_5(4x_6 - x_3) + (y_4 - y_2)(16x_6 - 4x_3) - y_3(4x_2 - x_5) \\ &\quad + (y_5 - y_3)(-4x_4) + (y_6 - y_2)(-16x_4) \end{aligned}$$

we can obtain the following formula for the centroid integral of u_x.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \{(u_1 - u_3)(y_5 - 4y_2) + (u_1 - u_5)(4y_6 - y_3) + (u_5 - u_3)(4y_4) \\ &\quad + (u_6 - u_4)(4y_5 - 16y_2) + (u_2 - u_6)(-16y_4) + (u_4 - u_2)(4y_3 - 16y_6)\} / D. \end{aligned}$$

This can be made zero in the following way for example,

$$u_1 = u_3 = u_5$$

$$u_2 = u_4 = u_6$$

and similarly

$$v_1 = v_3 = v_5$$

$$v_2 = v_4 = u_6.$$

As in the previous section, one way of achieving this just leads to uniform translation of the element and so is non-deforming. Two of the spurious modes, in the velocity space

$$(u_1, u_2, u_3, u_4, u_5, u_6, v_1, v_2, v_3, v_4, v_5, v_6),$$

are

$$(1, -1, 1, -1, 1, -1, 0, 0, 0, 0, 0, 0)$$

$$(0, 0, 0, 0, 0, 0, 1, -1, 1, -1, 1, -1).$$

There are two other spurious modes given by :-

$$(u_1, u_2, u_3, u_4, u_5, u_6) = \begin{cases} (-5, \frac{1}{4}, 4, -\frac{5}{4}, 1, 1) \\ (2, -\frac{3}{2}, 4, \frac{1}{2}, -6, -1), \end{cases}$$

with corresponding modes in v.

2.1.2 Quadratic Triangle and Vertex Quadrature

Let

$$
\begin{aligned}
D &= 6\,(-16x_6y_2 + 4x_6y_3 + 16x_2y_6 + 4x_5y_2 - 4x_2y_5 - 4x_3y_6 \\
&\quad + x_3y_5 - x_5y_3)
\end{aligned}
$$

$$
\begin{aligned}
E &= 6\,(16x_4y_6 - 16x_6y_4 + 4x_6y_3 - 4x_3y_6 + 3x_3y_5 - 3x_5y_3 \\
&\quad +12x_6y_5 + 12x_5y_4 - 12x_4y_5 - 12x_5y_6)
\end{aligned}
$$

$$
\begin{aligned}
F &= 6\,(-4x_5y_2 + 4x_2y_5 - 3x_3y_5 + 16x_4y_2 + 3x_5y_3 - 16x_2y_4 \\
&\quad +12x_2y_3 - 12x_4y_3 - 12x_3y_2 + 12x_3y_4)
\end{aligned}
$$

then the quadrature evaluated integral in this case is

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \{\{y_2(12u_1 + 4u_5 - 16u_6) + y_3(-3u_1 - u_5 + 4u_6) \\
&\quad +y_5(-4u_2 + 3u_1 + u_3) + y_6(16u_2 - 12u_1 - 4u_3)\}\,/D \\
&\quad + \{y_3(-u_1 - 3u_5 + 4u_6) + y_4(4u_1 - 16u_6 + 12u_5) \\
&\quad +y_5(-3u_1 + 3u_3 + 12u_6 - 12u_4) + y_6(-4u_3 - 12u_5 + 16u_4)\}\,/E \\
&\quad + \{y_2(-4u_5 - 12u_3 + 16u_4) + y_3(-3u_1 + 3u_5 - 12u_4 + 12u_2) \\
&\quad +y_4(4u_1 - 16u_2 + 12u_3) + y_5(4u_2 - u_1 - 3u_3)\}\,/F\}\,. \qquad (6)
\end{aligned}
$$

From eq. (6) it is quite straightforward to deduce that the only way the derivatives can be set to zero independently of $x_1, y_1, x_2, y_2 \ldots$ etc. is if $u_1 = u_2 = u_3 = u_4 = u_5 = u_6$. This is then just horizontal translation and is therefore quite allowable as it does not deform the triangle. This element/quadrature combination, then, has no spurious modes.

2.1.3 Quadratic Triangle and the Mid-Edge Rule

Let

$$\begin{aligned}
D &= 6(-2x_6y_3 + x_5y_3 + 2x_3y_6 - x_3y_5 + 2x_2y_3 - 2x_4y_3 - 2x_3y_2 + 2x_3y_4) \\
E &= 6(-2x_5y_2 + x_5y_3 + 2x_2y_5 - x_3y_5 - 2x_6y_5 - 2x_5y_4 + 2x_4y_5 + 2x_5y_6) \\
F &= 6(2x_6y_3 + 2x_5y_2 - x_5y_3 - 2x_2y_5 - 2x_3y_6 + x_3y_5 + x_2y_3 - 2x_4y_3 \\
&\quad - 2x_3y_2 + 2x_3y_4 - 2x_6y_5 - 2x_5y_4 + 2x_4y_5 + 2x_5y_6).
\end{aligned}$$

We can then write the integral for $\partial u/\partial x$ in this case as

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \{\{y_4(-2u_1 + 2u_3) + y_5(u_1 - u_3) + y_6(-2u_1 + 2u_3) \\
&\quad + y_3(u_1 + u_5 - 2u_6 - 2u_4 + 2u_2) + y_2(2u_1 - 2u_3)\}/D \\
&\quad + \{2y_2(u_1 - u_5) + y_3(u_5 - u_1) + 2y_4(u_1 - u_5) \\
&\quad + y_5(-u_1 - u_3 + 2u_2 + 2u_4 - 2u_6) + y_6(2u_5 - 2u_1)\}/E \\
&\quad + \{2y_2(u_5 - u_3) + y_3(-u_1 - u_5 + 2u_6 - 2u_4 + 2u_2) + 2y_4(u_3 - u_5) \\
&\quad + y_5(u_1 + u_3 - 2u_2 + 2u_4 - 2u_6) + 2y_6(u_5 - u_3)\}/F\}. \qquad (7)
\end{aligned}$$

It is clear from eq. (7) that to make this derivative vanish we must have $u_1 = u_3 = u_5$. Further inspection reveals that we must also have $u_1 = u_2 = u_3 = u_4 = u_5 = u_6$ and hence there are no spurious modes with this combination either.

2.2 Ten-Noded Cubic Triangle

The transformation of the general 10-noded cubic triangle onto the standard triangle is defined by:

$$\begin{aligned} t \;=\; & -\frac{1}{2}(3p+3q-2)(3p+3q-1)(p+q-1)t_1 \\ & +\frac{9}{2}p(3p+3q-2)(p+q-1)t_2 \\ & -\frac{9}{2}p(3p-1)(p+q-1)t_3+\frac{1}{2}p(3p-1)(3p-2)t_4 \\ & +\frac{9}{2}pq(3p-1)t_5+\frac{9}{2}pq(3q-1)t_6 \\ & +\frac{1}{2}q(3q-1)(3q-2)t_7-\frac{9}{2}q(3q-1)(p+q-1)t_8 \\ & +\frac{9}{2}q(3p+3q-2)(p+q-1)t_9-27pq(p+q-1)t_{10}, \end{aligned}$$

where the numbering of the nodes is anti-clockwise with node 10 being at the centroid of the element. The numbering for this element is shown in figure (4).

2.2.1 Cubic Triangle and Centroid Quadrature

Define

$$
\begin{aligned}
D = {} & 2(-3x_7y_3 + 3x_7y_2 + x_7y_4 - 3x_7y_5 + 3x_7y_9 + 9x_2y_8 \\
& - 9x_3y_6 + 3x_4y_8 - 3x_4y_9 - x_4y_7 - 9x_5y_6 + 9x_5y_2 \\
& - 9x_5y_8 + 9x_5y_8 + 3x_5y_7 + 9x_9y_6 + 9x_9y_8 - 3x_9y_7 + 9x_6y_3 \\
& - 9x_2y_5 + 9x_8y_3 - 9x_8y_2 - 3x_8y_4 + 9x_8y_5 - 9x_8y_9 \\
& - 9x_9y_3 + 3x_9y_4 - 9x_9y_5 + 9x_3y_2 + 9x_2y_6 + 3x_3y_7 \\
& + 9x_3y_9 - 9x_3y_8 - 3x_2y_7 + 3x_4y_6 - 3x_4y_2).
\end{aligned}
$$

Then the integral of the derivative is approximated by

$$
\begin{aligned}
\frac{\partial u}{\partial x} = {} & \{(3y_2 + 3y_9)(-u_7 - 3u_3 - 3u_5 + 3u_6 + u_4 + 3u_8) + (y_4 - 3y_3 - 3y_5 \\
& +3y_6 - y_7 + 3y_9)(-u_7 + 3u_6 + u_1 + 3u_8 - 3u_9 - 3u_2)\} / D \qquad (8)
\end{aligned}
$$

Unfortunately there are many ways of making (8) zero, eg.,

$$u_1 = u_4 = u_7 \begin{cases} u_5 = u_8 = \begin{cases} u_9 \\ u_2 \end{cases} \\ \text{and} \\ u_6 = u_3 = \begin{cases} u_2 \\ u_9 \end{cases} \\ \text{or} \\ u_5 = u_6 = \begin{cases} u_9 \\ u_2 \end{cases} \\ \text{and} \\ u_8 = u_3 = \begin{cases} u_2 \\ u_9 \end{cases} \end{cases}$$

There are so many spurious modes it does not seem worthwhile to list any more except perhaps for one. Since x_{10}, y_{10} and u_{10} make no appearance in these formulae there is a spurious mode given by

$$\begin{aligned} u_1 = u_2 = u_3 = u_4 = u_5 &= u_6 = u_7 = u_8 = u_9 = 0 \\ u_{10} &= 1. \end{aligned}$$

2.2.2 Cubic Triangle with Vertex Quadrature

Define

$$
\begin{aligned}
D &= 6(162x_8y_2 + 18x_8y_4 - 324x_9y_2 + 36x_4y_9 - 18x_4y_8 + 36x_2y_7 \\
&\quad - 162x_3y_9 + 324x_2y_9 - 162x_2y_8 + 4x_4y_7 - 18x_3y_7 + 81x_3y_8 \\
&\quad - 36x_9y_4 + 162x_9y_3 - 36x_7y_2 + 18x_7y_3 - 81x_8y_3)
\end{aligned}
$$

$$
\begin{aligned}
E &= 6(162x_8y_2 + 198x_4y_4 - 81x_9y_2 + 99x_4y_9 - 198x_4y_8 + 99x_2y_7 \\
&\quad - 162x_3y_9 + 81x_2y_9 - 162x_2y_8 + 121x_4y_7 - 198x_3y_7 + 324x_3y_8 \\
&\quad - 99x_9y_4 + 102x_9y_3 - 99x_7y_2 - 121x_7y_4 + 198x_7y_3 - 324x_8y_3)
\end{aligned}
$$

$$
\begin{aligned}
F &= -6(162x_8y_2 - 81x_9y_2 - 162x_3y_9 + 81x_2y_9 - 162x_2y_8 - 4x_4y_7 \\
&\quad - 324x_3y_6 + 162x_3y_5 - 36x_3y_4 + 324x_3y_8 + 162x_9y_3 + 4x_7y_4 \\
&\quad - 324x_8y_3 + 162x_6y_8 - 18x_6y_4 + 18x_2y_4 - 81x_6y_9 - 324x_5y_8 \\
&\quad - 18x_7y_5 - 18x_9y_7 + 36x_5y_4 + 162x_5y_9 - 36x_7y_8 + 36x_7y_6 \\
&\quad + 18x_7y_9 + 162x_2y_6 - 81x_2y_5 - 162x_8y_6 + 324x_8y_5 + 36x_8y_7 \\
&\quad + 324x_6y_3 - 243x_6y_5 - 36x_6y_7 - 162x_6y_2 - 162x_5y_3 + 243x_5y_6 + 18x_5y_7 \\
&\quad + 81x_5y_2 + 36x_4y_3 + 18x_4y_6 - 36x_4y_5 - 18x_4y_2 + 81x_9y_6 - 162x_9y_5).
\end{aligned}
$$

We can now write

$$
\frac{\partial u}{\partial x} = \{y_2(162u_8 - 36u_7 - 324u_9 + 198u_1) + y_3(-81u_8 + 18u_7 + 162u_9 - 99u_1)
$$

$$+ y_4(18u_8 - 4u_7 - 36u_9 + 22u_1) + y_7(-18u_3 - 22u_1 + 4u_4 + 36u_2$$

$$+ y_8(-18u_4 + 81u_3 + 99u_1 - 162u_2) + y_9(-162u_3 - 198u_1 + 36u_4 + 324u_2)\}/D$$

$$+ \{y_2(162u_8 - 99u_7 - 81u_9 + 18u_1) + y_3(-324u_8 + 198u_7 + 162u_9 - 36u_1)$$

$$+ y_4(198u_8 - 121u_7 - 99u_9 + 22u_1)$$

$$+ y_7(-22u_1 + 22u_4 - 198u_8 + 198u_6 - 99u_5 + 99u_9)$$

$$+ y_8(-36u_4 + 36u_1 + 324u_8 - 324u_6 + 162u_5 - 162u_9)$$

$$+ y_9(-18u_1 + 18u_4 - 162u_8 + 162u_6 - 81u_5 + 81u_9)\}/E$$

$$+ \{y_2(-162u_3 + 81u_2 + 99u_4 - 18u_1) + y_3(-198u_4 + 324u_3 - 162u_2 + 36u_1)$$

$$+ y_4(-4u_7 + 4u_1 - 18u_2 + 36u_3 + 18u_6 - 36u_5)$$

$$+ y_5(198u_4 + 54u_1 - 486u_3 + 243u_2 - 81u_6 + 18u_7 + 162u_5)$$

$$+ y_6(-99u_4 + 54u_1 + 486u_3 - 243u_2 + 162u_6 - 36u_7 - 324u_5)$$

$$+ y_7(-35u_3 - 4u_1 + 22u_4 + 18u_2)$$

$$+ y_8(-324u_3 - 36u_1 + 162u_2 - 162u_6 + 36u_7 + 324u_5)$$

$$+ y_9(162u_3 + 18u_1 - 81u_2 - 18u_7 + 81u_6 - 162u_5)\}/F.$$

Firstly we notice that u_{10} does not appear in this derivative and therefore possibly renders the scheme useless. The derivative can be made zero in the following ways. Letting $\mathbf{u} = (u_1, u_2, \ldots, u_{10})$,

$$\mathbf{u} = \begin{cases} (\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ -\ ) \\ (\ \frac{27}{13},\ \frac{27}{13},\ \frac{27}{13},\ \frac{27}{13},\ 1,\ -1,\ -\frac{27}{13},\ -1,\ 1,\ -\ ) \\ (\ \frac{449}{260},\ \frac{449}{260},\ \frac{449}{260},\ \frac{449}{260},\ 1,\ -\frac{7}{20},\ -\frac{14}{13},\ -\frac{7}{20},\ 1,\ -\ ) \\ (\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1\ ). \end{cases}$$

Again only the first mode (in conjunction with the fourth) represents a physically allowable mode.

2.2.3 Cubic Triangle and the Mid-Edge Rule

Define

$$\begin{aligned} D = \; & 6(-18x_8y_4 - 486x_8y_2 + 9x_9y_4 + 243x_9y_2 - 243x_9y_3 + 8x_7y_4 + 216x_7y_2 \\ & - 216x_7y_3 - 36x_2y_4 + 486x_8y_3 - 18x_6y_4 - 486x_6y_2 + 486x_6y_3 - 1458x_{10}y_3 \\ & - 9x_3y_4 + 243x_5y_2 + 54x_{10}y_4 + 1458x_{10}y_2 + 18x_4y_8 + 36x_4y_2 + 18x_4y_6 \\ & + 9x_4y_3 + 9x_5y_4 - 243x_5y_3 - 9x_4y_9 - 8x_4y_7 - 54x_4y_{10} - 9x_4y_5 \\ & + 486x_2y_8 - 243x_2y_9 - 486x_2y_6 - 216x_2y_7 - 1458x_2y_{10} + 1215x_2y_3 - 243x_2y_5 \\ & + 243x_3y_9 + 216x_3y_7 - 1215x_3y_2 - 486x_3y_6 + 1458x_3y_{10} - 486x_3y_8 + 243x_3y_5) \end{aligned}$$

$$\begin{aligned} E = \; & -6(1215x_8y_9 - 216x_8y_4 - 243x_8y_2 + 216x_9y_4 - 1215x_9y_8 + 9x_8y_7 + 36x_9y_7 \\ & - 243x_6y_9 + 243x_6y_8 - 1458x_{10}y_9 + 1458x_{10}y_8 - 9x_6y_7 + 243x_9y_2 - 54x_{10}y_7 \\ & - 486x_9y_3 + 18x_5y_7 + 486x_5y_9 - 486x_5y_8 + 8x_7y_4 + 9x_7y_2 - 18x_7y_3 \\ & + 486x_8y_5 - 1458x_8y_{10} - 243x_8y_6 - 18x_7y_5 + 54x_7y_{10} + 9x_7y_6 - 36x_7y_9 \\ & - 9x_7y_8 - 486x_9y_5 + 486x_8y_3 + 216x_4y_8 - 216x_4y_9 - 8x_4y_7 + 243x_2y_8 \\ & - 243x_2y_9 + 1458x_9y_{10} + 243x_9y_6 - 9x_2y_7 + 486x_3y_9 + 18x_3y_7 - 486x_3y_8) \end{aligned}$$

$$\begin{aligned} F = \; & -6(9x_8y_4 - 18x_9y_4 - 9x_8y_7 + 486x_6y_9 - 243x_6y_8 + 9x_{10}y_7 \\ & - 54x_{10}y_7 + 36x_5y_7 - 486x_5y_9 + 243x_5y_8 + 1215x_6y_5 - 1458x_6y_{10} - 1458x_{10}y_5 \\ & + 1458x_{10}y_6 + x_7y_4 - 1215x_5y_6 + 1458x_5y_{10} - 18x_7y_2 + 9x_7y_3 - 18x_2y_4 \\ & - 243x_8y_5 + 243x_8y_6 - 36x_7y_5 + 54x_7y_{10} - 9x_7y_6 - 18x_7y_9 + 9x_7y_8 \end{aligned}$$

$$+ 486x_9y_5 - 36x_6y_4 + 486x_6y_2 - 243x_6y_3 + 9x_3y_4 - 486x_5y_2 + 54x_{10}y_4$$

$$- 9x_4y_8 + 18x_4y_2 + 36x_4y_6 - 9x_4y_3 - 9x_5y_4 + 243x_5y_3 + 18x_4y_9 - x_4y_7$$

$$- 54x_4y_{10} + 9x_4y_5 - 486x_9y_6 - 486x_2y_6 + 18x_2y_7 + 486x_2y_5 - 9x_3y_7$$

$$+ 243x_3y_6 - 243x_3y_5).$$

We can obtain the derivative in this case as

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \{y_4(-18u_8 + u_1 + 9u_5 - 9u_3 + 54u_{10} - 18u_6 + 36u_2 + 8u_7 + 9u_9) \\
&+ (y_5 + 6y_{10} - 2y_6 + \frac{8}{9}y_7 + y_9 - 2y_8)(9u_1 + 243u_3 - 243u_2 - 9u_4) \\
&+ y_3(-36u_1 + 1215u_2 + 9u_4 - 243u_5 - 1458u_{10} + 486u_6 - 216u_7 - 243u_9 + 486u_8) \\
&+ y_2(-9u_1 - 1215u_3 + 36u_4 + 243u_5 + 1458u_{10} \\
&- 486u_6 + 216u_7 + 243u_9 - 486u_8)\}/D \\
&+ \{(y_2 - 2y_3 + \frac{8}{9}y_4 - 2y_5 + y_6 + 6y_{10})(9u_1 - 9u_7 - 243u_9 + 243u_8) \\
&+ y_7(54u_{10} + u_1 - 18u_3 + 9u_2 + 8u_4 - 9u_8 + 9u_6 - 18u_5 - 36u_9) \\
&+ y_8(-36u_1 + 486u_3 - 243u_2 - 216u_4 + 486u_5 + 9u_7 - 1458u_{10} - 243u_6 + 1215u_9) \\
&+ y_9(-486u_5 - 9u_1 - 486u_3 + 243u_2 + 216u_4 \\
&+ 36u_7 + 1458u_{10} - 1215u_8 + 243u_6)\}/E \\
&+ \{(-2y_2 + y_3y_8 - 2y_9 + 6y_{10})(9u_4 - 243u_5 - 9u_7 + 243u_6) \\
&+ y_4(-9u_8 - 8u_1 + 9u_5 - 9u_3 - 54u_{10} + 36u_6 + 18u_2 - u_7 + 18u_9) \\
&+ y_5(243u_8 + 216u_1 + 243u_3 - 486u_2 - 9u_4 - 486u_9 - 1215u_6 + 36u_7 + 1458u_{10}) \\
&+ y_6(-243u_8 - 216u_1 - 243u_3 + 486u_2 - 36u_4 + 486u_9 + 9u_7 - 1458u_{10} + 1215u_5) \\
&+ y_7(54u_{10} + 8u_1 + 9u_3 - 18u_2 + u_4 + 9u_8 - 9u_6 - 36u_5 - 18u_9)\}/F.
\end{aligned}$$

Firstly we notice that u_{10} does take an active part in this expression. The derivative can be zero only when:-

$$\mathbf{u} = \begin{cases} (\ 1\ ,\ 1\ ,\ 1\ ,\ 1\ ,\ 1\ ,\ 1\ ,\ 1\ ,\ 1\ ,\ 1\ ,\ 1\ ) \\ (\ 9\ ,\ 9\ ,\ 9\ ,\ 9\ ,\ -1\ ,\ 1\ ,\ 63\ ,\ 1\ ,\ -1\ ,\ -1\ ) \\ (\ 15\ ,\ -7\ ,\ -\frac{37}{5}\ ,\ \frac{21}{5}\ ,\ \frac{13}{5}\ ,\ 1\ ,\ -39\ ,\ -1\ ,\ 1\ ,\ -1\ ) \\ (\ -1\ ,\ 1\ ,\ \frac{3}{5}\ ,\ -\frac{59}{5}\ ,\ \frac{3}{5}\ ,\ 1\ ,\ -1\ ,\ -1\ ,\ -1\ ,\ 1\ ). \end{cases}$$

Again only the first mode is physical.

2.3 Ten-Noded Quadratic Tetrahedron

The quadratic triangle seems to be the most promising two-dimensional element and so we consider here, briefly, the three-dimensional equivalent. The local element numbering is shown in figure (5). The transformation is given by:-

$$\begin{aligned} t &= (2p+2q+2r-1)(p+q+r-1)t_1 + p(2p-1)t_2 + q(2q-1)t_3 + r(2r-1)t_4 \\ &\quad - 4p(p+q+r-1)t_5 + 4pqt_6 - 4q(p+q+r-1)t_7 - 4r(p+q+r-1)t_8 \\ &\quad + 4qrt_9 + 4prt_{10}. \end{aligned}$$

The algebra for this problem is even worse than that for two-dimensions so we shall just quote the results. They are all obtained in exactly the same way as the previous results.

<u>Centroid</u>:- The conditions for a zero derivative are

$$\begin{aligned} u_5 &= u_9 \\ u_6 &= u_8 \\ u_7 &= u_{10} \end{aligned}$$

There are 10 variables, 3 constraints plus one physical mode leaving 6 spurious modes. Including the y-velocity and z-velocities as well means that 18 out of 30 modes are spurious.

Vertex and Mid-Face Rule:- Both of these quadratures just have the one zero derivative mode given by $u_1 = u_2 = \ldots = u_{10}$.

3 Implementation

This theory is all well and good but unless it leads to an efficient scheme it is going to be of little use. In this section we consider the possibilities of using these elements in practice. The 10-noded cubic triangle would seem to be of little value, although perhaps a 9-noded reduced cubic (without the interior node) might be of more interest. This leaves the quadratic triangle. With centroid integration, the most efficient, 3 out of 6 of the modes are spurious. These could be damped after performing an analysis similar to that of Flanagan & Belytschko[6] perhaps. If these modes were damped too excessively though, the elements would be prone to locking as they only have the three non-spurious degrees of freedom, which is no better than the linear triangle. It would therefore seem more logical to use the quadratic triangle/vertex or mid-edge rule combination, since this rules out both locking and any spurious modes despite the extra expense involved. It should be noted that eqs. (6 ,7) have not been optimised for computation and the operation count can be reduced by factoring certain terms.

In many Lagrangian fluid dynamics codes mass lumping is used to avoid inverting the mass matrix, see Donéa et al[4] for example. However, with quadratic elements we do not have this option. Consider

$$\mathbf{M}\dot{\mathbf{u}} = \mathbf{b}, \tag{9}$$

where $\mathbf{M}$ is the mass matrix whose elements are given by

$$\mathbf{M}_{i,j} = \int_{\Omega} \rho \psi_i(\mathbf{x}) \psi_j(\mathbf{x}) d\Omega. \tag{10}$$

In (9), $\mathbf{b}$ is a vector representing the forces acting on an element and $\dot{\mathbf{u}}$ denotes the total nodal acceleration vector. In eq. (10) Ω is simply the domain we are integrating over, ψ_j are the element basis functions (piecewise linear, quadratic etc.) and ρ is the density. Mass lumping is achieved by replacing the matrix $\mathbf{M}$ by the diagonal matrix $\mathbf{M}^L$ whose elements are

$$\mathbf{M}^L_{i,j} = 0 \qquad i \neq j$$

$$\mathbf{M}^L_{i,i} = \sum_j \mathbf{M}_{i,j}.$$

We can write down a local mass matrix which involves the same calculation as the $\mathbf{M}$ in eq. (10) but with an integration domain consisting of just one element, E. For a regular quadratic triangular element this matrix is given by,

$$\mathbf{M_E} = \frac{\rho_E area(\triangle_E)}{180} \begin{pmatrix} 6 & -1 & -1 & 0 & -4 & 0 \\ -1 & 6 & -1 & 0 & 0 & -4 \\ -1 & -1 & 6 & -4 & 0 & 0 \\ 0 & 0 & -4 & 32 & 16 & 16 \\ -4 & 0 & 0 & 16 & 32 & 16 \\ 0 & -4 & 0 & 16 & 16 & 32 \end{pmatrix} \tag{11}$$

where $area(\triangle_E)$ is just the area of element E and ρ_E is the element density. We now note that if we lump the mass matrix of eq. (11) we get zeroes on the first three diagonals. (In practice it was found that when the global mass matrix was assembled and then lumped, this rarely led to zeroes but did give very small diagonal elements which then led to immense particle accelerations caused solely by rounding error. Although the centroid integrated quadratic, without damping, failed before the vertex integrated quadratic neither gave any meaningful results.) This therefore means that we are faced with the extra cost of inverting the full mass matrix.

Fortunately the mass matrix, $\mathbf{M}$, can be inverted accurately and cheaply. In Wathen[15] it was shown that with diagonal pre-conditioning the eigenvalues of $\mathbf{M}$ are all contained in the region $[0.3924, 2.0598]$ for the quadratic elements on triangles to be used here. Similar results hold for other types of element. This then implies that the pre-conditioned conjugate gradient method, see Golub & Van Loan[7] for example, is particularly efficient in solving eq. (9) in the sense that very few iterations are needed. If a vector machine is available each iteration can also be made extremely fast, see[1, 7, 14].

Having inverted the full mass matrix we have a more accurate scheme than if mass lumping had been used, since lumping the mass matrix reduces the order of the scheme. This means that if increased accuracy is not the goal then by using a more accurate scheme we can use fewer nodes to achieve the same degree of accuracy. This should go a very long way towards recouping the cost of inverting the full mass matrix. In this sense it would seem illogical to go to all the trouble of using quadratic elements, as opposed to linear elements, only to then lump the mass matrix and lose much of the accuracy gained.

4 Summary

In this paper we have extended the analysis of Margolin & Pyun[10] to higher order triangular elements. The 10-noded cubic element was very susceptible to hourglass modes and it seems that this is largely due to the presence of an interior node. The quadratic triangular element with centroid quadrature has three spurious modes which have been identified and could then be damped. Alternatively the vertex or mid-edge rule could be used which possess no spurious modes. Although these elements require more computational effort, especially as mass lumping cannot be used, this is made up for by the fact that the scheme is much more accurate and there are no problems with hourglassing or stiffness.

5 Acknowledgments

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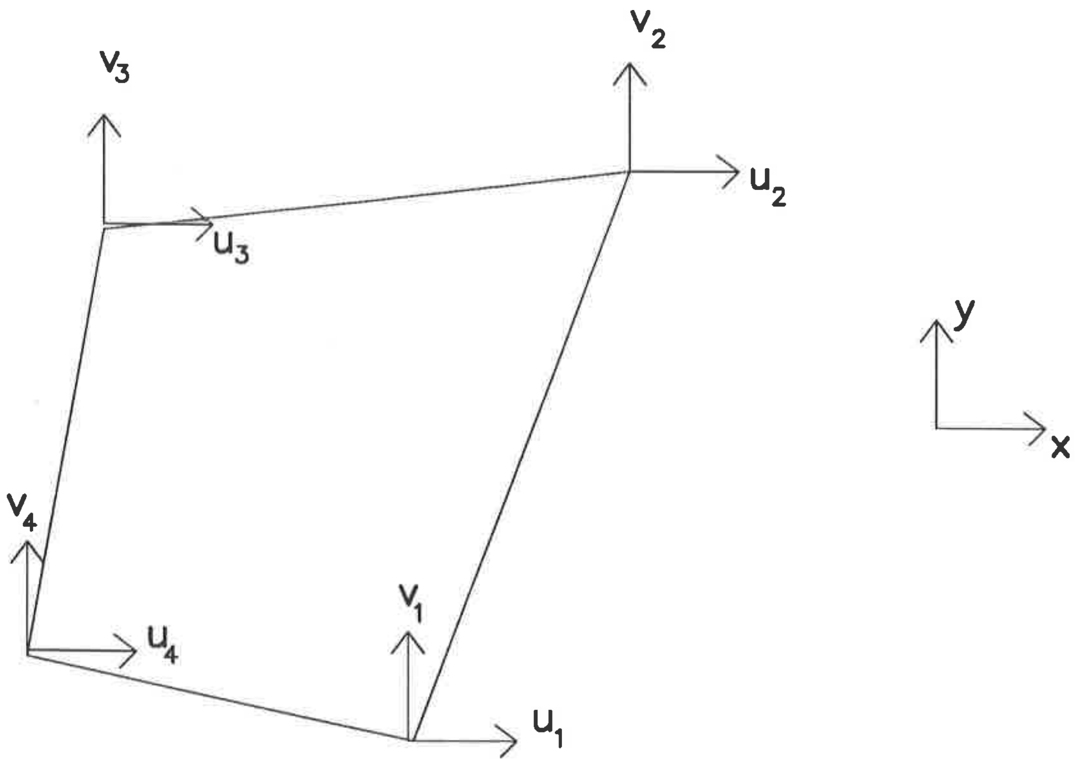

Figure 1: Velocity components on the bilinear quadrilateral element.

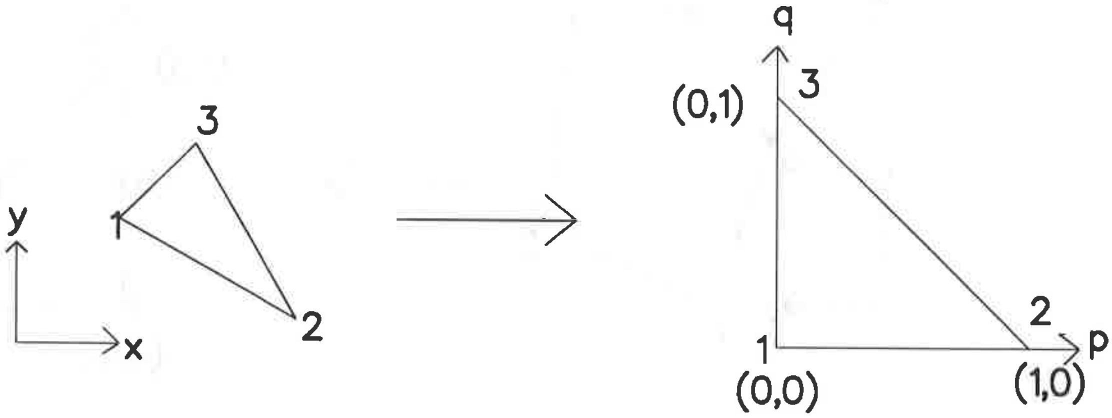

Figure 2: Transformation of a general triangle onto the standard element.

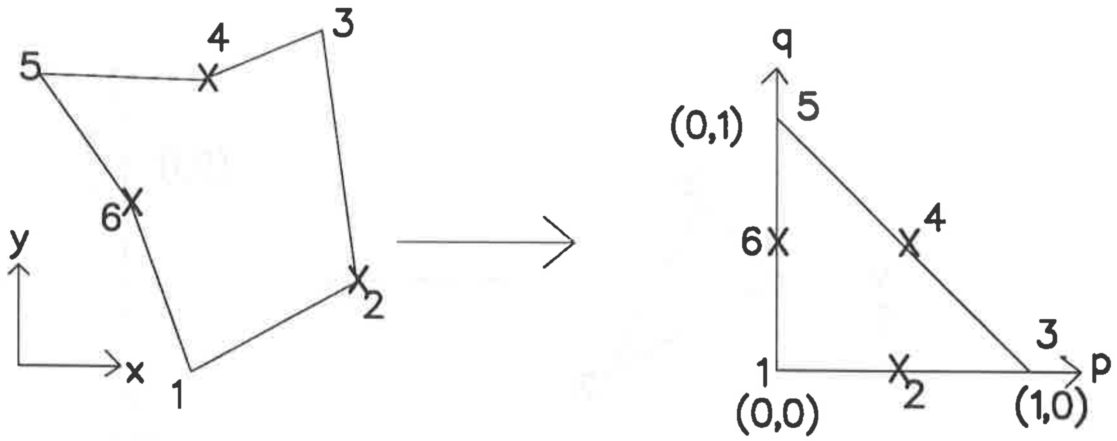

Figure 3: Transformation of the general quadratic triangle onto the standard element.

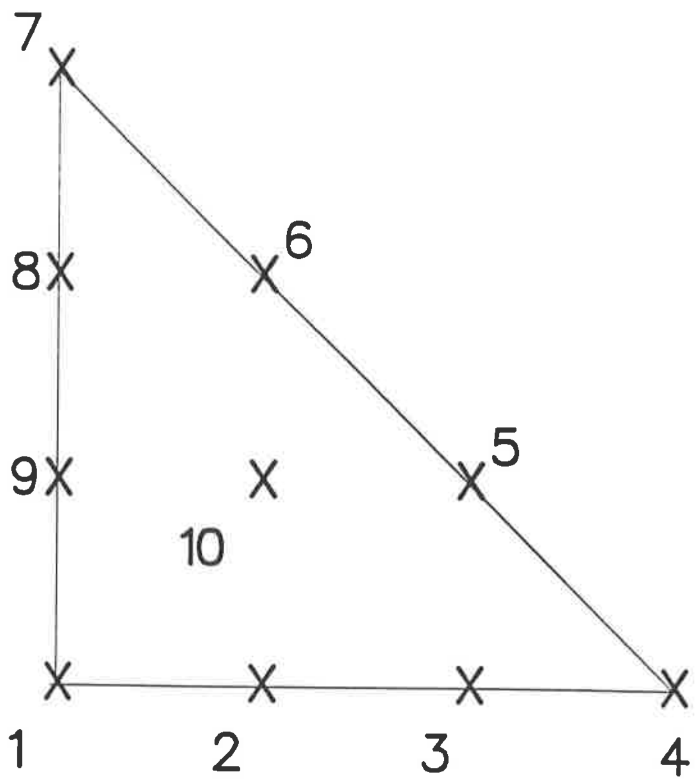

Figure 4: Local element numbering for the 10-noded cubic triangle.

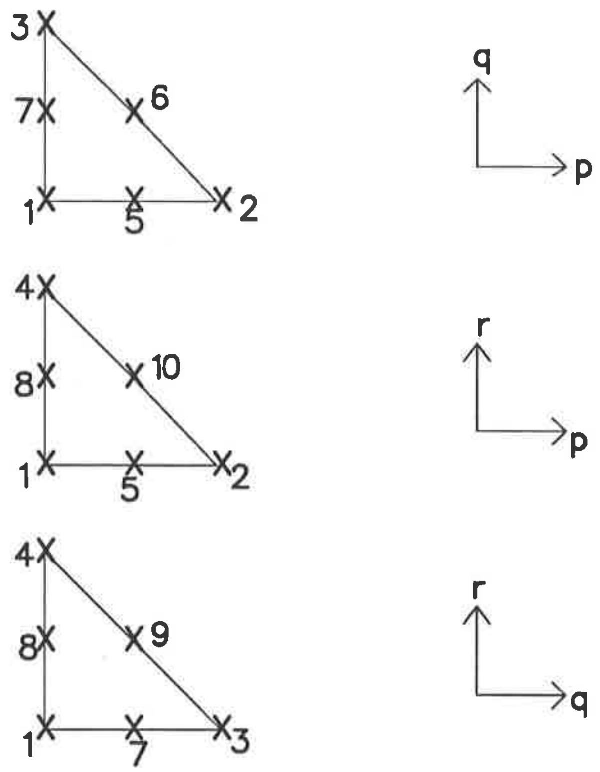

Figure 5: Local element numbering for the 10-noded tetrahedron.