ON THE SCHEMES OF JAMESON, DAVIS & ROE

M.G. EDWARDS

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Dept. of Mathematics, University of Reading

ABSTRACT

It is shown that, for the linear advection equation, the scheme of Jameson is almost identical to a member of a class of centrally based schemes recently introduced by Davis. Equivalence is shown when the Davis scheme is modified by halving the diffusion. By comparing the diffusion in the Davis (and a modified Davis) schemes with that in Roe's scheme, relationships are found between these schemes and that of Jameson.

1. INTRODUCTION

Recently Davis [1] proposed a class of centrally based schemes which are total variation decreasing (TVD) in the sense of Harten [2]. Such schemes extend the upwind schemes of Roe [3] and Sweby [4].

It is well known that the most successful scheme based on central differencing is that of Jameson [5] which produces results that other schemes would like to match.

The question arises as to whether there is a connection between the schemes of Jameson, Davis and Roe. In this report we show that, for the model linear advection equation, there is indeed a very close relationship, namely that in rough terms Jameson's scheme is a (slightly modified) Davis scheme. From this viewpoint it is then possible to show the connections between the three schemes of Jameson, Davis and Roe.

In §2 Jameson's scheme for the linear scalar wave equation is reviewed and its relationship with the Lax-Wendroff scheme is established, giving jusitification for some of Jameson's empirical rules. In §3 TVD schemes are defined and the schemes of Davis and Roe introduced. A strong connection between the diffusion in the schemes of Davis and Jameson is established here which is followed up in §4. Finally in §5 and §6 a comparison is made of the diffusion in the schemes of Davis and Roe both in the cases of smooth and shocked flow, which leads to relationships between all three schemes, and also to a modified Davis scheme which may be identified closely with the schemes of both Jameson and Roe.

2. JAMESON'S SCHEME

2.1 In reference [1] Jameson introduces Runge-Kutta (R-K) time-stepping to iterate the Euler equations to a steady state. He also presents a stability analysis of the scheme based on the R-K stability polynomial for the model advection diffusion equation. We apply the analysis here to the model advection equation.

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \qquad (2.1)$$

As the Jameson's numerical method uses central spatial differencing, (2.1) is discretized in space as

$$\frac{\partial u_i}{\partial t} + a \frac{(u_{i+1} - u_{i-1})}{2\Delta x} = 0 \qquad (2.2)$$

Let the Fourier transform of $u(x,t)$ be $\hat{u}(p,t)$. Taking the Fourier transform of (2.2) gives

$$\frac{d\hat{u}}{dt} + \frac{ai\sin\xi}{\Delta x}\ \hat{u} = 0 \qquad (2.3)$$

where $\xi = p\Delta x$.

Now the R-K stability polynomial for a 4th order method applied to

$$\frac{dy}{dt} = \lambda y \qquad \text{is}$$

$$p_4 = 1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{3!} + \frac{(\lambda h)^4}{4!} \qquad (2.4)$$

where $h = \Delta t$, using (2.3), (2.4) with $\lambda = -\frac{ai\sin\xi}{\Delta x}$.

Let the Courant number $\nu = \frac{a\Delta t}{\Delta x}$, so that $\lambda h = -i\nu\sin\xi$. Then

$$p_4 = 1 - i\nu\sin\xi - \frac{\nu^2(\sin\xi)^2}{2} + \frac{i(\nu\sin\xi)^3}{3!} + \frac{(\nu\sin\xi)^4}{4!} \qquad (2.5)$$

and $|p_4| = 1$ for $\nu\sin\xi = 2\sqrt{2}$. Moreover it can be shown that $\nu_{max} \leq 2\sqrt{2}$.

The Courant number ν therefore has a stability bound of $2\sqrt{2}$ for this scheme as given by Jameson in [5].

In Jameson's method for the Euler equations the scheme is augmented by the addition of artificial dissipative terms. By numerical experiment he selects a combination of second and fourth differences. The dissipation operator is defined by Jameson as follows;

$$\left.\begin{aligned} Dw &= d_{i+\frac{1}{2}} - d_{i-\frac{1}{2}} \\ d_{i+\frac{1}{2}} &= \frac{\Delta x}{\Delta t}\left[\varepsilon^1_{i+\frac{1}{2}}\,(u_{i+1} - u_i) - \varepsilon^2_{i+\frac{1}{2}}\,(u_{i+2} - 3u_{i+1} + 3u_i - u_{i-1})\right] \\ \varepsilon^1_{i+\frac{1}{2}} &= \max\,(\sigma_{i+1},\sigma_i)/4 \\ \varepsilon^2_{i+\frac{1}{2}} &= \max\,(0, \frac{1}{256} - \varepsilon_1) \\ \sigma_i &= \frac{|p_{i+1} - 2p_i + p_{i-1}|}{|p_{i+1} + 2p_i + p_{i-1}|} \end{aligned}\right\} \quad (2.6)$$

where p is the pressure and is always a positive physical quantity.

The dissipation is added to the differential equation so that

$$\frac{d(\Delta x w)}{dt} + Qw = Dw \ ,$$

where Q is the spatial operator for the Euler equations (see [1]).

From his computed results Jameson makes the following observations:-

1. In smooth regions of the flow the scheme is not sufficiently dissipative unless fourth differences are included.
2. Near shock waves it is found that the fourth differences tend to induce overshoots and are therefore switched off.

3. Dissipation is required to suppress the tendency for odd and even decoupling.

4. Although our aim is to try to see if there is a connection between centrally differenced TVD schemes and the Jameson scheme, it proves instructive to give first an alternative analysis of the method which sheds light on some of these observations.

2.2 Alternative Analysis of the R-K Method

We present here an alternative approach to analysing the application of R-K to the model advection equation, concentrating specifically on second order accuracy. Most schemes concentrate on this aspect, including the centrally based TVD scheme.

We start by looking at (2.4) and noting that second order R-K schemes are absolutely unstable. For the second order R-K polynomial is

$$p_2 = 1 - \nu i \sin\xi - \frac{(\nu \sin\xi)^2}{2}$$

and $|p_2|^2 = 1 + \frac{(\nu \sin\xi)^4}{4} > 1$, i.e. the scheme is unconditionally unstable. This is well known. Now let us look at the actual difference scheme produced by the second order R-K method.

Equation (2.1) is spatially discretized as

$$\frac{\partial u}{\partial t} = - a \frac{(u_{i+1} - u_{i-1})}{2\Delta x} = - \frac{a\Delta_0 u}{2\Delta x} = f(u) \quad ,$$

so that the application of second order R-K to $\frac{\partial u}{\partial t} = f(u)$ gives

$$u^{n+1} - u^n = \frac{\Delta t}{2} (f(u^n) + f(u^n + \Delta t f(u^n)))$$

where $$f(u^n) = - \frac{a\Delta_0 u^n}{2\Delta x} \quad .$$

Since $f(u^n + \Delta t f(u^n)) = -\frac{a\Delta_0 u^n}{2\Delta x} + \frac{a^2 \Delta t \Delta_0^2 u^n}{4\Delta x^2}$ the second order R-K method generates the difference scheme

$$u^{n+1} - u^n = -\frac{\nu\Delta_0 u^n}{2} + \frac{\nu^2 \Delta_0^2 u^n}{8}$$

$$u^{n+1} = u^n - \frac{\nu(u_{i+1} - u_{i-1})}{2} + \frac{\nu^2}{8}(u^n_{i+2} - 2u^n_i + u^n_{i-2}) \qquad (2.7)$$

We note that the diffusion term is decoupled and can therefore give rise to odd and even point decoupling. The stencil for the diffusion term is

1		-2		1
⊡	•	⊡	•	⊡
i-2	i-1	i	i+1	i+2

So far we have an unstable scheme. In fact Jameson goes to third and then fourth order accuracy in time before he achieves a stable scheme but the resulting Courant number is larger than the normal maximum of 1 produced by many second order schemes.

However, a simple and obvious way to stabilise this second order R-K scheme, at least for Courant numbers < 1, is to replace the uncoupled diffusion term in (2.7) with the more familiar diffusion term

$$\frac{\nu^2}{2}(u^n_{i+1} - 2u^n_i + u^n_{i-1})$$

from the Lax-Wendroff (LW) scheme. In practical terms this amounts to adding a term, τ_d say, to (2.7) where

$$\tau_d = \frac{\nu^2}{2}(u_{i+1} - 2u_i + u_{i-1}) - \frac{\nu^2}{8}(u_{i+2} - 2u_i + u_{i-2}) \qquad (2.8)$$

$$= \frac{-\nu^2}{8}(u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2})$$

$$= \frac{-\nu^2}{8} u_{xxxx}(\Delta x)^4 + O(\Delta x)^6$$

So we see that τ_d is precisely a fourth difference dissipative term.

We are now in a position to give at least some explanation of the peculiarities observed by Jameson and stated in §2.1.

1. The addition of the fourth order dissipative term to the second order R-K difference scheme has the effect of stabilising an unstable scheme for Courant number $\nu < 1$. Since the term is dissipative its effect on any scheme for solving the wave equation will in general be a stabilising one.

2. However at discontinuities LW type schemes are notorious for giving overshoots in solutions. It is therefore perhaps not surprising that this term has to be 'switched off' at shocks. The switch can clearly be seen in (2.5), where at a shock

$$\varepsilon^1_{i+\frac{1}{2}} \sim O(\tfrac{1}{4}) \quad \text{and} \quad \varepsilon^2_{i+\frac{1}{2}} = 0$$

3. By inspection the addition of the fourth difference term returns the scheme to the normal 3 point centred coupled scheme.

Although these remarks are consistent with the previous observations we still have no clear understanding of the shock capturing property of the scheme. In what follows it is assumed that in the Jameson scheme

(a) the third and fourth order time derivative terms in the R-K scheme serve only to enlarge the Courant number.

(b) the third and fourth order terms have no special shock capturing properties.

3. TOTAL VARIATION DECREASING (TVD) SCHEMES

3.1 First we define TVD and its significance for numerical schemes, and then give a general form for the Roe upwind TVD scheme and the more recent centrally based scheme of Davis.

Consider the initial value problem for a scalar conservation law.

$$u_t + f(u)_x = u_t + a(u)u_x = 0 \qquad t > 0$$

where

$$a(u) = \frac{df(u)}{du} \tag{3.1}$$

and

$$u(x,0) = u_0(x) \qquad -\infty < x < \infty$$

and where $u_0(x)$ is assumed to be of bounded total variation. A weak solution of the scalar initial value problem (3.1) has the following monotonicity property as a function of t (see ref. 2):

i) No new local extrema in x may be created

ii) The value of a local minimum is non-decreasing, the value of a local maximum is non-increasing.

The total variation of the solution to (3.1) is defined by

$$Tv(u(x,t)) = \sup \sum_k |u(x_{k+1},t) - u(x_k,t)|$$

where the supremum is taken overall partitions of the real line.
It follows from the above monotonicity property that the total variation in x, denoted by $Tv(u(x,t))$ is non-increasing.

i.e.

$$Tv(u(t_2)) \leqq Tv(u(t_1)) \qquad \forall t_2 \geqq t_1$$

Consider now explicit finite difference schemes in conservation form which approximate (3.1) and denote them by

$$u^{n+1} = L \cdot u^n$$

$$u^{n+1} = L\, u^n \tag{3.2}$$

A scheme of the form (3.2) is called Total Variation Diminishing if

$$TV(u^{n+1}) = TV(L \cdot u^n) \leq TV(u^n)$$

We now state a theorem and a lemma due to Harten [2]. (See also Lax-Jameson [6], Sweby [4]).

Theorem: A total variation diminishing scheme is monotonicity preserving.

This is one reason why TVD schemes are important since they cannot generate spurious oscillations. Now write the scheme (3.2) in the form

$$u_i^{n+1} = u_i^n - C_{i-\frac{1}{2}} \Delta u_{i-\frac{1}{2}}^n + D_{i+\frac{1}{2}}\, \Delta u_{i+\frac{1}{2}}^n$$

where

$$\Delta u_{i+\frac{1}{2}} = u_{i+1} - u_i$$

and

$$C_{i-\frac{1}{2}},\ D_{i+\frac{1}{2}} \text{ are functions of } u^n.$$

Lemma: If the coefficients of C and D of equation (3.3) satisfy the inequalities

$$C_{i+\frac{1}{2}} \geq 0$$

$$D_{i+\frac{1}{2}} \geq 0$$

$$0 \leq C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} \leq 1$$

then the scheme is TVD.

In [3] Roe presents a scheme of the form

$$u_i^{n+1} = u_i^n - \nu[1 - \tfrac{1}{2}(1-\nu)\phi_{i-\frac{1}{2}} + \tfrac{1}{2}(1-\nu)\phi_{i+\frac{1}{2}}/r_i^+]\Delta u_{i-\frac{1}{2}} \tag{3.5}$$

(using the notation of Davis [1]), where $\phi_{i\pm\frac{1}{2}}$ are flux limiters which are functions of the ratios

$$r_i^+ = \Delta u_{i-\frac{1}{2}}/\Delta u_{i+\frac{1}{2}}$$

and

$$r_i^- = \Delta u_{i+\frac{1}{2}}/\Delta u_{i-\frac{1}{2}} \qquad (3.5a)$$

(see also Sweby [4]).

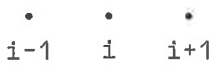
Application of (3.4) then reveals conditions for (3.5) to be TVD. Here $D = 0$ so that (3.4) reduces to $0 \leq C_{i+\frac{1}{2}} \leq 1$. For $0 \leq \nu \leq 1$ this is always satisfied

$$\text{if} \quad \begin{cases} 0 \leq \phi_{i-\frac{1}{2}} \leq 2 \\ 0 \leq \phi_{i+\frac{1}{2}}/r_i^+ \leq 2 \end{cases}$$

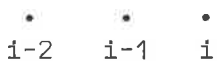
Then (3.5) is certainly TVD. For $\phi(r) = 1$ (3.5) reduces to the LW scheme and for $\phi(r) = r$ becomes the second order upwind Beam & Warming method.

The difference stencil for the Roe/Sweby scheme for $\nu > 0$ is

being the union of the stencil

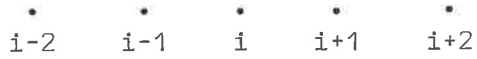


for the LW stencil and



for Beam-Warming. Similarly for $\nu < 0$ the stencil is

3.2 Recently Davis has introduced a centrally differenced based TVD scheme using the stencil of points



The Davis scheme is

$$u_i^{n+1} - u_i^n = -\frac{\nu}{2}(u_{i+1}^n - u_{i-1}^n) + \frac{\nu^2}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$
$$+ (k^+_{i+\frac{1}{2}}(r_i^+) + k^-_{i+\frac{1}{2}}(r^-_{i+1}))(u_{i+1}^n - u_i^n) \qquad (3.6)$$
$$- (k^+_{i-\frac{1}{2}}(r^+_{i-1}) + k^-_{i-\frac{1}{2}}(r_i^-))(u_i^n - u_{i-1}^n)$$

The Roe/Sweby scheme (3.5) can be identified in this form with

$$k^+_{i+\frac{1}{2}} = \begin{cases} \frac{\nu(1-\nu)}{2}(1 - \phi(r_i^+)) & a > 0 \\ 0 & a \leq 0 \end{cases}$$

$$k^-_{i+\frac{1}{2}} = \begin{cases} 0 & a \geq 0 \\ \frac{\nu(1+\nu)}{2}(\phi(r^-_{i+1}) - 1) & a < 0 \end{cases}$$

where the k's can be regarded as diffusion coefficients. Davis defines the particular forms

$$k^+_{i+\frac{1}{2}} = \frac{|\nu|}{2}(1 - |\nu|)(1 - \phi(r_i^+))$$

$$k^-_{i+\frac{1}{2}} = \frac{|\nu|}{2}(1 - |\nu|)(1 - \phi(r^-_{i+1}))$$

where r_i^+ and r_i^- have been defined earlier (3.5a) so that the Davis diffusion coefficient is

$$k^+_{i+\frac{1}{2}} + k^-_{i+\frac{1}{2}} = \frac{|\nu|}{2}(1 - |\nu|)(2 - \{\phi(r_i^+) + \phi(r^-_{i+1})\}) \qquad (3.7)$$

Note that the Davis diffusion always contains an upwind and a downwind

contribution, whereas the Roe/Sweby diffusion is always upwind.

3.3 We are now in a position to pursue the aim set down at the end of Section 2, that is to attempt to identify the shock capturing property of the Jameson scheme. We do this by relating it to the scheme of Davis. We continue the analysis from the second order R-K scheme. From equations (2.7), (2.8) we have the following symbolic relationship.

$$\text{2nd order R-K} = LW - \tau_d \tag{3.8a}$$

Note that the difference stencil of 2nd order R-K is the same as that of the Davis scheme and that the right hand side of (3.6) can be written

$$LW + k_{i+\frac{1}{2}}\Delta u_{i+\frac{1}{2}} - k_{i-\frac{1}{2}}\Delta u_{i-\frac{1}{2}} \tag{3.8b}$$

where

$$k_{i+\frac{1}{2}} = k^+_{i+\frac{1}{2}} + k^-_{i+\frac{1}{2}}$$

compare (3.8b) with the r.h.s. of (3.8a). The aim is now to rearrange the terms of $-\tau_d$ into an expression of the form $k_{i+\frac{1}{2}}\Delta u_{i+\frac{1}{2}} - k_{i-\frac{1}{2}}\Delta u_{i-\frac{1}{2}}$

Now

$$-\tau_d = \frac{\nu^2}{8}(u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}) \tag{3.9}$$

$$= \frac{\nu^2}{8}((u_{i+2}-u_{i+1}) + 3(u_i-u_{i+1}) + 3(u_i-u_{i-1}) + (u_{i-2}-u_{i-1}))$$

$$= \frac{\nu^2}{8}((u_{i+1}-u_i)(r^-_{i+1}-3) - (u_i-u_{i-1})(r^+_{i-1}-3))$$

Where $r^-_{i+1} = \Delta u_{i+\frac{3}{2}}/\Delta u_{i+\frac{1}{2}}$, $r^+_{i-1} = \Delta u_{i-\frac{3}{2}}/\Delta u_{i-\frac{1}{2}}$

$$= -\frac{\nu^2}{2}\left\{\left(\left(1 - \frac{r^-_{i+1}+1}{4}\right)\right)\Delta u_{i+\frac{1}{2}} - \left(1 - \frac{r^+_{i-1}+1}{4}\right)\right)\Delta u_{i-\frac{1}{2}}\right\}$$

$$= (\psi^-_{i+\frac{1}{2}}\Delta u_{i+\frac{1}{2}} - \psi^+_{i-\frac{1}{2}}\Delta u_{i-\frac{1}{2}}) \tag{3.10}$$

say where $\psi^-_{i+\frac{1}{2}} = -\frac{\nu^2}{2}\left(1 - \frac{(r^-_{i+1}+1)}{4}\right)$.

Recall that our aim is to derive a diffusion coefficient of the form

$$K_{i+\frac{1}{2}} = |\nu|\frac{(1-|\nu|)}{2}\left(2 - (\phi(r^+_i) + \phi(r^-_{i+1}))\right) \tag{3.11}$$

At this stage there is a certain freedom of choice as to how to identify $\psi^-_{i+\frac{1}{2}}$, $\psi^+_{i-\frac{1}{2}}$. Although we could add terms which would correspond to higher diffusion and first order accuracy, we instead concentrate on second order accuracy as a guide so that the resulting choice of K will apply in smooth regions of the flow.

We therefore restrict the choice of $\psi^-_{i+\frac{1}{2}}$, $\psi^+_{i-\frac{1}{2}}$ to use the flux limiters ϕ corresponding to the imposition of second order accuracy.

For second order accuracy it may be shown that

$$\phi(r) = 1 - \beta + \beta r$$

where β is an arbitrary constant. This is derived by taking a linear combination of LW and Beam-Warming such that the weights sum to one.

We now rewrite (3.11) as

$$K_{i+\frac{1}{2}} = |\nu|\frac{(1-|\nu|)}{2}\,2\beta\left(1 - \frac{(r^+_i + r^-_{i+1})}{2}\right) \tag{3.12}$$

Comparing (3.10), (3.12) we want to match the coefficients of $-\frac{\nu^2}{2}$.
Add and subtract terms $(A + Br^+_i)\left(+\frac{\nu^2}{2}\right)$ to $\psi^-_{i+\frac{1}{2}}$ and let

$$-\frac{\nu^2}{2}\left(1 - \left(A + \frac{1}{4} + Br^+_i + \frac{r^-_{i+1}}{4}\right)\right) = -\frac{\nu^2}{2}\left(2\beta\left(1 - \frac{r^+_i + r^-_{i+1}}{2}\right)\right).$$

From equating coefficients of r^-_{i+1}, r^+_i and constants respectively, $\beta = \frac{1}{4}$, $B = \frac{1}{4}$, $A = \frac{1}{4}$.

Similarly add and subtract $(A + Br^-_i)\left(+\frac{\nu^2}{2}\right)$ to $\psi^+_{i-\frac{1}{2}}$. Now expand (3.10) with these terms included then we obtain

$$-\frac{\nu^2}{2}\left[\left(1 - \frac{(2+r^+_i + r^-_{i+1})}{4}\right)\Delta u_{i+\frac{1}{2}} - \left(1 - \frac{(2 + r^+_{i-1} + r^-_i)}{4}\right)\Delta u_{i-\frac{1}{2}}\right] \tag{3.13}$$

$$-\frac{\nu^2}{2}\left[\left(\frac{1 + r^+_i}{4}\right)\Delta u_{i+\frac{1}{2}} - \left(\frac{1 + r^-_i}{4}\right)\Delta u_{i-\frac{1}{2}}\right]$$

the last term is identically zero using the definitions of r_i^+, r_i^-.

Since β is now defined we can write (3.13) in the form

$$-\frac{|\nu|^2}{2}\left((2-(\phi(r_{i+1}^-)+\phi(r_i^+)))\Delta u_{i+\frac{1}{2}}\right.$$
$$\left.-(2-(\phi(r_i^-)+\phi(r_{i-1}^+)))\Delta u_{i-\frac{1}{2}}\right) \qquad (3.14)$$

where

$$\phi(r)=\frac{(3+r)}{4}\ .$$

Comparison of (3.14) and (3.8b) shows that we must add and subtract to (3.14) a term

$$\frac{|\nu|}{2}\left(\alpha_{i+\frac{1}{2}}\Delta u_{i+\frac{1}{2}}-\alpha_{i-\frac{1}{2}}\Delta u_{i-\frac{1}{2}}\right)$$

where

$$\alpha_{i+\frac{1}{2}}=2-(\phi(r_i^+)+\phi(r_{i+1})) \qquad (3.14a)$$

giving

$$-\tau_d=|\nu|\frac{(1-|\nu|)}{2}\left(\alpha_{i+\frac{1}{2}}\Delta u_{i+\frac{1}{2}}-\alpha_{i-\frac{1}{2}}\Delta u_{i-\frac{1}{2}}\right) \qquad (3.15)$$

$$-\frac{|\nu|}{2}\left(\alpha_{i+\frac{1}{2}}\Delta u_{i+\frac{1}{2}}-\alpha_{i-\frac{1}{2}}\Delta u_{i-\frac{1}{2}}\right)$$

From (3.8a), (3.8b), (3.15) we have sybolically that

$$\text{2nd order R-K} \equiv \text{LW} + K_{i+\frac{1}{2}}\Delta u_{i+\frac{1}{2}} - K_{i-\frac{1}{2}}\Delta u_{i-\frac{1}{2}}$$
$$-\frac{|\nu|}{2}\left(\alpha_{i+\frac{1}{2}}\Delta u_{i+\frac{1}{2}}-\alpha_{i-\frac{1}{2}}\Delta u_{i-\frac{1}{2}}\right)$$

or

$$\text{2nd order R-K} + \frac{|\nu|}{2}\left(\alpha_{i+\frac{1}{2}}\Delta u_{i+\frac{1}{2}}-\alpha_{i-\frac{1}{2}}\Delta u_{i-\frac{1}{2}}\right)$$
$$\equiv \text{a 2nd order Davis} \qquad (3.16)$$

It can be shown that the Davis scheme is TVD if

$$|\nu| \leq \tfrac{1}{2}$$

$$\phi(r) \leq \min(2r, 1)$$

In practice the scheme is TVD for $|\nu| \leq 1$. The identity (3.16) is strictly true only for $\phi = \frac{3 + r}{4}$ which only corresponds to the Davis TVD region for

$$2r \leq \frac{3 + r}{4} \leq 1$$

but with the choice of $|\alpha|$ in place of α in (3.16) the Jameson scheme will approximate the Davis scheme for $\phi(r) = 1$, i.e. the smooth flow region.

Fig. (1) shows the part of the flux ratio plane for which the schemes correspond. In general it cannot be concluded that the (considered form of) Jameson scheme is always TVD.

4. THE DIFFUSION COEFFICIENT

We now examine the nature of the diffusion coefficient α in the scheme above and compare it with that of Jameson in (2.5). Note that if $\alpha_{i+\frac{1}{2}} \geq 0$ and for $\alpha_{i+\frac{1}{2}} = |\alpha_{i+\frac{1}{2}}|$.

From (3.14a)

$$|\alpha_{i+\frac{1}{2}}| = \frac{1}{4} |8 - (6 + r_i^+ + r_{i+1}^-)|$$

$$= \frac{1}{4} |(2\Delta u_{i+\frac{1}{2}} - (\Delta u_{i-\frac{1}{2}} + \Delta u_{i+\frac{3}{2}}))| / |\Delta u_{i+\frac{1}{2}}|$$

using (3.5a) for r_i^+, r_{i+1}^- . Thus

$$|\alpha_{i+\frac{1}{2}}| = \frac{1}{4} |\Delta^3 u_{i+\frac{1}{2}}| \; / \; |\Delta u_{i+\frac{1}{2}}|$$

$$= \frac{1}{4} \Delta x^2 \left(\left| \frac{\partial^3 u}{\partial x^3} \right| \Big/ \left| \frac{\partial u}{\partial x} \right| \right)_{i+\frac{1}{2}} + O(\Delta x)^4 \qquad (4.1)$$

So far we have considered the model equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$$

but in the Jameson code the devised scheme is used to solve the Euler equations. Moreover at a shock the method introduces artificial viscosity of order $\Delta x\, u_{xx}$. If we write down the momentum equation actually being solved by the scheme then it will be

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} = - \frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2}$$

to second order accuracy, where P is the pressure and μ is the coefficient of artificial viscosity. If pressure and viscous forces are assumed to be of the same order close to the shock then we may assume

$$P \cong \mu \frac{\partial u}{\partial x}$$

also assume that

$$\frac{\partial^2 P}{\partial x^2} \cong \mu \frac{\partial^3 u}{\partial x^3} \tag{4.2}$$

then (4.1) and (4.2) imply that

$$|\alpha_{i+\frac{1}{2}}| = (\Delta x)^2 \left(\left| \frac{\partial^2 P}{\partial x^2} \right| / |4P| \right)_{i+\frac{1}{2}}$$

To second order

$$(\Delta x)^2 \frac{\partial^2 P}{\partial x^2} = (P_{i+1} - 2P_i + P_{i-1})$$

$$4P = (P_{i+1} + 2P_i + P_{i-1})$$

so that we obtain

$$|\alpha_{i+\frac{1}{2}}| = \frac{|P_{i+1} - 2P_i + P_{i-1}|}{|P_{i+1} + 2P_i + P_{i-1}|} \tag{4.3}$$

Finally compare (2.5) with (4.3) in (3.16) we see that we have reproduced exactly twice the Jameson coefficient of the diffusion in this case.

5. COMPARISON BETWEEN THE SCHEMES OF JAMESON, DAVIS AND ROE

If we halve the Davis diffusion coefficient then

$$\frac{K_{i+\frac{1}{2}}}{2} = |\nu| \frac{(1 - |\nu|)}{2} \left\{ 1 - \frac{(\phi(r_i^+) + \phi(r_{i+1}^-)}{2} \right\}$$

$$\frac{\alpha_{i+\frac{1}{2}}}{2} = 1 - \left\{ \frac{(\phi(r_i^+) + \phi(r_{i+1}^-)}{2} \right\}$$

so that we now reproduce the Jameson diffusion coefficient exactly.

Now to remain consistent we replace (3.8) by

$$\text{2nd order} \quad R - K + \frac{\tau_d}{2} \equiv LW - \frac{\tau_d}{2} \tag{5.1}$$

and (3.16) by

$$\text{2nd order } R - K + \frac{\tau_d}{2} + \frac{|\nu|}{2} \left\{ \frac{\alpha_{i+\frac{1}{2}}}{2} \Delta u_{i+\frac{1}{2}} - \frac{\alpha_{i-\frac{1}{2}}}{2} \Delta u_{i-\frac{1}{2}} \right\} \tag{5.2}$$

$$\equiv LW + \frac{K_{i+\frac{1}{2}}}{2} \Delta u_{i+\frac{1}{2}} - \frac{K_{i-\frac{1}{2}}}{2} \Delta u_{i-\frac{1}{2}}$$

If α is identified by (3.14a) for all flux limiter functions then we may seek conditions which ensure a Jameson type scheme which will be TVD in the sense of Davis.

There is another consequence of dividing the Davis diffusion coefficient by two, other than obtaining equality with Jameson. The Davis scheme can be regarded as a modified Roe scheme which relies on the Roe scheme for its fundamental basis i.e. knowing what form the flux limiting functions ϕ should take. In devising his scheme Davis tried to remain as close as possible to Roe, so that he would produce comparable results for discontinuities.

Now we look at the difference between the diffusion coefficients for smooth flow and show that this difference results in a second order truncation between Roe and Davis.

The Roe diffusion coefficient can be written in terms of (3.6) as

$$R^{+}_{i+\frac{1}{2}} = (\nu + |\nu|)(1 - |\nu|)(1 - \phi(r^{+}_{i}))/4$$
$$R^{-}_{i+\frac{1}{2}} = (|\nu| - \nu)(1 - |\nu|)(1 - \phi(r^{-}_{i+1}))/4 \qquad (5.3)$$

we denote this more generally by

$$R_{i+\frac{1}{2}} = R^{+}_{i+\frac{1}{2}} + R^{-}_{i+\frac{1}{2}}$$

and this always gives the Roe upwind diffusion $\forall\ \nu$. The Davis coefficient is

$$K_{i+\frac{1}{2}} = |\nu|(1 - |\nu|)(2 - (\phi(r^{+}_{i}) + \phi(r^{-}_{i+1})))/2$$

Denote the difference between the diffusion coefficients by

$$e = K_{i+\frac{1}{2}} - R_{i+\frac{1}{2}}$$
$$= \frac{(1 - |\nu|)}{2}\ |\nu| \left[\left[1 - \frac{(\phi(r^{+}_{i}) + \phi(r^{-}_{i+1}))}{2} \right] - \frac{\nu(\phi(r^{-}_{i+1}) - \phi(r^{+}_{i}))}{2|\nu|} \right]$$

First consider the form of $\phi(r)$ giving second order accuracy as in §3, in particular $\phi(r) = \frac{(3 + r)}{4}$. Take $\nu > 0$ for all x values: then, after some cancellation

$$e = (1 - |\nu|)|\nu|(1 - r^{-}_{i+1})/8 \quad .$$

This results in a difference in diffusion of

$$e_{i+\frac{1}{2}}\Delta u_{i+\frac{1}{2}} - e_{i-\frac{1}{2}}\Delta u_{i-\frac{1}{2}}$$
$$= (1 - |\nu|)|\nu|((\Delta u_{i+\frac{1}{2}} - \Delta u_{i+\frac{3}{2}}) - (\Delta u_{i-\frac{1}{2}} - \Delta u_{i+\frac{1}{2}}))/8$$
$$= - |\nu|(1 - |\nu|)\ \frac{\Delta x^3}{8}\ \frac{\partial^3 u}{\partial x^3} + O(\Delta x^5)$$

The corresponding truncation error is

$$- \frac{a\Delta x^2}{8}\ (1 - |\nu|)\ \frac{\partial^3 u}{\partial x^3}$$

This shows that to second order accuracy there is a direct relationship between the schemes of Jameson, Davis and Roe, and that the Jameson scheme is a special case of the Davis formulation for the indicated region of the flux ratio plane Fig. (1).

§6 LIMITERS

So far we have considered smooth flow conditions where second order accurate approximations can be made. In Section 4 we pointed out that at a shock the schemes become first order accurate and that an artificial viscosity is added. This occurs as the TVD conditions are in danger of being violated. The scheme goes to first order accuracy to maintain the TVD condition, so preventing oscillations.

We restrict attention to flux limiters $\phi(r)$ of the form $\phi(r) = 1 - \beta + \beta r$ for second order accuracy in smooth regions. When the TVD condition is threatened $\phi(r)$ takes the form $\phi(r) = \gamma r$ for small r and

$$\phi(r) = 0 \qquad r < 0$$

Assuming that $\nu > 0$ for all x, the difference between the diffusion coefficient$ (c.f. §5) is

$$e = (1 - |\nu|)\,|\nu|\,(1 - \gamma r^{-}_{i+1})/2$$

This results in a difference of diffusion of

$$e_{i+\frac{1}{2}}\,\Delta u_{i+\frac{1}{2}} - e_{i-\frac{1}{2}}\,\Delta u_{i-\frac{1}{2}} =$$

$$(1 - |\nu|)\,|\nu|\,((\Delta u_{i+\frac{1}{2}} - \Delta u_{i-\frac{1}{2}}) - \gamma(\Delta u_{i+\frac{3}{2}} - \Delta u_{i+\frac{1}{2}}))/2$$

$$= (1 - |\nu|)\,|\nu|((1 - \gamma)(\Delta u_{i+\frac{1}{2}} - \Delta u_{i-\frac{1}{2}}) + \gamma((\Delta u_{i+\frac{1}{2}} - \Delta u_{i-\frac{1}{2}}) - \gamma(\Delta u_{i+\frac{3}{2}} - \Delta u_{i+\frac{1}{2}}))$$

$$= (1 - |\nu|)\,|\nu|\,((1 - \gamma)\,\Delta x^2 u_{xx} - \gamma \Delta x^3 u_{xxx} + 0\Delta x^4)/2 \quad .$$

Compared with the Roe scheme the Davis scheme now adds a truncation error of

$$(1 - |\nu|)\,a\,(\Delta x(1 - \gamma)u_{xx} - \gamma u_{xxx}\Delta x^2)/2$$

which is clearly an extra first order diffusion term for $\gamma \leq 1$. In particular

for $r < 0$ then Davis adds $(1 - |\nu|)\,|\nu|\,\Delta x u_{xx}/2$ to the standard first order upwind diffusion doubling the diffusion!

Finally consider the difference between the Roe diffusion coefficient and the modified Davis diffusion coefficient Km where we halve the Davis diffusion. Then

$$Km_{i+\frac{1}{2}} = |\nu|(1 - |\nu|)(1 - (\phi(r_i^+) + \phi(r_{i+1}^-))/2)/2$$

and

$$e_m = Km_{i+\frac{1}{2}} - R_{i+\frac{1}{2}}$$

$$= (1 - |\nu|)\,|\nu|\,(-\nu(\phi(r_{i+1}^-) - \phi(r_i^+)))/4|\nu|$$

For $\nu > 0$ and second order accurate ϕ the extra truncation is still second order. But for $\nu > 0$ and $\phi(r) = \gamma r$, corresponding to a diffusive scheme, the difference in diffusion is

$$e_{m_{i+\frac{1}{2}}}\ \Delta u_{i+\frac{1}{2}} - e_{m_{i-\frac{1}{2}}}\ \Delta u_{i-\frac{1}{2}} =$$

$$- \nu(1 - \nu)\gamma((\Delta u_{i+\frac{3}{2}} - \Delta u_{i-\frac{1}{2}}) - (\Delta u_{i+\frac{1}{2}} - \Delta u_{i-\frac{3}{2}}))/4$$

$$= - \nu(1 - \nu)\,\frac{\gamma}{4}\,u_{xxx}\,\Delta x^3 + O(\Delta x^5).$$

So the modified Davis scheme has the same diffusion as the Roe scheme to order $O(x^2)$. (For $r < 0$ no extra diffusion is added).

It would appear that, for the model advection equation, at least for $\nu > 0$, Jameson, modified Davis and Roe are equal to second order accuracy. But it should be noted that whereas the modified Davis scheme and Roe scheme can tune their Flux Limiters, should the α of Section 4 become negative the Jameson Scheme continues with $|\alpha|$ for the coefficient of its diffusion.

RESULTS

All results are for the linear advection equation using square wave initial data and mesh ratio $\Delta t/\Delta x = 0.9$.

Figs. 2A, 2B show the result of the Davis scheme for 70 steps and 280 steps respectively.

Figs. 3A, 3B show the result of the R-K 2nd order plus diffusion [from equation (3.16)] for 70 steps and 280 steps respectively.

CONCLUSION

The considerations above lead us on to experimenting with various modifications of the Davis & Jameson schemes. This results in new schemes which can be proved to be TVD for special regions of the flux ratio plane. The details will be given in a follow-up report.

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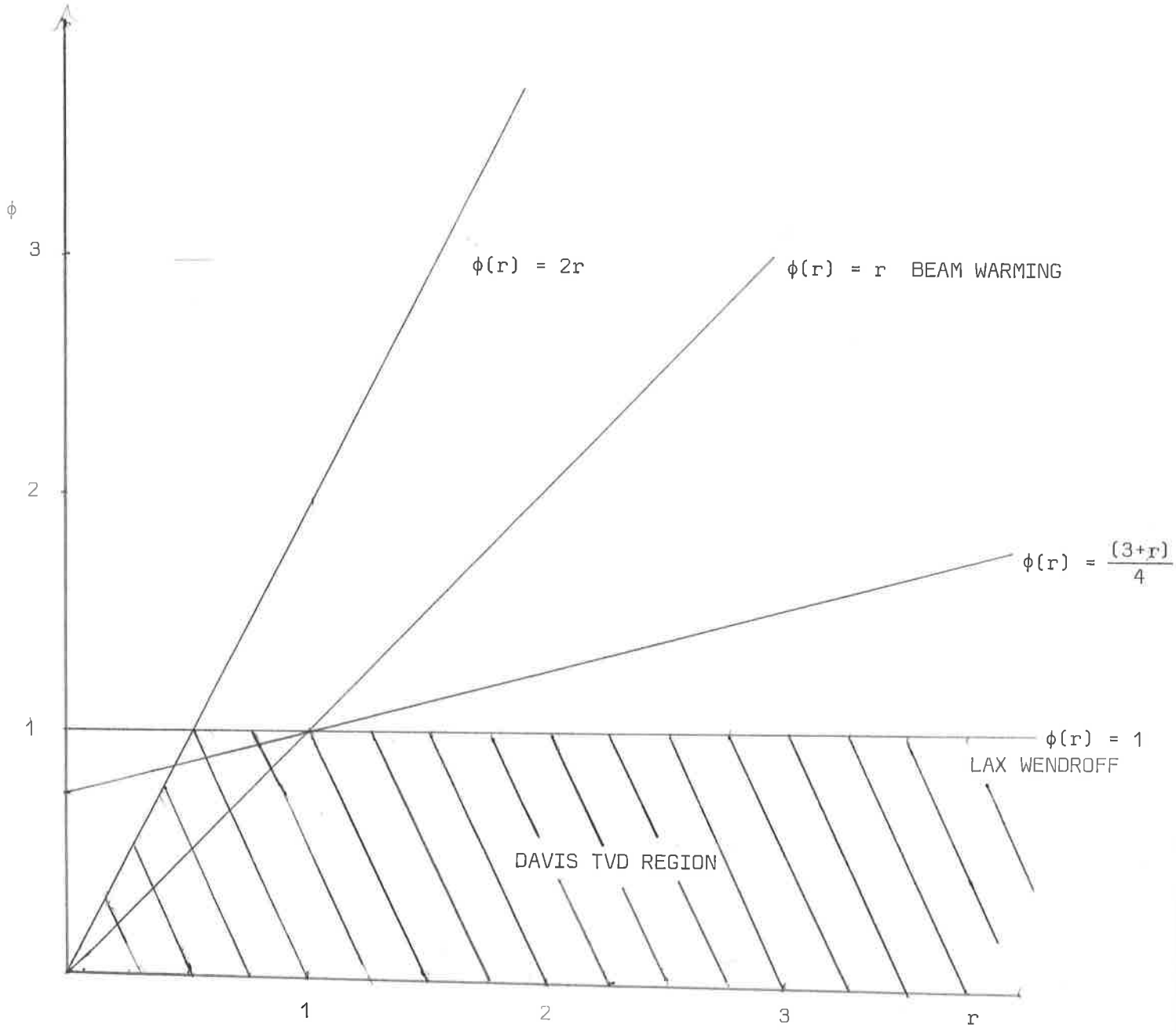

FLUX RATIO PLANE

FIG. 1

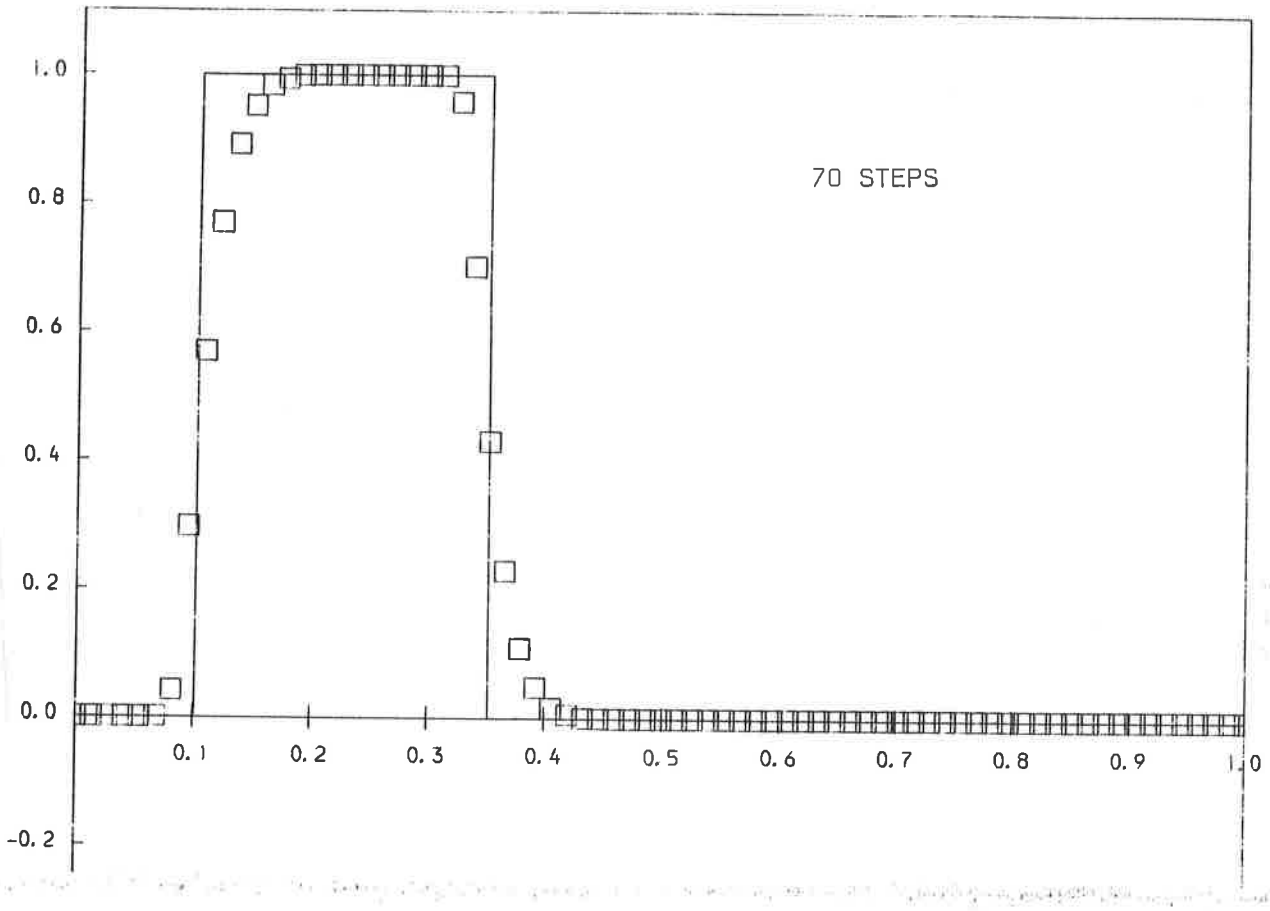

FIG. 2A

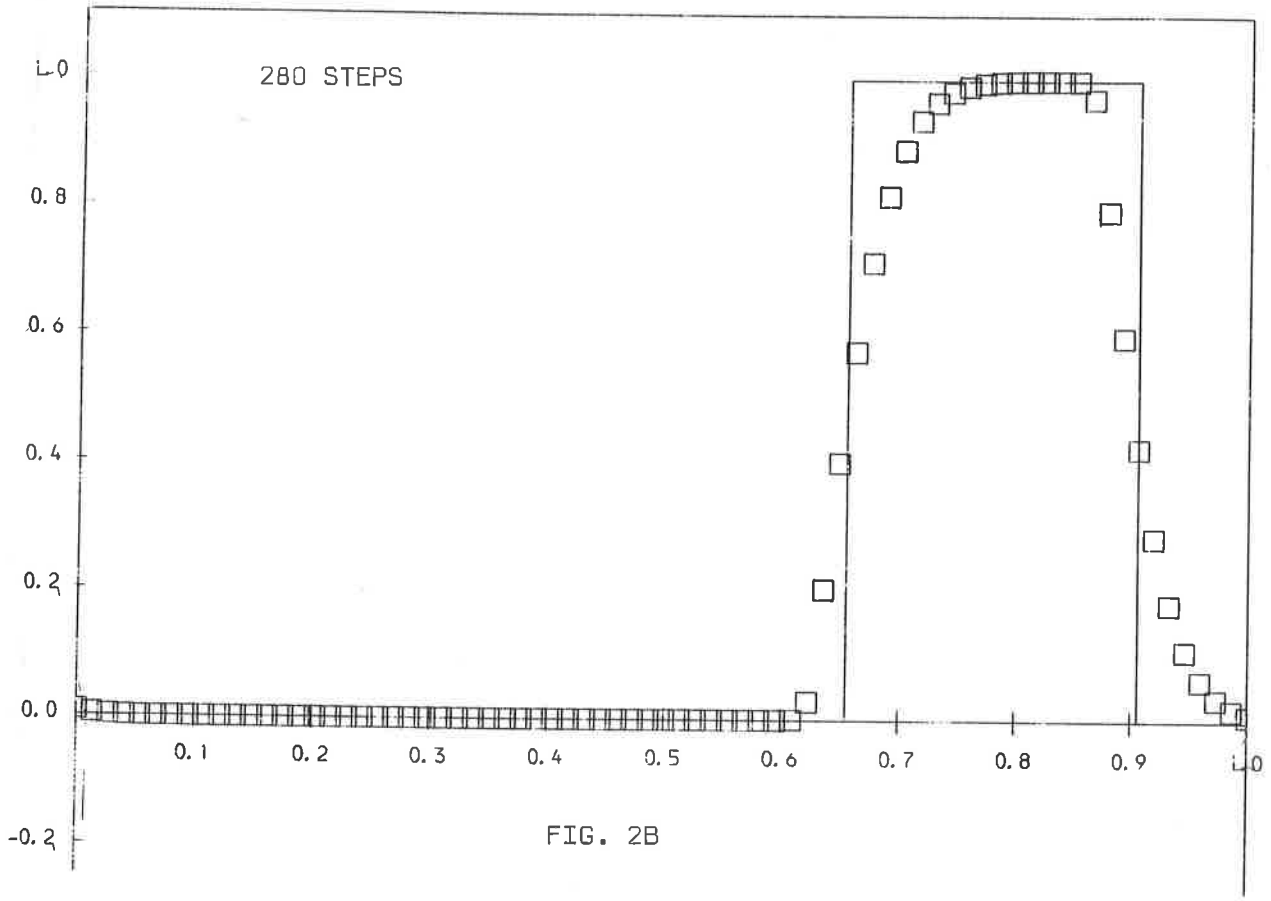

FIG. 2B

DAVIS

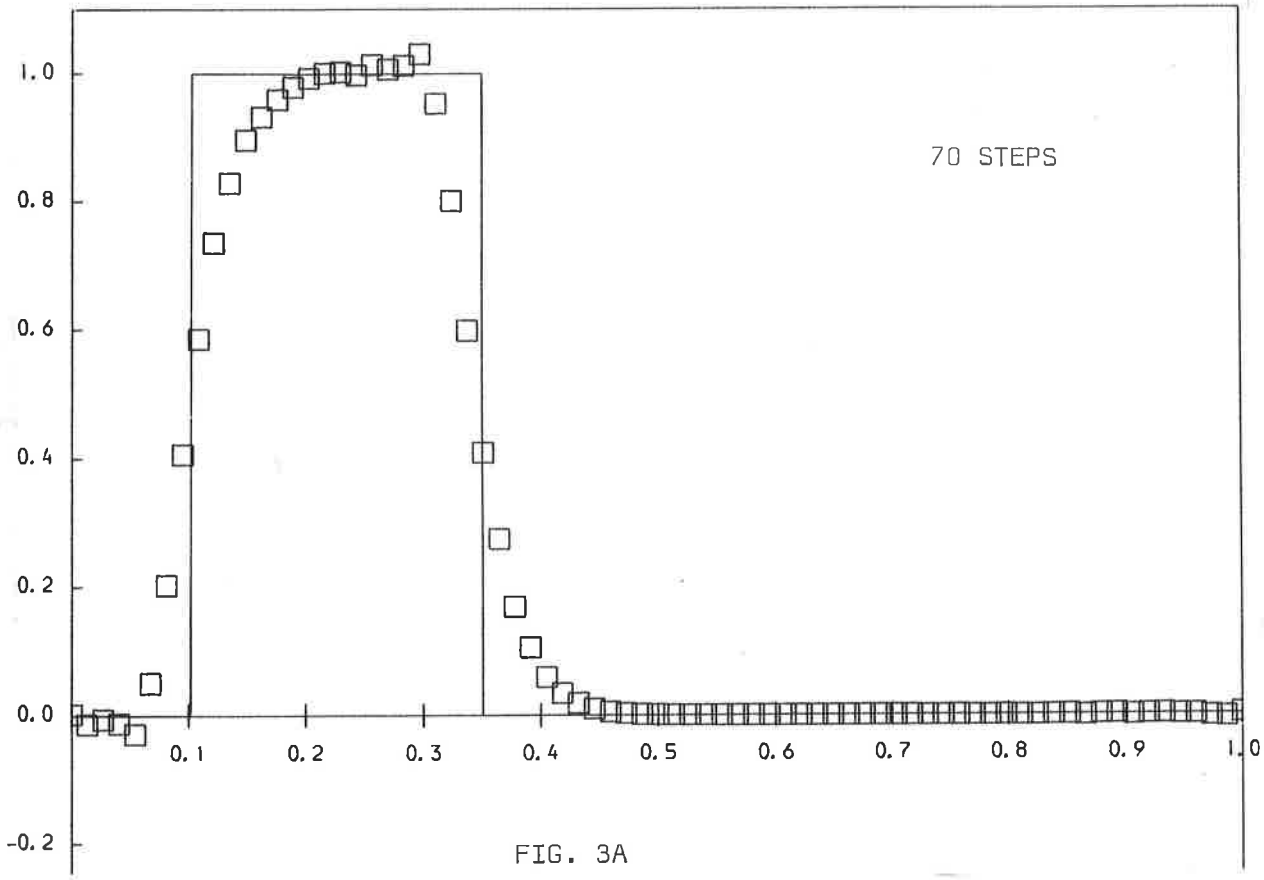

FIG. 3A

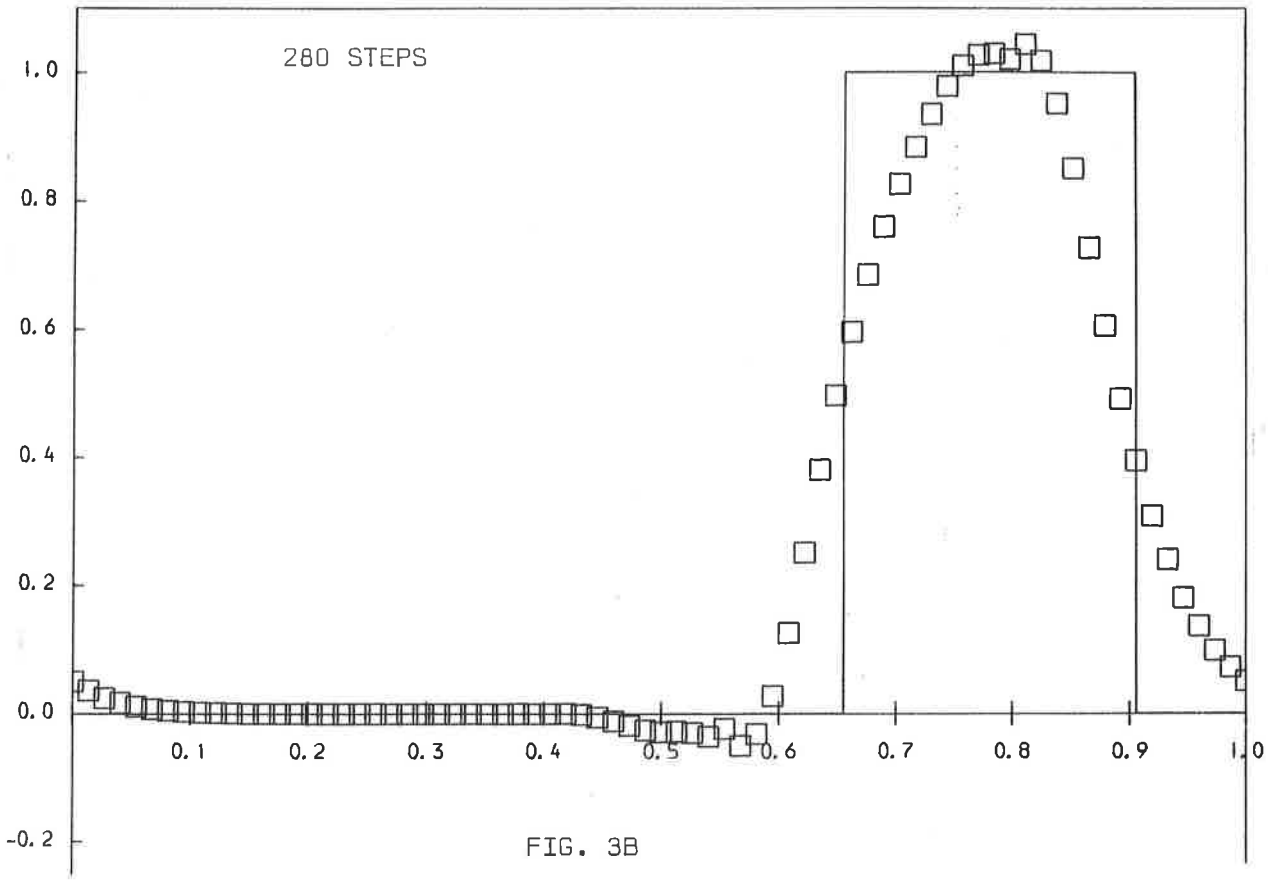

FIG. 3B

SECOND ORDER R-K + DIFFUSION