U N I V E R S I T Y   O F   R E A D I N G

FOUNDATIONS FOR THE STUDY OF SYSTEMS OF CONSERVATION LAWS WITH LIMITING DIFFUSION

Peter C. Samuels

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M A T H E M A T I C S   D E P A R T M E N T

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Abstract

This report represents a personal attempt to provide an overall framework for discussing the inviscid (or nondiffusive) limit of general conservation laws. The first chapter provides motivation for this enterprise via several paradigms from fluid mechanics. In the second chapter, it is first necessary to derive a more general definition of piecewise continuity. Then, structures are introduced for both diffusive and nondiffusive conservation laws in different forms. As a corollary, a formal definition of diffusivity is derived in the next chapter, along with a demonstration of its validity for the Navier-Stokes' equations. A model of a single two-dimensional steady diffusive conservation law is then constructed, somewhat analogous to Burgers' equation. Finally, in the fourth chapter, a survey is given of the work of others in attempting to provide the appropriate function space for describing nondiffusive conservation laws and various issues concerning the nondiffusive limit are discussed.

0. Introduction

My research objectives are to gain understanding into the behaviour of fluid flow at the weak end of shock waves and to improve the stability of numerical schemes within this region. However, in analysing the structure of the equations characteristic of the flow in this region, I have found it necessary to analyse the general structure of conservation equations; and, in attempting to characterise the behaviour of the shock tip in the presence of limiting viscosity, I have been drawn towards providing an overall framework for discussing the nondiffusive limit.

Having found no single reference which treats these fundamental problems in a unified manner (although [1], [2] and [3] come close), I decided to write a report on the subject myself.

I would like to also add the following two justifications for the work of this report: firstly, that it will provide the groundwork for subsequent reports; and secondly, that it may well have useful spin-offs, such as the construction of the appropriate model equations for this flow region.

1. **Paradigms From Physical Systems**

1.1 The Concept of Conservation

In many physical applications such as fluid mechanics, electromagnetism and quantum mechanics, there is the general set-up of a quantity (u say) varying in space ($\underline{x}$) and time (t) in such a way that, given a volume (V(t)) which varies over time in a known manner, the amount of the quantity u within the volume changes in a way related to the flux ($\underline{f}(u)$) of the quantity out of the surface (S(t)) enclosing the volume V(t). This is the principle of conservation.

For example, in fluid mechnaics, for the conservation of mass, the quantity is density (ρ) and the flux is the momentum density ($\rho\underline{q}$) with the conservation law being

$$\frac{d}{dt}\int_V \rho dv + \int_S \rho\underline{q}\cdot\underline{ds} = 0 \qquad (1.1)$$

where V and S are fixed over time.

1.2 The Concept of Diffusion

Many physical applications also contain higher order effects which dominate only where quantities change value rapidly (such as at boundary layers or shocks). These diffusive effects have the effect of damping down rapid changes. For example, in fluid mechanics, viscosity opposes the formation of discontinuous shock waves and limits them to steep transition fronts. The process is modelled by Burgers' equation (see [4], chapter 4).

1.3 The Concept of Source Phenomena

Diffusion and conservation may not describe all dominant effects in the behaviour of dynamical systems. Other effects, such as those due to a non-Euclidean geometry, rotation, chemistry and/or nonequilibrium thermodynamics are collectively described as source phenomena and are treated by adding certain 'right-hand side' terms to the equations of motion.

2. Description of the Generalised Structure

2.0 Introduction

Before the generalised structure can be formulated, it is necessary to describe the space of functions appropriate for the conserved quantities. This space must allow continuous discontinuity surfaces within the domain of an arbitrary dimension. In studying the literature I was only able to come across either nonconstructive function spaces, (e.g. Sobolev spaces and the like, see [5]), on which the position of discontinuities were unknown, or ad hoc definitions of constructive function spaces, (e.g. a countable union of non-intersecting shock curves in two dimensions, see [6] and [7]).

It became apparent to the author that what was required was something more fundamental, namely a definition of piecewise continuity in several dimensions. After playing around with the formulation of this space, its definition came out inductively, along with the definition of generalised flat intervals.

2.1 Definition of Piecewise Continuity and Flat Intervals in Several Dimensions

Let Δ be an arbitrary finite open compact subset of $\mathbb{R}^N$ of dimension N. $(N > 0)$.

Let

$$\mathbb{R}^N_K = \{\underline{x} \in \mathbb{R}^N : \ \|\underline{x}\|_\infty < K\} , \tag{2.1}$$

the N-dimensional hypercube of semi-length $K > 0$.

Hence $\mathbb{R}^N_K$ is a possible choice of Δ for all $K > 0$.

Let $\oplus$ be the disjoint union operator.

Let $\mathfrak{R}$ be the range function.

Let $C^{\infty}[X \rightarrow Y]$ be the space of continuous, infinitely differentiable functions from X to Y.

The definition of piecewise continuous functions will be achieved in two steps. Firstly the space of shock discontinuities $(\mathcal{S}^n(\Delta))$ and the space of generalised intervals $(\mathcal{I}^n)$ are defined inductively.

Initially, $\mathcal{S}^0(\Delta)$ and $\mathcal{I}^1$ are defined as follows:

<u>Definition 2.1</u>

$$s^0 \in \mathcal{S}^0(\Delta) \Leftrightarrow \exists\, M_0 \in \mathbb{N}, \quad x_1, \ldots, x_{M_0} \in \Delta$$

$$\text{such that } s^0 = \bigoplus_{m=1}^{M_0} \{x_m\} . \tag{2.2}$$

So $x_1, \ldots, x_{M_0}$ are distinct.

<u>Definition 2.2</u>

$$I^1 \in \mathcal{I}^1 \Leftrightarrow \exists\, x_l, x_r \in \mathbb{R}^1_K \text{ such that } x_l < x_r \text{ and}$$

$$I^1 = (x_l, x_r) \text{ for some } K > 0 . \tag{2.3}$$

These definitions are fairly simple. The inductive definitions follow analogously with a few alterations:

<u>Definition 2.3</u>

$$\forall n < N, \quad s^n \in \mathcal{S}^n(\Delta) \Leftrightarrow \exists\, M_n, \quad s^n_1, \ldots, s^n_{M_n} \subset \Delta \text{ such that}$$

$$s^n = \bigoplus_{m=1}^{M_n} s^n_m$$

and, $\forall\, m \leq M_n$,

(i) $\partial s^n_m \in \bar{\mathcal{S}}^{n-1}(\Delta)$;

(ii) $\exists\ I^n \in \mathcal{J}^n$, $\psi \in C^\infty[I^n \longrightarrow \Delta]$ such that ψ is 1 - 1 and $\mathcal{R}(\psi) = \mathfrak{s}^n_m$.

(2.4)

where $\bar{\mathcal{S}}^n(\Delta)$ is the space of the closures of elements of $\mathcal{S}^n(\Delta)$, i.e.

$$\mathfrak{s}^n \in \mathcal{S}^n(\Delta) \Leftrightarrow \bar{\mathfrak{s}}^n \in \bar{\mathcal{S}}^n(\Delta) \ . \tag{2.5}$$

Thus elements of $\mathcal{S}^n(\Delta)$ are disjoint unions of smooth surfaces with dimension n.

There is also the parallel definition of $\mathcal{J}^n$:

<u>Definition 2.4</u>

$\forall n$, $I^n \in \mathcal{J}^n \Leftrightarrow$

(i) $\exists\ K > 0$ such that $I^n \subset \mathbb{R}^n_K$;

(ii) ∂I^n is simply-connected;

(iii) $\partial I^n \in \bar{\mathcal{S}}^{n-1}(\mathbb{R}^n_K)$;

(iv) I^n is open. (2.6)

This inductive definition follows since $\bar{\mathcal{S}}^{n-1}(\mathbb{R}^n_K)$ is well-defined and N is an arbitrary integer, (so the restriction $n < N$ is unimportant). Examples of $\mathfrak{s}^0$, $\mathfrak{s}^1$ and $\mathfrak{s}^2$ are given on figure 1, examples of I^1, I^2 and I^3 are given in figure 2. It is clear that the boundary of an element of $\mathcal{S}^1(\Delta)$ will be an element of $\mathcal{S}^0(\Delta)$, and the boundary of an element of $\mathcal{S}^2(\Delta)$ will be the disjoint union of an element of $\mathcal{S}^1(\Delta)$ with an element of $\mathcal{S}^0(\Delta)$. This leads to the following lemma:

Lemma 2.1

For all Δ satisfying the conditions in the above definition, $\forall\ N \in \mathbb{N}$, $\forall\ n < N$, $\forall\ \mathfrak{a}^n \in \mathcal{S}^N(\Delta)$ $\exists\ \mathfrak{a}^0 \in \mathcal{S}^0(\Delta)$, $\mathfrak{a}^1 \in \mathcal{S}^1(\Delta), \ldots, \mathfrak{a}^{n-1} \in \mathcal{S}^{n-1}(\Delta)$ such that

(i) $\forall\ m \le n$, $\quad \bar{\mathfrak{a}}^m = \bigoplus_{k=0}^{m} \mathfrak{a}^k$;

(ii) $\forall\ m \le n$, $\quad \partial\mathfrak{a}^m = \bigoplus_{k=0}^{m-1} \mathfrak{a}^k$.

The last background definition required prior to the final definition of piecewise continuity is that of the jump function, [•]. This is defined in the usual way to be the difference in the value of a function on either side of a surface of discontinuity. Notice that the jump of a jump is not defined because surfaces of discontinuity will typically either peter out (giving a continuous decrease in jump strength to zero) or they will meet together in groups of three (so two quantities cannot be identified). It may be assumed that jumps do not jump in value at these points, but this assumption is not necessary for the definition of piecewise continuity.

Let Δ be defined as before. Then $pwC^m(\Delta)$, the space of piecewise continuous function with m continuous derivatives with domain Δ is defined as follows:

Definition 2.5

$u \in pwC^m(\Delta) \Leftrightarrow \exists\ \mathfrak{a}^{N-1} \in \mathcal{S}^{N-1}(\Delta)$ such that

(i) $\quad u \in C^m(\Delta \backslash \bar{\mathfrak{a}}^{N-1})$;

(ii) $\quad [u] \in C^m(\mathfrak{a}^{N-1})$. $\qquad$ (2.7) where $C^m(X)$ is the space of continuous m times differentiable functions with domain X. Note that u is undefined on $\bar{S}^{N-1} \backslash S^{N-1}$. This set corresponds to the set of irregular points of u.

Another useful definition is that of a piecewise continuous surface. Let Σ_n^N be the space of piecewise continuous surfaces of dimension n contained within $\mathbb{R}_K^N$ for some $k > 0$. Then Σ_n^N is defined as follows:

Definition 2.6

$$\bar{S}^n \in \Sigma_n^N \Leftrightarrow \quad \text{(i)} \ \exists\, K > 0 \ \text{such that} \ \bar{S}^n \in \bar{\mathscr{S}}^n(\mathbb{R}_K^N) \ ;$$

$$\text{(ii)} \ \bar{S}^n \ \text{is simply-connected.}$$

Gauss' divergence theorem and Stokes' curl theorem may now be stated within this formalism:

Lemma 2.2 (Gauss' Divergence Theorem)

$\forall\, N \in \mathbb{N}$, $\forall\, \bar{S}^{N-1} \in \Sigma_{N-1}^N$ closed (i.e. topologically equivalent to a sphere), containing the volume V^N,

$$\forall\, \underline{f} \in C^1[\bar{V}^N]^N \ ,$$

$$\int_{V^N} \nabla\cdot\underline{f}\,dv = \int_{\bar{S}^{N-1}} \underline{f}\cdot\underline{ds} \ . \tag{2.8}$$

Lemma 2.3 (Stokes' Curl Theorem)

$\forall\, \bar{S}^2 \in \Sigma_2^3$ open $\forall\, \underline{f} \in C^1[\bar{S}^2]^3$,

$$\int_{\bar{S}^2} (\nabla \wedge \underline{f}).\underline{ds} = \int_{\partial\bar{S}^2} \underline{f}.\underline{dr} \ . \tag{2.9}$$

Note: $\partial\bar{S}^2$ is only the outer boundary of $\bar{S}^2$ and does not contain the inner lines of discontinuity of slope because $\bar{S}^n \in \bar{\mathscr{S}}^n(\mathbb{R}^N_K)$ and not $\mathscr{S}^n(\mathbb{R}^n_K)$. The curl theorem has been written with $N = 3$ as it is seldom necessary to consider higher (or lower) dimensional generalisations.

2.2 The Generalised Diffusive Structure

2.2.1 Integral Form

Before the integral form can be defined, more notation needs to be introduced. By convention, superfixes will be used for dependent variables and suffices for independent variables.

Let there be N space dimensions parameterised by $\underline{x} = (x_1,\ldots,x_N) \in \mathbb{R}^N$. Let the time dimension be parameterised by $t \in \mathbb{R}_+ = [0,\infty]$. Let Δ be a subset of $\mathbb{R}^N$ obeying the same conditions as in the previous section. Define the M conserved variables $\underline{u} = (u^1,\ldots,u^M)$ by:

$$\forall\ i \leq M\ , \quad u^i \in C^1[\mathbb{R}_+ \times \Delta \longrightarrow \mathbb{R}]\ , \quad \text{with} \quad u^i = u^i(t,\underline{x})\ . \tag{2.10}$$

Define the M^2 flux vector components f^i_j $(i,j = 1,\ldots,M)$ by:

$$\forall\ i,j \leq M, \quad f^i_j \in C^\infty[\mathbb{R}^M \longrightarrow \mathbb{R}]\ , \quad \text{with} \quad f^i_j = f^i_j(\underline{u})\ . \tag{2.11}$$

Let there be α constant scale values for the diffusion coefficients forming the vector $\underline{d} = (d_1,\ldots,d_\alpha)$. Let there be αM^2N^2 components to the rank 5 viscosity tensor V^{il}_{jkm} $(i,l = 1,\ldots,M;\quad j,k = 1,\ldots,N;$ $m = 1,\ldots,\alpha)$ defined by

$$\forall\ i,l \le M,\quad \forall\ j,k \le N,\quad \forall\ m \le \alpha\ ,$$
$$V^{il}_{jkm} \in C^{\infty}[\mathbb{R}^M \longrightarrow \mathbb{R}]\ ,\quad \text{with}\quad V^{il}_{jkm} = V^{il}_{jkm}(\underline{u})\ . \tag{2.12}$$

Finally, let there be M source terms $S^1,\ldots,S^M$ defined by

$$\forall\ i \le M\ ,\quad S^i \in C^0[\mathbb{R}^M \times \mathbb{R}_+ \times \Delta \longrightarrow \mathbb{R}]\ ,\quad \text{with}$$
$$S^i = S^i(\underline{u};t,\underline{x}) \tag{2.13}$$

Definition 2.7

The integral form (I) of the equations of motion with diffusion is defined as

$$(I) \Leftrightarrow \forall\ T \in \mathcal{I}^1,\quad D \in \mathcal{I}^N \text{ such that } T \subset \mathbb{R}_+,\quad D \subset \Delta\ ,$$

$$\forall\ i \le M\ ,\qquad \int_{\partial(T\times D)} (u^i,\underline{F}^i)\cdot(n_t,\underline{n})ds = \int_{T\times D} S^i dtdv \tag{2.14}$$

where $$F^i_j = f^i_j - V^{il}_{jkm}\frac{\partial u^l}{\partial x_k} d_m \qquad \forall\ i \le m,\quad \forall\ j \le N\ , \tag{2.15}$$

(using the summation convention); and $(n_t,\underline{n})$ is the unit normal to the space-time hypersurface $\partial(T \times D)$ - see figure 3.

This definition could, of course, be generalised to the case of an arbitrary space-time volume in $\mathbb{R}_+ \times \Delta$ or specialised to the case of steady state. The former leads to greater theoretical complications but may be useful for moving boundary problems.

A more conventional integral form may be derived from definition 2.7. Observe that $T = (t_1, t_2)$ for some $t_1 < t_2$, $t_1, t_2 \in \mathbb{R}_+$. Also $\partial(T\times D) = \partial T\times D \oplus T\times\partial D$, with $\partial T = \{t_1, t_2\}$. On ∂T, $(n_t, \underline{n}) = (\pm 1, \underline{0})$ and on ∂D, $(n_t, \underline{n}) = (0, \underline{n}^*)$ where $\underline{n}^*$ is the scaled unit normal to ∂D. The integral equation definition 2.7 can then be rewritten

$$\int_D \left\{ u^i \big|_{t_2} - u^i \big|_{t_1} \right\} dv + \int_{t_1}^{t_2} \int_{\partial D} \underline{F}^i \cdot \underline{ds}\, dt = \int_{t_1}^{t_2} \int_D S^i dv \quad . \qquad (2.16)$$

Thus, since D is constant, in the limit $t_2 - t_1 \longrightarrow 0$ we obtain

$$\frac{\partial}{\partial t} \int_D u^i dv + \int_{\partial D} \underline{F}^i \cdot \underline{ds} = \int_D S^i dv \quad . \qquad (2.17)$$

The integral form follows naturally from physical arguments. The strong and weak forms may now be derived easily.

2.2.2 Strong Form

Definition 2.8

The strong form (S) of the equations of motion with diffusion requires the stronger condition

$$\forall\, i \leq M\ , \quad u^i \in C^2[\mathbb{R}_+ \times \Delta \longrightarrow \mathbb{R}]$$

and may be written:

$$(S) \iff \forall\, t \in \mathbb{R}_+ \quad \forall\, \underline{x} \in \Delta \quad \forall\, i \leq M\ ,$$

$$\frac{\partial u^i}{\partial t} + \nabla \cdot \underline{F}^i = S^i \quad . \qquad (2.18)$$

2.2.3 Weak Form

Let $C_0^1[X \longrightarrow \mathbb{R}]$ be the space of continuous once-differentiable functions from X to $\mathbb{R}$ which are zero on ∂X.

Definition 2.9

The weak form (W) of the equations of motion requires only the original weak condition $\forall\ i \leq M,\quad u^i \in C^1[\mathbb{R}_+ \times \Delta \longrightarrow \mathbb{R}]$ and may be written:

$$(W) \Leftrightarrow \forall\ T \in \mathcal{I}^1,\ D \in \mathcal{I}^N \text{ such that } T \subset \mathbb{R}_+,$$

$$D \subset \Delta,\ \forall i \leq M,\ \forall \phi^i \in C_0^1[\overline{T\times D} \longrightarrow \mathbb{R}],$$

$$\cdot \int_{T\times D} \left[u^i \frac{\partial \phi^i}{\partial t} + \underline{F}^i \cdot \nabla \phi^i + S^i \phi^i\right] dtdv = 0 \qquad (2.19)$$

(not summed over i).

2.2.4 Equivalence Proof

If the stronger continuity condition is assumed, the three forms of the equation of motion can be proved to be equivalent. Summation convention will not be used on this subsection.

Theorem 2.4

If $\forall\ i \leq M,\quad u^i \in C^2[\mathbb{R}_+ \times \Delta \longrightarrow \mathbb{R}]$, then

$$(I) \Leftrightarrow (S) \Leftrightarrow (W)\ .$$

Proof

(I) $\Leftrightarrow$ (S):

$X \in \mathcal{S}^n$, $X \subset Y \Rightarrow \partial X \in \mathcal{S}^{n-1}(Y)$ by the definitions of $\mathcal{S}^n$ and $\mathcal{S}^{n-1}$, provided Y satisfies the same conditions as Δ. Hence $\partial(T \times D) \in \Sigma_N^{N+1}$, so we may apply Lemma 2 (the divergence theorem) to the left-hand side of equation (2.14).

Now the space-time divergence of $(u^i, \underline{F}^i)$ is $\frac{\partial u^i}{\partial t} + \nabla \cdot \underline{F}^i$. Hence we infer

$$(I) \Rightarrow \forall\ T \in \mathcal{S}^1,\ D \in \mathcal{S}^N \text{ st } T \subset \mathbb{R}_+,\ D \subset \Delta,$$

$$\forall\ i \leq M,\quad \int_{T\times D} \left[\frac{\partial u^i}{\partial t} + \nabla \cdot \underline{F}^i\right] dt dv = \int_{T\times D} S^i dt dv . \tag{2.20}$$

Hence, as T and D are quantified over all of $\mathbb{R}_+$ and Δ, we may apply Lagrange's lemma and infer

$$(I) \Leftrightarrow \forall\ t \in \mathbb{R}_+\ \forall\ \underline{x} \in \Delta,\quad \forall\ i \leq M,$$

$$\frac{\partial u^i}{\partial t} + \nabla \cdot \underline{F}^i = S^i , \tag{2.21}$$

since the continuity of $\underline{u}$ implies this equation is well-defined, i.e. (I) $\Leftrightarrow$ (S).

(S) $\Leftrightarrow$ (W):

Let
$$J^i = \int_{T\times D} \left\{\frac{\partial}{\partial t} (\phi^i u^i) + \nabla \cdot (\phi^i \underline{F}^i)\right\} dt dv . \tag{2.22}$$

As already shown, $\partial(T \times D) \in \Sigma_N^{N+1}$, so the divergence theorem is applicable:

$$\Rightarrow J^i = \int_{\partial(T\times D)} (\phi^i u^i, \phi^i \underline{F}^i)\cdot(n_t, \underline{n})ds . \qquad (2.23)$$

But $\phi \in C_0^1[\overline{T\times D} \longrightarrow \mathbb{R}]$, hence $J^i = 0$. (2.24)

Also, by the product rule,

$$J^i = \int_{T\times D} \left\{\phi_t^i u^i + \underline{F}^i\cdot\nabla\phi^i + \phi^i\left[\frac{\partial u^i}{\partial t} + \nabla\cdot\underline{F}^i\right]\right\} dtdv . \qquad (2.25)$$

Thus, combining (2.24) and (2.25),

$$\int_{T\times D} \left\{u^i\phi_t^i + \underline{F}^i\cdot\nabla\phi^i\right\} dtdv = - \int_{T\times D} \phi^i\left[\frac{\partial u^i}{\partial t} + \nabla\cdot\underline{F}^i\right]dtdv . \qquad (2.26)$$

Adding $\int_{T\times D} \phi^i S^i dtdV$ to each side we obtain the equivalence

$$\int_{T\times D} \left\{u^i\phi_t^i + \underline{F}^i\cdot\nabla\phi^i + S^i\phi^i\right\} dtdv = 0 \iff$$

$$\int_{T\times D} \phi^i\left\{\frac{\partial u^i}{\partial t} + \nabla\cdot\underline{F}^i - S^i\right\} dtdv = 0 . \qquad (2.27)$$

Also, by Lagrange's lemma, as ϕ is also an arbitrary function within $\mathbb{R}_+ \times \Delta$, we have the identity:

$$(S) \iff \forall\ T \in \mathscr{I}^1,\ D \in \mathscr{I}^N \text{ st } T \subset \mathbb{R}_+,$$

$$D \subset \Delta,\ \forall\ i \leq M,\ \forall\ \phi^i \in C_o^1[\overline{T\times D} \longrightarrow \mathbb{R}],$$

$$\int_{T\times D} \phi^i\left\{\frac{\partial u^i}{\partial t} + \nabla\cdot\underline{F}^i - S^i\right\} dtdv = 0 . \qquad (2.28)$$

But this is just a quantification of the right-hand side of (2.27). Hence (S) is equivalent to the same quantification of the left-hand side of (2.27) - which is precisely (W).

■

2.2.5 The Nondiffusive Limit System

Here we are concerned with the behaviour of the solution as a function of the diffusion coefficients, $\underline{d}$, in the limit $\underline{d} \longrightarrow \underline{0}$. First of all, the system must be constructed.

Assume the continuity hypothesis of theorem 2.4, i.e. $\forall\ i \leq M$, $u^i \in C^2[\mathbb{R}_+ \times \Delta \longrightarrow \mathbb{R}]$. Hence theorem 2.4 holds, so the integral, strong and weak forms are all equivalent.

Let (B) stand for consistent sufficient boundary conditions. Thus we assume it is theoretically possible to solve the problem of, given (B) and (I) (or (S) or (W)), to determine $\underline{u}$ over the domain $\mathbb{R}_+ \times \Delta$.

Definition 2.10

$$\Sigma\ (\underline{d}) \Longleftrightarrow (B) \wedge (I) \tag{2.29}$$

$$\Sigma\ (\underline{d}) \vdash \forall\ i \leq M,\quad u^i = u^i_*(t,\underline{x};\underline{d})\ . \tag{2.30}$$

Where '$\vdash$' is the symbol for logical inference, '$\wedge$' is the logical 'and', and $\underline{u}_*$ is the solution vector. $\Sigma(\underline{d})$ represents the system of equations with boundary conditions dependent on $\underline{d}$. Clearly $\Sigma(\underline{d}) \Leftrightarrow (B) \wedge (S)$ and $\Sigma(\underline{d}) \Leftrightarrow (B) \wedge (W)$.

The nondiffusive limit system may now be written as

$$\lim_{\underline{d} \to \underline{o}} \Sigma(\underline{d}) \,, \tag{2.31}$$

with corresponding solution vector

$$\lim_{\underline{d} \to \underline{o}} \underline{u}_{*}(t,\underline{x};\underline{d}) \,. \tag{2.32}$$

The question of whether $\Sigma(\underline{o})$ (i.e., the nondiffusive system) is equivalent to $\lim_{\underline{d} \to \underline{o}} \Sigma(\underline{d})$ will be addressed in §3.5 and §4.1.1.

2.3 The Generalised Nondiffusive Structure

2.3.1 Integral Form

The integral and weak forms could be defined with $T \times D$ having the same constraints as for the diffusive structure. However, as will become apparent when discussing the equivalence proofs, it is more convenient to restrict $T \times D$ such that either u is smooth within $T \times D$ or $T \times D$ is partitioned into two regions by a single smooth space-time shock, Γ.

Formally, consider $T \in \mathcal{I}^1$, $D \in \mathcal{I}^N$ such that $T \subset \mathbb{R}$, $D \subset \Delta$. A more restrictive class of piecewise continuous, m times differentiable functions, $pwC^m_*[T{\times}D \longrightarrow \mathbb{R}_+]$ is defined as follows:

Definition 2.11:

$$u \in pwC^m_*[T{\times}D \longrightarrow \mathbb{R}] \iff u \in C^m[T \times D \longrightarrow \mathbb{R}]$$

or $\exists\ I_n \in \mathcal{I}^n$, $\psi \in C^\infty[I^n \longrightarrow T \times D]$ such that

i) ψ is $1-1$;

ii) $\mathfrak{R}(\psi) = \Gamma(T) = \{\Gamma(t) \text{ such that } t \in T\}$;

iii) $\forall\ t \in T, \quad D = D^-(t) \oplus \Gamma(t) \oplus D^+(t)$ such that $D^-(t)$, $D^+(t) \in \mathcal{I}^N$ (see Figure 4) ;

iv) $u \in C^m[D^-(T) \longrightarrow \mathbb{R}]$;

$u \in C^m[D^+(T) \longrightarrow \mathbb{R}]$;

$[u] \in C^m[\Gamma(T) \longrightarrow \mathbb{R}]$;

$$\text{where } D^{\pm}(T) = \{D^{\pm}(t) : t \in T\} . \tag{2.33}$$

Definition 2.12

The integral form of the equations of motion without diffusion requires the condition

$$\forall\ i \leq M, \quad u^i \in pwC^0[\mathbb{R}_+ \times \Delta \longrightarrow \mathbb{R}]$$

and is defined by

$(I^*) \Leftrightarrow \forall\ T \in \mathcal{I}^1,\ D \in \mathcal{I}^N$ such that $T \subset \mathbb{R}_+$, and $\forall\ i \leq M$, $u^i \in pwC^0_*[T \times D \longrightarrow \mathbb{R}]$,

$$\forall\ i \leq M, \quad \int_{\partial(T\times D)} (u^i,\ \underline{f}^i)\cdot(n_t, \underline{n})ds = \int_{T\times D} S^i dtdv . \tag{2.34}$$

2.3.2 Strong Form with Jump Conditions

The strong form may now be defined analogously to the case with diffusion apart from the fact that it does not hold over curves of discontinuity. Along these curves we have jump conditions holding - generalisations of the Rankine-Hugoniot jump conditions. At the intersections and ends of these curves, we have no information - as in the case of the definition of piecewise continuous functions. A similar definition of these and the weak forms is given in [1] pp.1-3.

Definition 2.13

The strong form and jump conditions of the equations of motion without diffusion require the condition $\forall\ i \le M,\ \ u^i \in pwC^1[\mathbb{R}_+ \times \Delta \longrightarrow \mathbb{R}]$ and are defined by

$$(S^*) \Leftrightarrow \forall\ t \in \mathbb{R}_+,\ \forall\ \underline{x} \in \Delta \backslash \overline{\mathcal{S}}^{N-1}(t),\ \forall\ i \le M,\ \frac{\partial u^i}{\partial t} + \nabla \cdot \underline{f}^i = S^i; \qquad (2.35)$$

$$(J) \Leftrightarrow \forall\ t \in \mathbb{R}_+,\ \forall\ \underline{x} \in \mathcal{S}^{N-1}(t),\ \forall\ i \le M,\ ([u^i],[\underline{f}^i]) \cdot (n_t,\underline{n}) = 0; \qquad (2.36)$$

where $\mathcal{S}^N = \{\mathcal{S}^{N-1}(t) : t \in \mathbb{R}_+\}$ is the discontinuity surface for $\underline{u}$ and $(n_t,\underline{n})$ is its space-time normal.

Note

(J) may be rewritten as

$\forall\ i \le M,\ \ s = \dfrac{[\underline{f}^i] \cdot \underline{n}^*}{[u^i]}$, where s is the shock speed and $\underline{n}^*$ is now the unit space normal. This is the usual form of the unsteady Rankine-Hugoniot jump conditions.

2.3.3 Weak Form

Definition 2.14

The weak form without diffusion only requires the condition $\forall\ i \le M,\quad u^i \in pwC^0[\mathbb{R}_+ \times \Delta \longrightarrow \mathbb{R}]$ and is defined by $(W^*) \Leftrightarrow \forall\ T \in \mathcal{I}^1$, $D \in \mathcal{I}^N$ such that $T \subset \mathbb{R}_+$, $D \subset \Delta$ and $\forall\ \phi^i \in C_o^1[T\times D \longrightarrow \mathbb{R}]$,

$$\int_{T\times D} \{u^i \phi_t^i + \underline{f}^i \cdot \nabla\phi^i + S^i\phi) dtdV = 0 \qquad (2.38)$$

(again, not summed over i).

2.3.4 Equivalence Proof

If, again, the stronger continuity condition is assumed, the three forms of the equation of motion can be proved to be equivalent. Summation convention is again not used in this subsection.

Theorem 2.5

If $\forall\ i \le M,\quad u^i \in pwC^1[\mathbb{R}_+ \times \Delta \longrightarrow \mathbb{R}]$, then

$$(I^*) \Longleftrightarrow (S^*) \wedge (J) \Longleftrightarrow (W^*)$$

('$\wedge$' is the logical 'and' symbol).

(Note: the equivalence of $(S^*) \wedge (J)$ and (W^*) is discussed for one dimensional flow in [8] pp.246-248).

Proof

$(I^*) \Longleftrightarrow (S^*) \wedge (J)$:

If $\underline{u}$ is continuous in D then the equivalence of the integral and strong forms is identical to that given in theorem 2.4 apart from replacing $\underline{F}^i$ with $\underline{f}^i$. We may thus limit ourselves to considering the case where $\underline{u}$ is discontinuous in D. Let $\Gamma^+(T)$, $\Gamma^-(T)$ be the two sides of $\Gamma(T)$ corresponding to $D^+(T)$, $D^-(T)$ respectively, see figure 4.

Now

$$\int_{\partial(T\times D)} \equiv \int_{\partial D^+(T)} + \int_{\partial D^-(T)} - \int_{\Gamma^+(T)} + \int_{\Gamma^-(T)} \tag{2.39}$$

and, for a bounded function,

$$\int_{T\times D} \equiv \int_{D^+(T)} + \int_{D^-(T)} . \tag{2.40}$$

Thus the integral equation in the definition of (I^*) may be rewritten as

$$\left\{\int_{\partial D^+(T)} + \int_{\partial D^-(T)} - \int_{\Gamma^+(T)} + \int_{\Gamma^-(T)}\right\}(u^i, \underline{f}^i)\cdot(n_t, \underline{n})ds$$

$$= \left\{\int_{D^+(T)} + \int_{D^-(T)}\right\}S^i dtdv . \tag{2.41}$$

Since $\Gamma(T)$ is a smooth surface, $\partial D^+(T)$, $\partial D^-(T) \in \Sigma_N^{N+1}$, so the divergence theorem is again applicable, since u^i and $\underline{f}^i$ are continuous within $D^+(T)$ and $D^-(T)$. Also,

$$\int_{\Gamma^+(T)} \psi ds - \int_{\Gamma^-(T)} \psi ds = \int_{\Gamma(T)} [\psi]ds \tag{2.42}$$

for all functions ψ.

Hence we obtain the equivalent form:

$$\int_{D^+(T)} (u^i_t + \nabla\cdot\underline{f}^i - S^i)dtdv + \int_{D^-(T)} (u^i_t + \nabla\cdot\underline{f}^i - S^i)dtdv$$

$$= \int_{\Gamma(T)} ([u^i], [\underline{f}^i])\cdot(n_t, \underline{n})ds \ . \qquad (2.43)$$

Thus we have $(S^*) \wedge (J) \Rightarrow (I^*)$ by simple integration.

Clearly $D^+(T)$ and $D^-(T)$ may be deformed to include any connected region in $T\times D\setminus\bar{\mathcal{G}}^N(T)$, given an initial $\Gamma(T) \subset \bar{\mathcal{G}}^N(T)$. Hence $(I^*) \Rightarrow (S^*)$, by Lagrange's lemma. Also, by considering limitingly thin domains $D^+(T)$, $D^-(T)$ we derive the condition

$$\int_{\Gamma(T)} ([u^i], [\underline{f}^i])\cdot(n_t, \underline{n})ds = 0 \qquad (2.44)$$

and as $\Gamma(T)$ is also arbitrary, again by Lagrange's lemma, we infer $(I^*) \Rightarrow (J)$.

<u>$(S^*) \wedge (J) \Leftrightarrow (W^*)$:</u>

Again, the case where $\underline{u}$ is continuous in D gives the equivalence of the strong and weak forms by an argument identical to that of theorem 2.4, and we may again limit ourselves to the case of $\underline{u}$ being discontinuous in D.

Let

$$J^i_\pm = \int_{D^\pm(T)} \left\{\frac{\partial}{\partial t} (u^i\phi^i) + \nabla\cdot(\underline{f}^i\phi^i)\right\}dtdv \ . \qquad (2.45)$$

As already shown, $\partial D^+(T)$ and $\partial D^-(T) \in \Sigma_N^{N+1}$, so the divergence theorem is applicable to both of these integrals

$$\Longrightarrow J^i_\pm = \int_{\partial D^\pm(T)} \phi^i(u^i, \underline{f}^i)\cdot(n_t, \underline{n})dS . \tag{2.46}$$

But as ϕ^i has compact support on $\partial(T\times D)$,

$$J^i_+ = \int_{\Gamma^+(T)} \phi^i(u^i, \underline{f}^i)\cdot(n_t, \underline{n})ds , \tag{2.47}$$

and

$$J^i_- = -\int_{\Gamma^-(T)} \phi^i(u^i, \underline{f}^i)\cdot(n_t, \underline{n})ds . \tag{2.48}$$

Applying (2.42) we obtain

$$J^i_+ + J^i_- = \int_{\Gamma(T)} \phi^i([u^i], [\underline{f}^i])\cdot(n_t, \underline{n})ds , \tag{2.49}$$

since ϕ^i is continuous across $\Gamma(T)$. Applying the product rule to (2.45) gives

$$J^i_\pm = \int_{D^\pm(T)} \left\{u^i\phi^i_t + \underline{f}^i\cdot\nabla\phi^i + \phi^i(u^i_t + \nabla\cdot\underline{f}^i)\right\}dtdv . \tag{2.50}$$

As ϕ^i is continuous across $\Gamma(T)$, we have

$$\left\{\int_{D^+(T)} + \int_{D^-(T)}\right\} (u^i\phi^i_t + \underline{f}^i\cdot\nabla\phi^i)dtdv$$

$$= \int_{T\times D} u^i\phi^i_t + \underline{f}^i\cdot\nabla\phi^i dtdv , \tag{2.51}$$

and

$$\left\{ \int_{D^+(T)} + \int_{D^-(T)} \right\} S^i\phi^i dtdv = \int_{T\times D} S^i\phi^i dtdv . \tag{2.52}$$

Equations (2.49),...,(2.52) combine to give

$$\int_{T\times D} (u^i\phi^i_t + \underline{f}^i\cdot\nabla\phi^i + S^i\phi)dtdv = \int_{\Gamma(T)} \phi^i([u^i], [\underline{f}^i])\cdot(n_t, \underline{n})ds$$

$$- \left\{ \int_{D^+(T)} + \int_{D^-(T)} \right\} \phi^i(u^i_t + \nabla\cdot\underline{f}^i - S^i)dtdV . \tag{2.53}$$

So equations (2.38), (2.53) and the assumption $u^i \in pwC^1[\mathbb{R}_+\times\Delta \longrightarrow \mathbb{R}_+]$ $\forall\ i \leq M$ give $(W^*) \Leftrightarrow \forall\ T \in \mathcal{I}^1$, $D \in \mathcal{I}^N$ such that $T \subset \mathbb{R}_+$, $D \subset \Delta$ and $\forall\ i \leq M$, $u^i \in pwC^1_*[\overline{T\times D} \longrightarrow \mathbb{R}]$, $\forall\ i \leq M$ $\forall\ \phi^i \in C^1_o[T\times D \longrightarrow \mathbb{R}]$,

$$\int_{\Gamma(T)} \phi^i([u^i], [\underline{f}^i])\cdot(n_t, \underline{n})ds - \left\{ \int_{D^+(T)} + \int_{D^-(T)} \right\} \phi^i(u^i_t+\nabla\cdot\underline{f}^i - S^i)dtdv = 0 \tag{2.54}$$

with the possibility $\Gamma(T) = \emptyset$ - the empty set.

From (2.54) we clearly infer $(S^*) \wedge (J) \Rightarrow (W^*)$ as the logical quantifications are satisfiable.

But also, $D^+(T)$ and $D^-(T)$ may be deformed arbitrarily or may become limitingly thin, hence

$$(W^*) \Rightarrow \forall\ (t,\underline{x}) \in \mathbb{R}_+\times\Delta\backslash\mathcal{I}^N(T) ,$$

$$u^i_t + \nabla\cdot\underline{f}^i - S^i = 0 \text{ by Lagrange's lemma,}$$

i.e. $(W^*) \Rightarrow (S^*)$

and $(W^*) \Longrightarrow \forall$ T, D as above,

$$\int_{\Gamma(T)} \phi^i([u^i], [\underline{f}^i]) \cdot (n_t, \underline{n}) ds = 0 . \qquad (2.55)$$

But as $\Gamma(T)$ and ϕ^i are arbitrary, (2.55) gives $(W^*) \Longrightarrow (J)$ again, by Lagrange's lemma.

■

2.3.5 Connectivity Conjectures

(W^*) and (I^*), although they are defined only for domains containing simple discontinuities, are logically equivalent to definitions defined over any countable union of domains containing simple discontinuities. It would seem, therefore, that they are applicable to any shock geometry as long as the shock configuration does not contain shock ends or shock collisions. It may be that even these restrictions are unnecessary as they only occur at isolated irregular points. However, these issues do not seem to be very imporrtant, so I have not pursued work on them any further.

2.4 Relationship to Tonti's Work

2.4.1 Description of Tonti's Structure

In his paper ([2]), Tonti describes a general structure for physical systems. His basic idea was to classify variables representing quantities into four types and relate each type with two others forming a closed system, see figure 5. A simple example is given in figure 6. This scheme was then shown to be applicable to a wide variety of physical theories, (see figure 7), and a mathematical theory of the general structure and its dual was formulated.

2.4.2 Attempt at a synthesis

The natural idea is of course to represent the general diffusive and nondiffusive equations of motion for a system of conservation laws within Tonti's structure. However, after considerable effort, I have found this to be impossible. It seems that Tonti's structure needs to be extended in order to relate it to conservation laws. Figures 8 and 9 show these extended structures for the diffusive and nondiffusive conservation laws respectively.

3. Further Formal Definitions

3.0 Introduction

It is well known that the incorporation of negative diffusion coefficients leads to ill-posed problems. However, for a system of diffusive conservation laws to remain well-posed, it shall be necessary to derive a stronger condition than simply non-negativity of the scale values for the diffusion coefficients (it is more akin to a generalised parabolo-ellipticity). A suitable definition of diffusivity is therefore formulated and then validated for the Navier-Stokes' equations.

The structure for a generalised system of conservation laws with diffusion may be used to construct a model equation for two-dimensional steady flow. This is the next formal definition provided.

Further to this, it is shown that the invariant property of the formal definition of diffusivity also holds for a stronger formal diffusivity condition. However, this second definition does ot seem to have the same applicability.

The next subject to be tackled is the admissibility of solutions to nondiffusive systems. Existing work in this field concentrates on simple cases, so the intention here is merely to conject on the general structure of formal admissibility criteria.

Further to this, a general formal definition of hyperbolicity for nondiffusive systems is derived, along with some comments on another feature of the system structure.

Finally, the possible formalisation of the concept of antidiffusion is discussed.

3.1 Diffusivity

For the formulation, we will need the strong form of the equations of motion with diffusion. Equations (2.15) and (2.18) give

$$(S) \Leftrightarrow \forall\ t \in \mathbb{R}_+ \quad \forall\ \underline{x} \in \Delta \quad \forall\ i \leq M\ ,$$

$$\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x_j}\left\{f^i_j(\underline{u}) - V^{il}_{jkm}(\underline{u})\,\frac{\partial u^l}{\partial x_k}\,d_m\right\} = S^i(\underline{u};t,\underline{x})\ , \tag{3.1}$$

(using summation convention).

A formal definition of diffusivity in the general spirit of [8] will be given. An alternative formulation could be along the lines of [9]. In [8], Garebedian concentrates on the highest order terms in the system. In (3.1) this gives us

$$V^{il}_{jkm}(\underline{u})d_m\,\frac{\partial^2 u^l}{\partial x_j \partial x_k}\ . \tag{3.2}$$

Now, consider a change of co-ordinates which preserves the time variable:

$$(t,\underline{x}) \longrightarrow (t,\underline{\xi})\ , \quad \text{with}\ \underline{\xi} = \underline{\xi}(t,\underline{x})\ . \tag{3.3}$$

Let

$$\underline{v}(t,\underline{\xi}) = \underline{u}(t,\underline{x})\ . \tag{3.4}$$

The term (3.2) may now be rewritten as

$$V^{il}_{jkm}(\underline{u})\ d_m\,\frac{\partial^2 v^l}{\partial \xi_p \partial \xi_q}\,\frac{\partial \xi_p}{\partial x_j}\,\frac{\partial \xi_q}{\partial x_k}\ . \tag{3.5}$$

Let M^{il} be the matrices given by

$$(M^{il})_{jk} = V^{il}_{ikm}(\underline{u})d_m \, , \tag{3.6}$$

and let N^{il} be the matrices given by

$$(N^{il})_{pq} = V^{il}_{jkm} \, d_m \frac{\partial \xi_p}{\partial x_j} \frac{\partial \xi_q}{\partial x_k} \, . \tag{3.7}$$

Thus the highest order term in (3.1) is now represented by

$$M^{il}_{jk} \frac{\partial^2 u^l}{\partial x_j \partial x_k} \tag{3.8}$$

and

$$N^{il}_{pq} \frac{\partial^2 v^l}{\partial \xi_p \partial \xi_q} \, . \tag{3.9}$$

But equations (3.6) and (3.7) give the relationships:

$$N^{il}_{pq} = M^{il}_{jk} \frac{\partial \xi_p}{\partial x_j} \frac{\partial \xi_q}{\partial x_k} \tag{3.10}$$

$$= (\nabla \underline{\xi})_{pj} (M^{il})_{jk} (\nabla \underline{\xi})_{qk} \, . \tag{3.11}$$

Thus

$$N^{il} = \nabla \underline{\xi} \, M^{il} (\nabla \underline{\xi})^T \, .$$

Hence, as $\nabla \underline{\xi}$, N^{il} and M^{il} are all square matrices, taking determinants g (3.12) gives

$$\det N^{il} = J(\underline{\xi})^2 \det M^{il} \, , \tag{3.13}$$

where $J(\underline{\xi})$ is the Jacobian of the transformation (3.3). When the transformation is well-defined,

$$0 < J(\underline{\xi}) < \infty \qquad \forall\ \underline{x} \in \Delta \qquad \forall\ t \in \mathbb{R}_+ . \tag{3.14}$$

This leads us to the formal definition of diffusivity:

Definition 3.1

A conservation law satisfying equation (3.1) is diffusive if

$$\forall\ t \in \mathbb{R}_+ , \qquad \forall\ \underline{x} \in \Delta , \qquad \forall\ i,l \leq M ,$$

$$\det M^{il} \geq 0 \tag{3.15}$$

where M^{il} is defined as in equation (3.6).

Lemma 3.1

The definition of diffusivity is well-defined since the sign of $\det M^{il}$ is invariant under any well-defined co-ordinate transformation (as already shown).

3.2 Validation for the Steady Navier-Stoke's Equations

3.2.1 One Dimension

The one-dimensional steady Navier-Stoke's equations are

$$\frac{d}{dx}\left[\underline{f}(\underline{u}) - \mu\underline{f}_v(\underline{u}) - \kappa\underline{f}_c(\underline{u})\right] = 0 \tag{3.16}$$

where

$$\underline{u} = \begin{bmatrix} \rho \\ u \\ T \end{bmatrix} \tag{3.17}$$

$$\underline{f} = \begin{bmatrix} \rho u \\ p(\rho,T) + \rho u^2 \\ u\left[\dfrac{\gamma p(\rho,T)}{\gamma - 1} + \frac{1}{2}\rho u^2\right] \end{bmatrix} \tag{3.18}$$

$$\underline{f}_v = \frac{4}{3}\begin{bmatrix} 0 \\ du/dx \\ udu/dx \end{bmatrix} \tag{3.19}$$

$$\underline{d}_c = \begin{bmatrix} 0 \\ 0 \\ dT/dx \end{bmatrix} \tag{3.20}$$

Within the previous formalism, we set

$$\underline{d} = \begin{bmatrix} \mu_o \\ \kappa_o \end{bmatrix} , \tag{3.21}$$

where μ_o, κ_o are scale values for μ and κ. In the one-dimensional case, the viscosity tensor is only rank 3 and is represented by

$$V_m^{il}(\underline{u}) .$$

By comparing (3.16) with (3.1) in this case, we obtain the relation:

$$\underline{V}_m^l(\underline{u}) \frac{du^l}{dx} d_m = \mu\underline{f}^v(\underline{u}) + \kappa\underline{f}^c(\underline{u}) , \tag{3.22}$$

where

$$(\underline{V}_m^l)^i = V_m^{il} . \tag{3.23}$$

Also, in this one-dimensional case, (3.6) reduces to

$$M^{il} = V_m^{il}(\underline{u})d_m \tag{3.24}$$

where M^{il} are now 1×1 matrices.

But (3.19) and (3.20) show that

$$\underline{V}_m^{il}(\underline{u})\,\frac{du^l}{dx}\,d_m = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{3}\mu & 0 \\ 0 & \frac{4}{3}\mu u & \kappa \end{bmatrix} \begin{bmatrix} d\rho/dx \\ du/dx \\ dT/dx \end{bmatrix} . \tag{3.25}$$

Hence the matrix of matrices (M^{il}) is given by

$$(M^{il}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{3}\mu & 0 \\ 0 & \frac{4}{3}\mu u & \kappa \end{bmatrix} . \tag{3.26}$$

Thus the formal diffusivity condition (3.15) holds provided $\forall\ t \in \mathbb{R}_+$ $\forall\ \underline{x} \in \Delta$,

$$\left.\begin{array}{l} \mu \geq 0 \\ u \geq 0 \\ \kappa \geq 0 \end{array}\right\} . \tag{3.27}$$

The only one of these conditions that may cause problems in validation is the condition $u \geq 0$. This will hold because the flow must be unidirectional in one dimension and by the normal choice of variables u is non-negative.

3.2.2 Two Dimensions

The two-dimensional steady Navier-Stokes' equations are

$$\frac{\partial \underline{F}}{\partial x} + \frac{\partial \underline{G}}{\partial y} = \underline{0} , \tag{3.28}$$

where

$$\underline{F} = \underline{f} - u\underline{f}_v - \kappa\underline{f}_c \tag{3.29}$$

$$\underline{G} = \underline{g} - \mu\underline{g}_v - \kappa\underline{g}_c \tag{3.30}$$

$$\underline{u} = (\rho, u, v, T)^T \tag{3.31}$$

$$\underline{f}(\underline{u}) = \begin{bmatrix} \rho u \\ p(\rho,T) + \rho u^2 \\ \rho uv \\ u\left[\frac{\gamma}{\gamma-1} p(\rho,T) + \frac{1}{2}\rho(u^2+v^2)\right] \end{bmatrix} \tag{3.32}$$

$$\underline{g}(\underline{u}) = \begin{bmatrix} \rho v \\ \rho uv \\ p(\rho,T) + \rho v^2 \\ v\left[\frac{\gamma}{\gamma-1} p(\rho,T) + \frac{1}{2}\rho(u^2+v^2)\right] \end{bmatrix} \tag{3.33}$$

$$\underline{f}_v(\underline{u}) = \begin{bmatrix} 0 \\ \frac{2}{3}(2u_x - v_y) \\ u_y + v_x \\ \frac{2}{3}u(2u_x - v_y) + v(u_y + v_x) \end{bmatrix} \tag{3.34}$$

$$\underline{g}_v(\underline{u}) = \begin{bmatrix} 0 \\ u_y + v_x \\ \frac{2}{3}(2v_y - u_x) \\ u(u_y + v_x) + \frac{2}{3}v(2v_y - u_x) \end{bmatrix} \tag{3.35}$$

$$\underline{f}_c(\underline{u}) = (0,0,0,T_x)^T \tag{3.36}$$

$$\underline{g}_c(\underline{u}) = (0,0,0,T_y)^T \, . \tag{3.37}$$

Conversion to the previous formalism is given by the following equations:

$$F^i = F_1^i , \qquad G^i = F_2^i . \tag{3.38}$$

$$f^i = f_1^i , \qquad g^i = f_2^i . \tag{3.39}$$

Equation (2.15) gives the relationships

$$F_1^i = f_1^i - V_{1km}^{il} d_m \frac{\partial u^l}{\partial x_k} , \tag{3.40}$$

$$F_2^i = f_2^i - V_{2km}^{il} d_m \frac{\partial u^l}{\partial x_k} . \tag{3.41}$$

$\underline{d}$ is again defined as in equation (3.21).

Hence we have

$$\mu f_v^i + \kappa f_c^i = V_{1km}^{il} d_m \frac{\partial u^l}{\partial x_k} , \tag{3.42}$$

$$\mu g_v^i + \kappa g_c^i = V_{2km}^{il} d_m \frac{\partial u^l}{\partial x_k} , \tag{3.43}$$

equation (3.6) then gives

$$\mu f_v^i + \kappa f_c^i = (M^{il})_{1k} \frac{\partial u^l}{\partial x_k} , \tag{3.44}$$

$$\mu g_v^i + \kappa g_c^i = (M^{il})_{2k} \frac{\partial u^l}{\partial x_k} . \tag{3.45}$$

Now, putting $i = 1$ in equations (3.44) and (3.45), we observe both left-hand sides are zero. Hence

$$\forall l, \qquad M^{1l} = 0 . \tag{3.46}$$

(Note M^{il} are now 2×2 matrices).

Also, as there are no ρ_x and ρ_y terms in the fluxes on the left-hand sides of equations (3.44) and (3.45) we also have:

$$\forall i, \qquad M^{i1} = 0 . \tag{3.47}$$

There are now only nine matrices to find - namely M^{il} for $2 \leq i,l \leq 4$. These we found by setting $i = 2,3,4$ in turn in equations (3.44) and (3.45):

<u>i = 2</u>

$$(3.44) \Rightarrow \frac{2}{3}\mu(2u_x - v_y) = (M^{22})_{11}u_x + (M^{22})_{12}u_y + (M^{23})_{11}v_x + (M^{23})_{12}v_y + (M^{24})_{11}T_x + (M^{24})_{12}T_y . \tag{3.48}$$

$$(3.45) \Rightarrow \mu(u_y + v_x) = (M^{22})_{21}u_x + (M^{22})_{22}u_y + (M^{23})_{21}v_x + (M^{23})_{22}v_y + (M^{24})_{21}T_x + (M^{24})_{22}T_y . \tag{3.49}$$

Hence (3.48), (3.49) $\Rightarrow$

$$M^{22} = \begin{bmatrix} \frac{4}{3} & 0 \\ 0 & \mu \end{bmatrix} \tag{3.50}$$

$$M^{23} = \begin{bmatrix} 0 & -\frac{2}{3}\mu \\ \mu & 0 \end{bmatrix} \tag{3.51}$$

$$M^{24} = 0 \tag{3.52}$$

$$(3.50) \Rightarrow \det M^{22} = \frac{4}{3}\mu^2 \tag{3.53}$$

$$(3.51) \Rightarrow \det M^{23} = \frac{2}{3}\mu^2 . \tag{3.54}$$

<u>i = 3</u>

$$(3.44) \Rightarrow \mu(u_y + v_x) = (M^{32})_{11}u_x + (M^{32})_{12}u_y$$

$$+ (M^{33})_{11}v_x + (M^{33})_{12}v_y + (M^{34})_{11}T_x + (M^{34})_{12}T_y \; . \qquad (3.55)$$

$$(3.45) \Rightarrow \frac{2}{3}\mu(2v_y - u_x) = (M^{32})_{21}u_x + (M^{32})_{22}u_y$$

$$+ (M^{33})_{21}v_x + (M^{33})_{22}v_y + (M^{34})_{21}T_x + (M^{34})_{22}T_y \; . \qquad (3.56)$$

(3.55), (3.56) $\Rightarrow$

$$M^{32} = \begin{bmatrix} 0 & \mu \\ -\frac{2}{3}\mu & 0 \end{bmatrix} , \qquad (3.57)$$

$$M^{33} = \begin{bmatrix} \mu & 0 \\ 0 & \frac{4}{3}\mu \end{bmatrix} , \qquad (3.58)$$

$$M^{34} = 0 \; , \qquad (3.59)$$

$$(3.57) \Rightarrow \det M^{32} = \frac{2}{3}\mu^2 \; . \qquad (3.60)$$

$$(3.58) \Rightarrow \det M^{33} = \frac{4}{3}\mu^2 \; . \qquad (3.61)$$

<u>i = 4</u>

$$(3.44) \Rightarrow \kappa T_x + \frac{2}{3}\mu u(2u_x - v_y) + \mu v(u_y + v_x) = (M^{42})_{11}u_x$$
$$+ (M^{42})_{12}u_y + (M^{43})_{11}v_x + (M^{43})_{12}v_y + (M^{44})_{11}T_x + (M^{44})_{12}T_y . \qquad (3.62)$$

$$(3.45) \Rightarrow \kappa T_y + \mu u(u_y + v_x) + \frac{2}{3}\mu v(2v_y - u_x) = (M^{42})_{21}u_x$$
$$+ (M^{42})_{22}u_y + (M^{43})_{21}v_x + (M^{43})_{22}v_y + (M^{44})_{21}T_x + (M^{44})_{22}T_y . \qquad (3.63)$$

(3.62), (3.63) $\Longrightarrow$

$$M^{42} = \begin{bmatrix} \frac{4}{2}\mu u & \mu v \\ -\frac{2}{3}\mu v & \mu u \end{bmatrix} \tag{3.64}$$

$$M^{43} = \begin{bmatrix} \mu v & -\frac{2}{3}\mu u \\ \mu u & \frac{4}{3}\mu v \end{bmatrix} \tag{3.65}$$

$$M^{44} = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} . \tag{3.66}$$

$$(3.64) \Longrightarrow \det M^{42} = \frac{2}{3}\mu^2(2u^2 + v^2) \tag{3.67}$$

$$(3.65) \Longrightarrow \det M^{43} = \frac{2}{3}\mu^2(u^2 + 2v^2) \tag{3.68}$$

$$(3.66) \Longrightarrow \det M^{44} = \kappa^2 . \tag{3.69}$$

Hence the formal definition of diffusivity is validated for the two-dimensional steady Navier-Stokes' equations as the determinants of all the matrices M^{il} are non-negative.

It is suspected that the three-dimensional steady flow equations may also be validated. Steady flow has been chosen for ease of computation, since $\underline{u}$ is here not defined as the conserved variables for unsteady flow, but as a simpler set of primitive variables.

3.3 A Two-Dimensional Single Equation Model of Diffusion

As stated in the abstract, it is hoped that this model equation will have a use similar to Burgers' equation. It is relevant, however, to two-dimensional steady flow rather than one-dimensional unsteady flow.

We start by taking the diffusive strong form of the equations of motion given by equations (2.15) and (2.18):

$$\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x_j}\left\{f^i_j(\underline{u}) - V^{il}_{jkm}(\underline{u})\,\frac{\partial u^l}{\partial x_k}\,d_m\right\} = S^i(\underline{u};t,\underline{x}) . \qquad (3.70)$$

The following simplifications are made:

$$\left.\begin{array}{ll} \text{i)} & N = 2 \;; \\ \text{ii)} & \partial/\partial t \equiv 0; \\ \text{iii)} & \alpha = 1 \;; \\ \text{iv)} & V^{il}_{jkm}(\underline{u}) \text{ is constant;} \\ \text{v)} & S^i = 0 \;; \\ \text{vi)} & M = 1 \;. \end{array}\right\} \qquad (3.71)$$

Let

$$f^1_1(u) = f(u), \text{ and } f^1_2(u) = g(u) . \qquad (3.72)$$

Let V^{11}_{jk1} be given by the constant matrix

$$V = \begin{bmatrix} A & B \\ B & C \end{bmatrix} . \qquad (3.73)$$

Note V is symmetric as, in this simplified case, the ordering of $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial x_k}$ is unimportant.

Equation (3.6) gives the relation

$$M_{jk} = d\,V_{jk} . \qquad (3.74)$$

Thus the diffusivity condition here is, assuming $d > 0$,

$$\left.\begin{array}{l} \det V \geq 0 \\ \text{i.e. } B^2 \leq AC \end{array}\right\} \qquad (3.75)$$

The equation of motion is now:

$$\frac{\partial}{\partial x}\left[f(u) - d\,V_{11}u_x - d\,V_{12}u_y\right] + \frac{\partial}{\partial y}\left[g(u) - d\,V_{21}u_x - d\,V_{22}u_y\right] = 0 \qquad (3.76)$$

i.e.

$$f(u)_x + g(u)_y = d(Au_{xx} + 2Bu_{xy} + Cu_{yy}) \; . \qquad (3.77)$$

- a model equation for two-dimensional steady flow.

3.4 Strong Diffusivity

The previously stated definition of diffusivity was based upon the invariance of the sign of the characteristic matrices multiplying the highest order derivatives within the system. It is a generalisation of the hyperbolic/parabolic/elliptic classification of a single second order partial differential equation. However, as will be shown, positive (negative) definiteness (semi-definiteness) is invariant under arbitrary co-ordinate transformations.

Definition 3.2

A conservation law satisfying equation (3.1) is strongly diffusive if $\forall\ t \in \mathbb{R}_+$, $\forall\ \underline{x} \in \Delta$, $\forall\ i,l \leq M$,

$$M^{il} \text{ is positive semi-definite}$$

i.e.

$$\forall \underline{y} \in \mathbb{R}^m, \qquad \underline{y}^T M^{il} \underline{y} \geq 0 . \tag{3.78}$$

<u>Lemma 3.2</u>

Strong diffusivity is well-defined as the condition (3.78) is invariant under a change of co-ordinates.

<u>Proof</u>

Equation (3.12) gives $N^{il} = \nabla\underline{\xi}\, M^{il}(\nabla\underline{\xi})^T$.

Let

$$P = \nabla\underline{\xi} . \tag{3.79}$$

If the co-ordinate transformation $\underline{x} \longrightarrow \underline{\xi}(t,\underline{x})$ is well-defined, P is non singular.

So we may define

$$\underline{z} = (P^T)^{-1}\underline{y} . \tag{3.80}$$

Then, (3.78) gives

$$(P^T\underline{z})^T M^{il} (P^T\underline{z}) \geq 0$$

$$\text{i.e.} \qquad \underline{z}^T P M^{il} P^T \underline{z} \geq 0$$

$$\text{i.e.} \qquad \underline{z}^T N^{il} \underline{z} \geq 0 . \tag{3.81}$$

Also, as P is invertible, $\text{span}\{\underline{y}\} = \text{span}\{\underline{z}\}$.

Hence the lemma is proved.

■

Note

Equations (3.51) and (3.57) show that the two-dimensional Navier-Stokes' equations do not obey this condition.

3.5 Diffusive Admissiblity

Diffusive admissiblity is the process of using a sequence of diffusive systems, $\Sigma(\underline{d})$, in order to choose a specific solution to the corresponding nondiffusive system, $\Sigma(\underline{o})$.

It will be necessary to introduce a new form of norm in order to characterise this convergence process. The paradigm illustrating this necessity is the way in which a sequence of tanh curves do not converge to their limiting function (a step function) in the normal sense. In order to overcome this, the norm described combines the dependent and independent variables in a way which would seem to be analogous to the use of Lagrangian variables. (An alternative norm is the total variation norm).

Two versions of the admissibility criterion are presented, one stronger than the other. They are, however, both weaker forms of convergence than uniform or pointwise convergence as already explained. It should be noted that there may be viscosity tensors for which the solution does not converge in the weak limit and there also may be solutions of $\Sigma(\underline{o})$ which are not limits of viscous solutions.

Once these criteria have been defined, a related condition on the viscosity tensor is described. First of all, this new norm is defined. It could be called a nondimensionalised cartesian combination norm, but that's a bit of a mouthful!

Definition 3.3

The norm $\|\cdot\|_{T,L,\underline{U}}: \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}^M \rightarrow \mathbb{R}_+$ is defined by:

$$\|(t,\underline{x},\underline{v})\|_{T,L,\underline{U}} = \left[\left(\frac{t}{T}\right)^2 + \left(\frac{|\underline{x}|}{L}\right)^2 + \sum_{i=1}^{M}\left(\frac{v^i}{U^i}\right)^2\right]^{\frac{1}{2}} \tag{3.82}$$

where $T, L, U^1, \ldots, U^M > 0$.

As in §2.2.5, let $\underline{u}_*(t,\underline{x};d)$ represent the solution to $\Sigma(\underline{d})$. We may now define the two forms of diffusive admissibility.

Definition 3.4

The undirected diffusively admissible solution, $\underline{u}_*(t,\underline{x};\underline{0})$ to $\Sigma(\underline{o})$ for the viscosity tensor $V^{il}_{jkm}(\underline{u})$ is that which obeys the condition:

$\forall\ T, L, U^1, \ldots, U^M > 0$, $\forall\ \epsilon > 0$, $\exists\ d(\epsilon) < d_o$ such that $\forall\ \underline{d}_1, \underline{d}_2$ such that $|\underline{d}_1|, |\underline{d}_2| < d(\epsilon)$, $\forall\ (t_1, \underline{x}_1)$, $\exists\ (t_2, \underline{x}_2)$ such that

$$\|(t_1, \underline{x}_1, \underline{u}_*(t_1, \underline{x}_1; \underline{d}_1)) - (t_2, \underline{x}_2, \underline{u}_*(t_2, \underline{x}_2; \underline{d}_2))\|_{T,L,\underline{U}} < \epsilon, \tag{3.83}$$

for some $d_o > 0$.

Definition 3.5

The directed diffusively admissible solution, $\underline{u}_*(t,\underline{x};\underline{o})$, to $\Sigma(\underline{o})$ for the direction $\hat{\underline{d}}$ $(|\hat{\underline{d}}| = 1)$ and the viscosity tensor $V^{il}_{jkm}(\underline{u})$ is that which obeys the condition:

$\forall\ T,L,U^1,\ldots,U^M > 0, \quad \forall\ \epsilon > 0, \quad \exists\ d(\epsilon) < d_o$ such that

$\forall\ d_1,d_2 < d(\epsilon), \quad \forall\ (t_1,\underline{x}_1)\ \exists\ (t_2,\underline{x}_2)$ such that

$$\|(t_1,\underline{x}_1,\underline{u}_*(t_1,\underline{x}_1;\ d_1\hat{\underline{d}})) - (t_2,\underline{x}_2,\underline{u}_*(t_2,\underline{x}_2;\ d_2\hat{\underline{d}}))\|_{T,L,\underline{U}} < \epsilon, \tag{3.84}$$

for some $d_o > 0$.

It should be noted that these two conditions are problem dependent. Independent conditions may be constructed by considering model problems such as plane wave solutions to the Riemann problem (see [11]).

In order to define the related condition on the viscosity tensor, it will be necessary to define some new concepts and variables. Starting from definition 2.8, the strong form (without quantification) of the equations of motion is

$$\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x_j}\left\{f^i_j(\underline{u}) - V^{il}_{jkm}(\underline{u})\,\frac{\partial u^l}{\partial x_k}\,d_m\right\} = S^i(\underline{u};t,\underline{x}) \ . \tag{3.85}$$

Now, let $A_j(\underline{u})$ be the flux matrices defined by

$$\frac{\partial \underline{f}_j}{\partial \underline{u}} = A_j(\underline{u}) \tag{3.86}$$

where

and

$$\left.\begin{aligned} (\underline{f}_j)_i &= f^i_j \\ (\underline{u})_i &= u^i \end{aligned}\right\} . \tag{3.87}$$

Now, consider a change of variables $\underline{u} \longrightarrow \underline{v}_j(\underline{u})$ which has the property of diagonalising the flux matrix $A_j(\underline{u})$. So, if $P_j(\underline{u})$ is defined by

$$P_j(\underline{u}) = \frac{\partial \underline{v}_j}{\partial \underline{u}} \, . \tag{3.88}$$

we require

$$P_j(\underline{u})A_j(\underline{u})P_j^{-1}(\underline{u}) = \Lambda_j(\underline{u}) \, , \tag{3.89}$$

where $\Lambda_j(\underline{u})$ is a diagonal matrix and (3.89) is not summed over j.

The related condition on the viscosity tensor corresponds to the directed form of the admissibility criterion. It is therefore necessary to collapse the rank of the viscosity tensor. Let $W_{jk}(\underline{u})$ be the matrix with the property

$$(W_{jk})^{il} = V^{il}_{jkm}(\underline{u})\hat{d}_m \tag{3.90}$$

for all $j,k \leq N$.

<u>Definition 3.6</u>

The related condition on the set of viscosity matrices $W_{jk}(\underline{u})$ is: for all transformations $\underline{u} \longrightarrow \underline{v}_j(\underline{u})$ satisfying the condition (3.89),

$$\forall \; j,k \quad P_j(\underline{u})W_{jk}(\underline{u})P_k^{-1}(\underline{u}) \text{ is positive definite,} \tag{3.91}$$

(not summed over j or k).

Motivation for this definition comes from Pego's result ([11]), showing that, on the case $N = 1$, this related condition, when holding in a neighbourhood of a fixed value $\underline{u}_o$, ensures that the directed diffusively admissible condition for the corresponding viscosity tensor is equivalent to the entropy condition for the non-diffusive solution for weak shock solutions to the Riemann problem. The entropy condition will be discussed, along with other conditions, in §4.2.

Note

From equation (3.90),

$$(W_{jk})^{il} = V^{il}_{jkm}\hat{d}_m ,$$

but equation (3.6) gave

$$(M^{il})_{jk} = V^{il}_{jkm}d_m .$$

Thus

$$|\underline{d}|(W_{ik})^{il} = (M^{il})_{ik} . \qquad (3.92)$$

Also the diffusivity conditions related to M^{il} and the corresponding condition on V^{il}_{jkm} related to W_{jk}. So we see that in some sense these are duals of each other.

Also the admissibility conjecture can be seen intuitively by noting that the matrices $P_j(\underline{u})$ have the effect of diagonalising the respective flux matrices $A_j(\underline{u})$ in the sense of (3.89). A diagonalised flux matrix corresponds to a decoupled system, so it is natural to expect the right-hand side to be at least positive semi-definite in this situation. Hence the definition 3.6.

3.6 Hyperbolicity

We start here by again quoting the general structure of a nondiffusive system of conservation laws. Equation (2.35) gave

$$\frac{\partial u^i}{\partial t} + \frac{\partial}{\partial x_j} (\underline{f}_j^i(\underline{u})) = S^i(\underline{u};t,\underline{x}) .$$

Using equation (3.86), this may be written in the alternative form:

$$\frac{\partial \underline{u}}{\partial t} + A_j(\underline{u}) \frac{\partial \underline{u}}{\partial x_j} = \underline{S}(\underline{u};t,\underline{x}) . \tag{3.93}$$

We now quote Majda's definition of hyperbolicity (see [1], p 10).

Definition 3.7

For an arbitrary unit vector, $\underline{\omega}. \in \mathbb{R}^N$, we define the following matrix

$$A(\underline{u};\underline{\omega}) = A_j(\underline{u})\omega_j ; \tag{3.94}$$

the system of conservation laws (3.93) is then said to be hyperbolic if and only if

$$\forall \ \underline{w} , \quad A(\underline{u};\underline{\omega}) \text{ has real eigenvalues.} \tag{3.95}$$

An extension to this definition would be to also constrain the eigenvalues to be distinct (this is discussed at length for he case of a single space dimension in [16]). However, it is not clear whether this constraint is satisfiable in the multi-dimensional case. In any case, a useful lemma is the following:

Lemma 3.3

The hyperbolicity condition, with or without the extra condition on the eigenvalues, is invariant under an arbitrary well-defined time independent co-ordinate transformation.

Proof

We consider the arbitrary co-ordinate transformation given by equations (3.3) and (3.4), except that we now have

$$\underline{\xi} = \underline{\xi}(\underline{x}) . \qquad (3.96)$$

The equation system (3.93) may now be transformed to

$$\frac{\partial \underline{v}}{\partial t} + \tilde{A}_j(\underline{v}) \frac{\partial \underline{v}}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_j} = \tilde{\underline{S}}(\underline{v};\underline{\xi},t) , \qquad (3.97)$$

where

$$\left.\begin{array}{l} \tilde{A}_j(\underline{v}(\underline{\xi}(\underline{x}),t) \equiv A_j(\underline{u}(\underline{x},t)) \\ \tilde{\underline{S}}(\underline{v}(\underline{\xi}(\underline{x}),t);\underline{\xi}(\underline{x}),t) = \underline{S}(\underline{u}(\underline{x},t);\underline{x},t) \end{array}\right\} , \qquad (3.98)$$

Let

$$B_k = \tilde{A}_j \frac{\partial \xi_k}{\partial x_j} . \qquad (3.99)$$

The corresponding hyperbolicity condition on the system in $\underline{v}$ is

$\forall\ \underline{\omega}$ such that $|\underline{\omega}| = 1$,

$B(\underline{\omega}) = B_k \omega_k$ has real (distinct eigenvalues).

But,

$$B(\underline{\omega}) = \tilde{A}_j \frac{\partial \xi_k}{\partial x_j} \omega_k ,$$

i.e.,

$$B(\underline{\omega}) = \tilde{A}_j((\nabla \underline{\xi})^T \underline{\omega})_j . \tag{3.100}$$

As $\underline{x} \mapsto \underline{\xi}$ is a well-defined transformation, $\nabla\underline{\xi}$ is always finite and invertible. Thus

$$\forall\ \underline{x}, \quad \text{span}\{\underline{\omega}\} = \text{span}\{(\nabla\underline{\xi})^T\underline{\omega}\} . \tag{3.101}$$

Hence the hyperbolicity condition for $B(\underline{\omega})$ is inferred from that for $A(\underline{\omega})$.

■

Note that $B(\underline{\omega})$ does not in general depend on $\underline{\xi}$ and t only through $\underline{v}(\underline{\xi},t)$, see equation (3.99). Hence the structure given by equation (3.93) is only invariant for a restricted class of co-ordinate transformations with

$$\nabla\underline{\xi} = \nabla\underline{\xi}(\underline{v}(\underline{\xi}(\underline{x},t),t)) . \tag{3.102}$$

Furthermore, in the case when $\underline{\xi} = \underline{\xi}(\underline{x})$, the structure is only invariant for linear transformations (as can be easily shown).

3.7 Antidiffusion

As already discussed, one useful measure of diffusion is the signs of the determinants of the characteristic matrices M^{il}. However, there are other possibilities. Entropy can be used to measure diffusivity because diffusion is a dominating process within shock waves and entropy increases through a shock while otherwise remaining constant. Pike ([13]) has shown that taking Pia's formula for viscosity ([14]) leads to a spiked entropy profile for the one-dimensional steady Navier-Stokes' equations, see figure 10. One may infer that this corresponds to a region of antidiffusion lying within an outer region of diffusion.

Another way of viewing the problem is by using the method of characteristics. For a single equation in two dimensions (be it space-time or space-space), the characteristic surface may exhibit a Riemann-Hugoniot (or cusp) catastrophe corresponding to the weak end of the shock wave, see [7] and [15]. Normally, the overturned characteristic manifold is fitted with a discontinuity curve corresponding to the Rankine-Hugoniot jump conditions. However, if this is not assumed, the three-valued function region could be interpreted as a region of antidiffusion.

Formal results concerning shock formation and catastrophe theory will be given in a later report.

4. A Survey of the Mathematical Treatment of Conservation Laws with Limiting Diffusion

4.0 Introduction

The mathematical treatment of conservation laws is compelled to possess a close relationship with continuum physics in order to maintian its applicability. This relationship is summerised below.

As already mentioned, nonlinear conservation laws permit the formation of shock waves in finite time. The mathematical treatment of these shocks in a nondiffusive setting necessitates the introduction of discontinuous function spaces. This introduction, however, leads to the possibility of the non-uniqueness of solutions (see [16], p.8). Hence, some means of distinguishing between possible solutions is required. The ones that are selected are generally called admissible solutions. Thus admissibility criteria need to be established and compared.

One selection criterion, as already mentioned in §3.5, is to admit solutions which are limits of the corresponding diffusive system. This would seem to be a natural resolution, but the issue is not this simple as there are other 'natural' choices (for example, those that correspond in some way to physical entropy), and also, the diffusive admissibility criterion is too weak on its own in the case of systems (see [11]).

In terms of the mathematical theory itself, there are two things to note. Firstly, it is often necessary to derive a function space for the solution which is, in some sense, invariant (usually invariant in time), the invariance property in this context is commonly called regularity. Secondly, there would seem to be a tension between the (usually meansure theoretic) nonconstructive analysis of these systems and the derivation of constructive algorithms. Obviously, both endeavours have their place, but there often seems to be a bias to one or the other within accounts in the literature.

4.1 Admissibility Criteria

4.1.0 Introduction

An excellent summary of the different admissibility criteria is given by Dafermos in [3]. He shows all the standard possible criteria, their inter-relationship, and their relationship to physics. The emphasis here in this paper is on the nondiffusive limit, so the corresponding admissibility criterion is treated first and separately.

4.1.1 Non-Diffusive Limit Admissibility

The formulation for this problem was given in §3.5. The only relevant points here would seem to be as follows: firstly, that the criterion depends upon the choice of viscosity tensor and the sequence of diffusion scale coefficients; and secondly, that the criterion is often equivalent to the entropy criterion (see below and [17]).

4.1.2 Alternative Admissibility Criteria

The three main alternative criteria to the nondiffusive limit criterion are: the entropy criterion, the shock criterion and the entropy rate criterion. The entropy criterion is a model of physical entropy and the second law of thermodynamics. This basically means the existence of an entropy function that increases across shock waves (i.e. in the direction of the flow).

The shock admissibility criterion is based upon the concept of characteristics converging onto (rather than diverging from) a shock wave. Two alternative forms of the criterion are given by Lax ([16]) and Liu ([18]).

The entropy rate criterion is based on the idea that entropy is not only increasing, but that it is increasing as fast as it can under its constraints. This criterion only seems to have been discussed by Dafermos in [3] and does not seem to be used much in practise.

4.2 Issues Concerning Convergence

4.2.0 Introduction

This section describes the analytical and numerical convergence of systems of conservation laws and the convergence of shock waves and their corresponding jump conditions in the nondiffusive limit.

4.2.1 Analytical Convergence of Solutions

For a single equation on one dimension, Oleĭnik ([19]) has shown (in an extensive study) the total variational convergence for all finite time under reasonable assumptions.

The case of two conservation laws (again in one dimension) was first tackled by Conley & Smoller in [11] for the Riemann problem. They give conditions on the matrix of diffusion coefficients (similar to definition 3.6 in this simplified setting) in order to guarantee convergence and a counter-example which does not converge for a wide class of data.

The general case of several conservation laws has been studied by DiPerna ([20]) and Pego ([12]). Pego's work also covers the general case of several space dimensions. He derives conditions and counter-examples analogous to Conley and Smoller's. One of his counter-examples seems to add weight to the thesis that undirected convergence is impossible but directed convergence is sometimes possible (cf. definitions 3.4 and 3.5 in §3.5).

Two existence theorems for systems of conservation laws with dissipation are given by Hoff in [21].

4.2.2 Numerical Convergence of Solutions

There are at least three factors relevant to this problem. Firstly; the process of discretizing a continuous system may introduce dissipative effects (see [22], pp. 108-119). This may also be viewed in the opposite way - certain diffusive terms may be added to a nondiffusive system which have an exact discretization (called the modified equation method).

Secondly, it has been found that the incorporation of artificial diffusion terms into a numerical scheme can improve stability and convergence (see [23]).

Thirdly, other exponents stress the relevance of the physical origins of diffusive terms within a system of equations as the numerically approximated nondiffusive system will often itself be an analytical approximation of a diffusive system (although they have different time scales). It is argued that making these dissipative terms more 'physical' will lead to improved stability and convergence necessarily as the physical process is stable (see [24]).

4.2.3 Convergence of Shock Waves

The issue in question here is how (if at all) do shock wave profiles in systems with diffusion converge to discontinuous profiles in the corresponding system without diffusion. Whitham ([4], §4) has studied the case of viscous shock convergence for Burgers' equation. Exact solutions do not seem to have been studied in this way for more complicated systems (see, however, a later report in this series for generalizations of the Cole-Hopf transformation). In view of this fact and the difficulty of locating shock waves analytically (as they are a feature internal to the flow), it appears that little other work has been done in this field. Haberman ([15] and [25]) sets up a framework for analysing this issue but is concerned with a slightly different problem.

4.2.4 Convergence of Jump Conditions

More work has been carried out in this field than in that of the previous section. The one-dimensional locally steady case has been studied by Courant and Friedrichs ([17] pp. 135-137). They show how the Navier-Stokes' equation lead to the appropriate form of the Rankine-Hugoniot jump conditions in this case. However Dulikravich et.al. ([26]) claim that the Rankine-Hugoniot jump conditions are only possible if Stokes' hypothesis is enforced.

Whitham ([4], §4) shows the convergence of the jump conditions, again for Burgers' equation. Haberman ([15] and [25]) gives convergence for the weak end of shocks in space-time. Initial analysis by the author would seem to suggest that weak shocks in space behave the same way (even more than for strong shocks). More about this will be said in a later report.

4.3 The Two Approaches to Regularity Without Diffusion

4.3.1 The Nonconstructive Approach

Within the nonconstructive approach to regularity there are two distinct sub-approaches. The first approach consists of requiring the function space to be the class of piecewise continuous (usually infinitely differential) functions and then attempting to discern what conditions are necessary on the flux functions and the initial data to ensure regularity. Examples in the case of a sigle equation are the work of Schaeffer ([7]) and Dafermos ([27]). According to DiPerna ([6]), a C^{∞} condition on the initial data is insufficient to guarantee pwC^{∞} behaviour for all time.

The second approach is to use a weaker function space than the space of piecewise continuous functions and employ measure theoretic arguments to ensure regularity and to obtain insight into the behaviour of shock waves within the domain.

It is well accepted (e.g., see [5], [6]) that the weakest space exhibiting regularity and worthy of analysis is the space of functions of bounded variation in the sense of Cesari (see [28]). In this case, DiPerna ([6]) cites that the two-dimensional space-time domain may be partitioned into three basic regions for a single equation, namely those of continuity, discontinuity and irregularity. These three regions contain successively one less dimension in terms of their Hausdorff measure. The region of discontinuity may itself be partitioned into an at least countable union of Lipschitz continuous curves (in the case when Δ is infinite). Also, the appropriate form of the Rankine-Hugoniot jump conditions is obeyed along these curves. Nothing is known about the behaviour of the function in the regions of irregularity, but this is not serious as these are just isolated points at the ends of the shock curves in this case. These results do not seem to have been generalised to higher dimensional problems or to systems.

Other stronger function spaces that have been analysed are integer Sobolev spaces (see [1] and [29]), non-integer Sobdev spaces (also called Besov spaces, see [5] and [29]) and uniformly local Sobolev spaces ([1]).

A new method relevant to these problems is the method of compensated compactness (see [30]) for an extensive overview). I can't make head or tail of it myself ('so it really is a bit of a Tartar!' - M.J. Baines).

4.3.2 The Constructive Approach

As is well known, the two basic approaches to providing algorithms to solve systems of conservation laws are shock capturing and shock fitting. The most important shock capturing method lending itself to analysis is Glimm's random choice method (as the method uses piecewise constant data, it has some features of both shock capturing and shock fitting), see [31].

In shock fitting algorithms it is always assumed that the exact solution is a piecewise continuous function of space and time. An exact solution of a simple model characteristic equation shows that the initial formation of shockwaves is an example of the cusp catastrophe (see [15] and [32]) and the ensuing discontinuity may be fitted by using the Rankine-Hugoniot jump conditions with logical consistency. In practice, this is the only method employed for the propagation of discontinuous shock fronts (yet catastrophe theory is not generally used in practice to describe the initial breaking of waves). This method works well in cases of approximate radial expansion (where the surface will propagate normally). However, in certain circumstances (for example a rotating bow shock), this method may well lack accuracy and it would be expedient to define a shock velocity in some constructive sense in order to predict the new position of the shock wave. I intend to address both these deficiencies in a later report.

5. Conclusions

In this report a formal basis for discussion concerning systems of conservation laws in several dimensions is given. This has led to equivalence proofs for the different forms of expressing the equations of motion and formal definitions such as those of diffusivity and admissibility. Finally, a survey of work done on the mathematical treatment of conservation laws with limiting diffusion has been presented, with several areas I intend to follow up in later reports.

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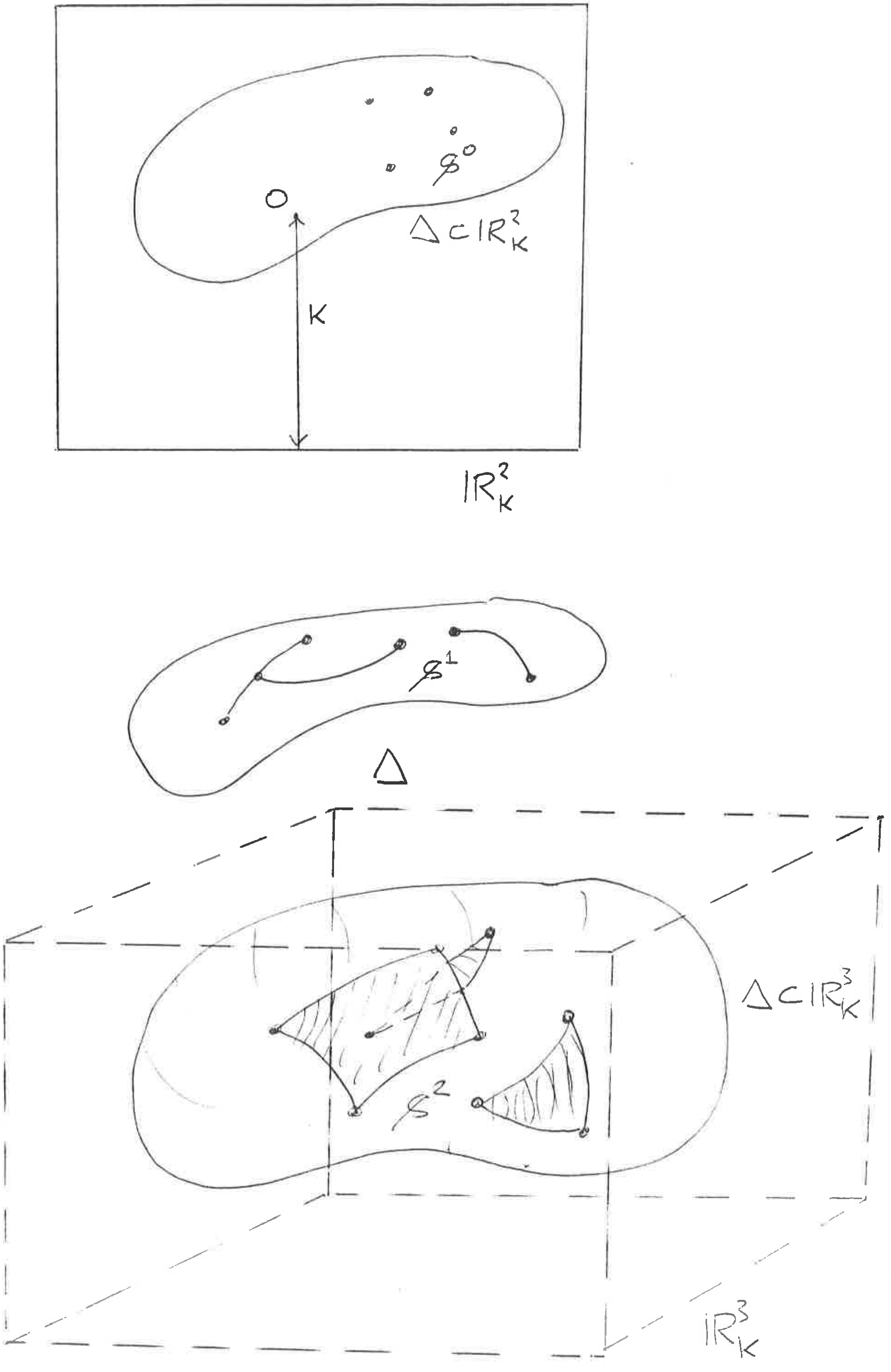

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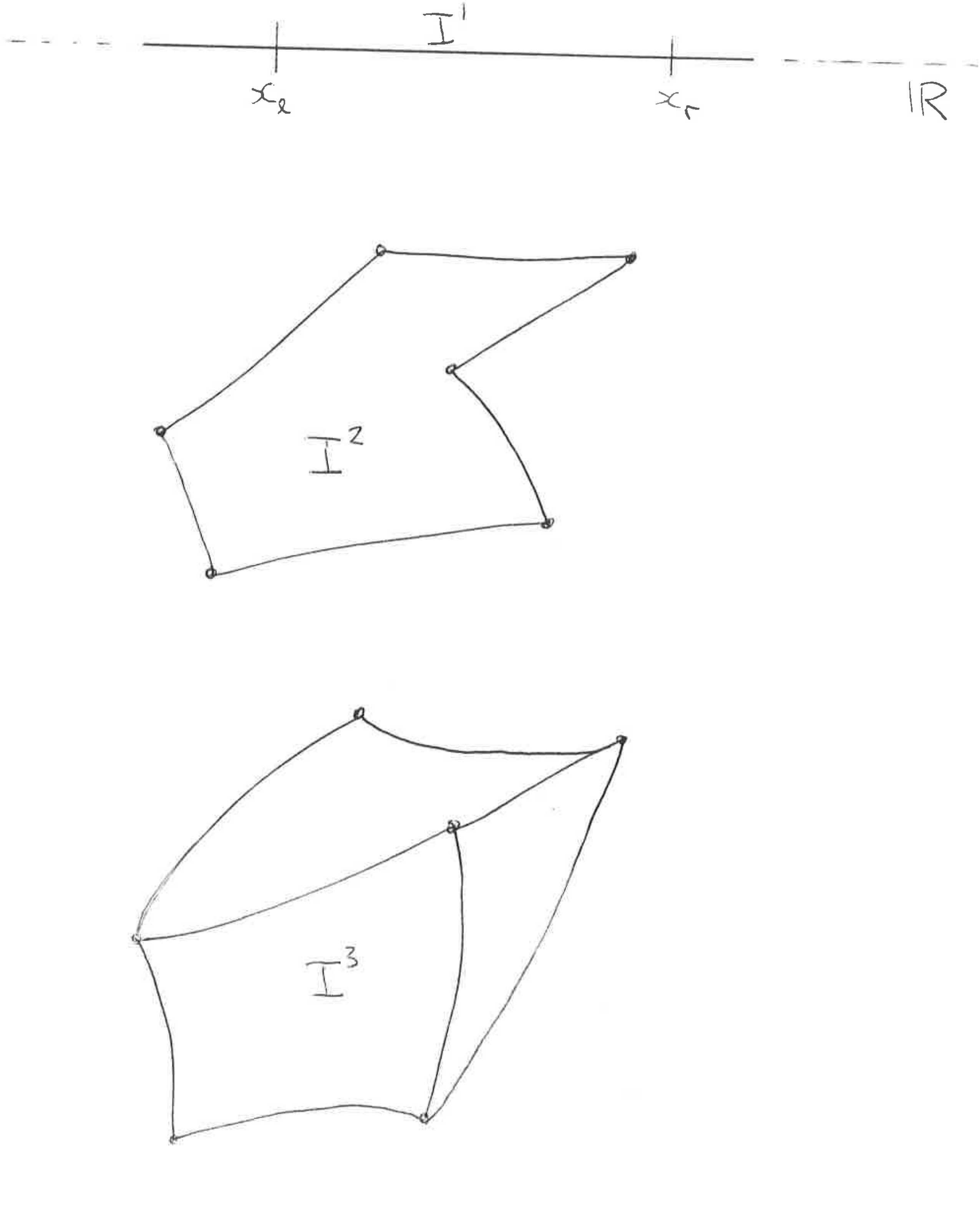
Figure 1.



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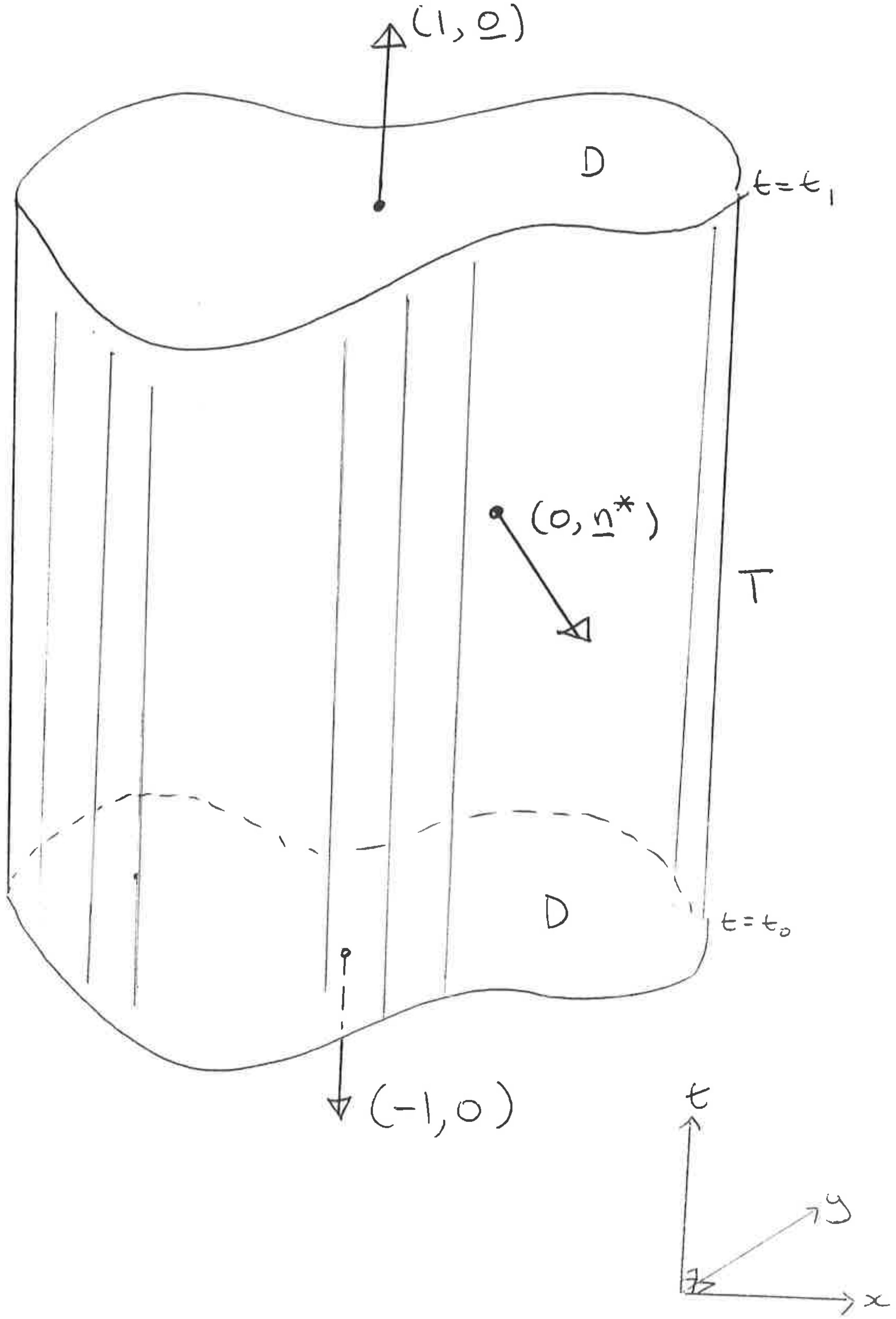

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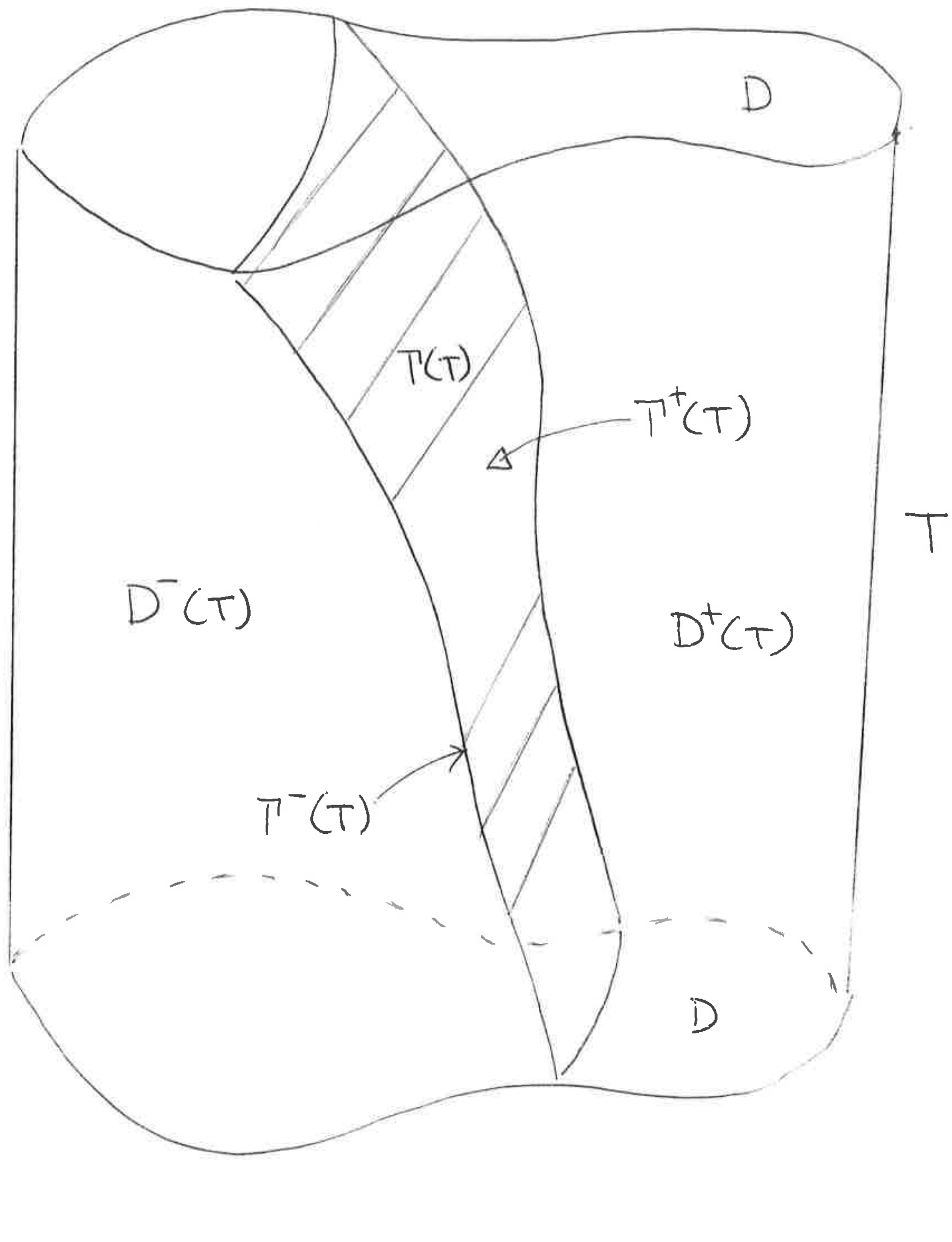

Figure 4.

Tonti's Diagram for the Structure of Physical Theories

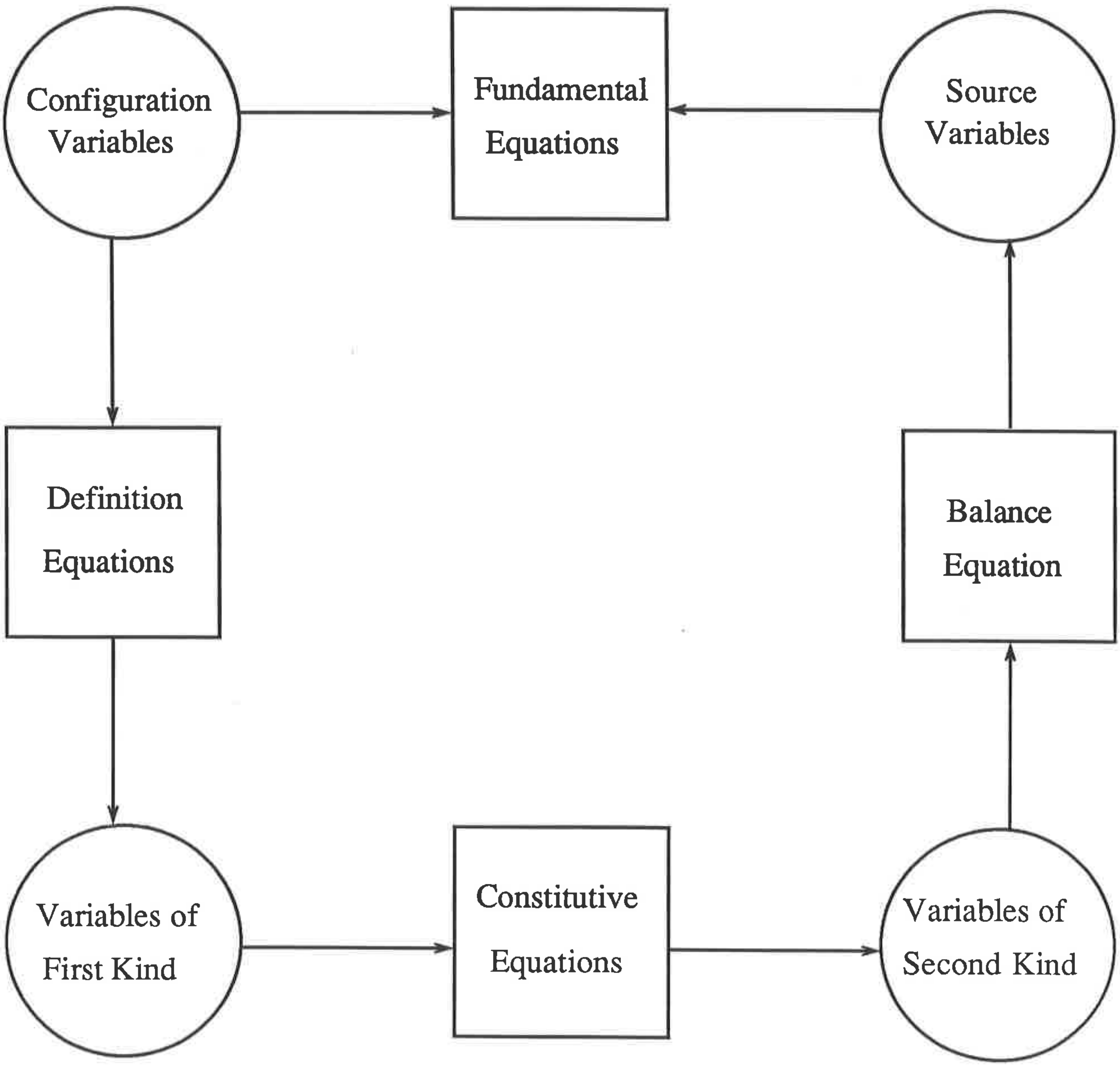

Figure 5

Tonti's Diagram for Maxwell's Equations

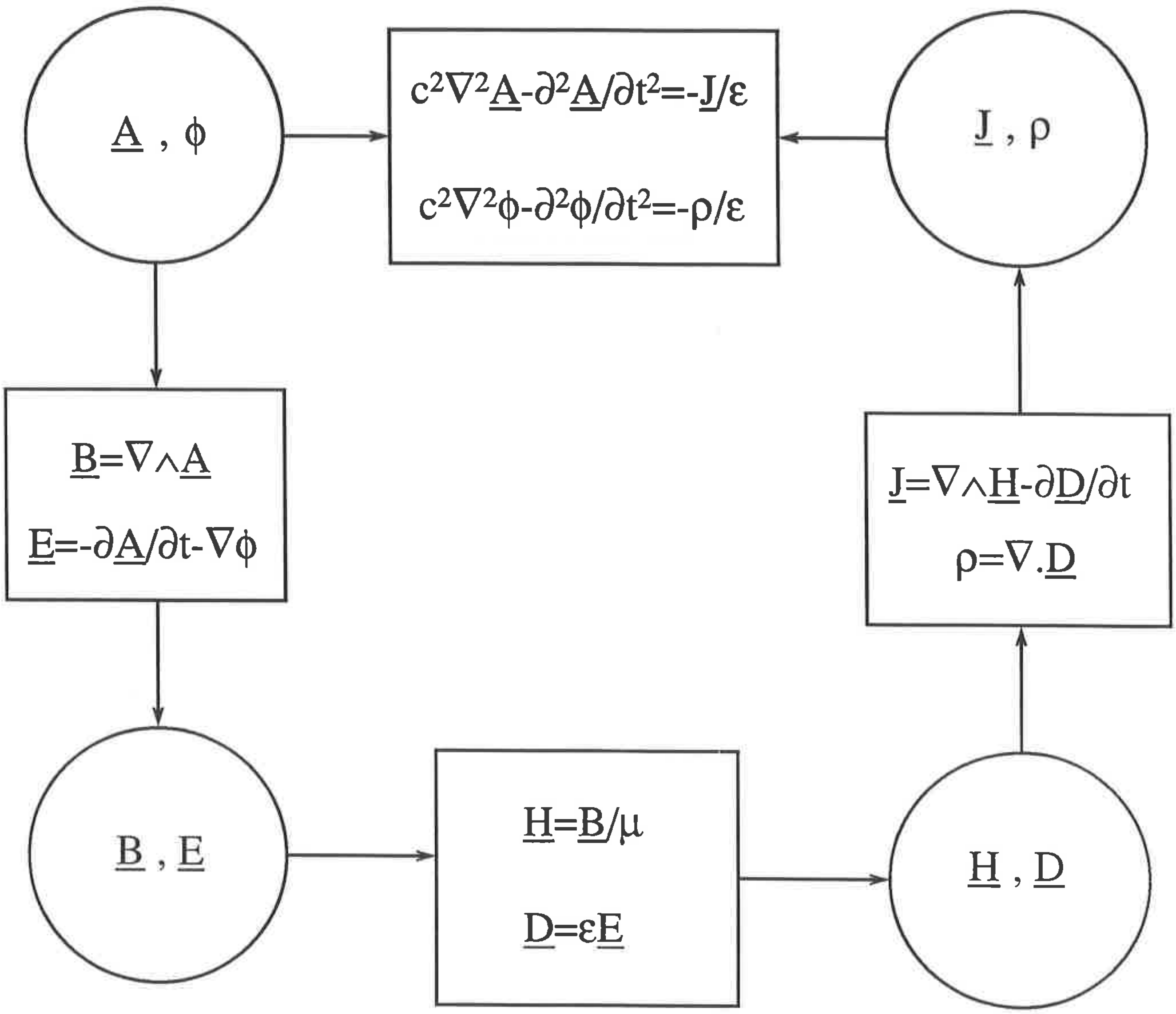

Figure 6

Tonti's Structure for some Physical Theories

Theory	Definition Equations	Constitutive Equations	Balance Equations
Classical Particle Dynamics	$v_i = \frac{dx_i}{dt}$	$p_i = mv_i$	$f_i = \frac{dp_i}{dt}$
Relativistic Particle Dynamics	$V^\alpha = \frac{dX^\alpha}{d\tau}$	$P_\alpha = m_0 g_{\alpha\beta} V^\beta$	$F_\alpha = \frac{dP_\alpha}{d\tau}$
Electro-magnetism	$B_i = \varepsilon_{ijk}\partial A_k/\partial x_j$ $E_i = -\partial A_i/\partial t - \partial\phi/\partial x_i$	$H_i = B_i/\mu$ $D_i = \varepsilon E_i$	$J_i = \varepsilon_{ijk}\partial H_k/\partial x_j - \partial D_i/\partial t$ $\rho = \partial D_i/\partial x_i$
Elastodynamics	$2e_{ij} = \partial u_i/\partial x_j + \partial u_j/\partial x_i$ $v_i = \partial u_i/\partial t$	$\sigma_{ij} = C_{ijkl}e_{kl}$ $p_i = \rho v_i$	$f_i = -\partial\sigma_{ij}/\partial x_j + \partial p_i/\partial t$
Heat Conduction	$p_i = -\partial T/\partial x_i$ $X = \partial T/\partial t$	$Q_i = \lambda(T)p_i$ $\Phi = c_v\rho X$	$\Sigma = \partial Q_i/\partial x_i + \partial\Phi/\partial t$
Quantum Mechanics	$v_i = \partial\psi/\partial x_i$ $\chi = -\partial\psi/\partial t$	$q_i = \hbar^2 v_i/2m$ $\phi = i\hbar\chi$	$\sigma = \partial q_i/\partial x_i + \partial\phi/\partial t$

Figure 7

Tonti's Diagram Extended for Diffusive Conservation Laws

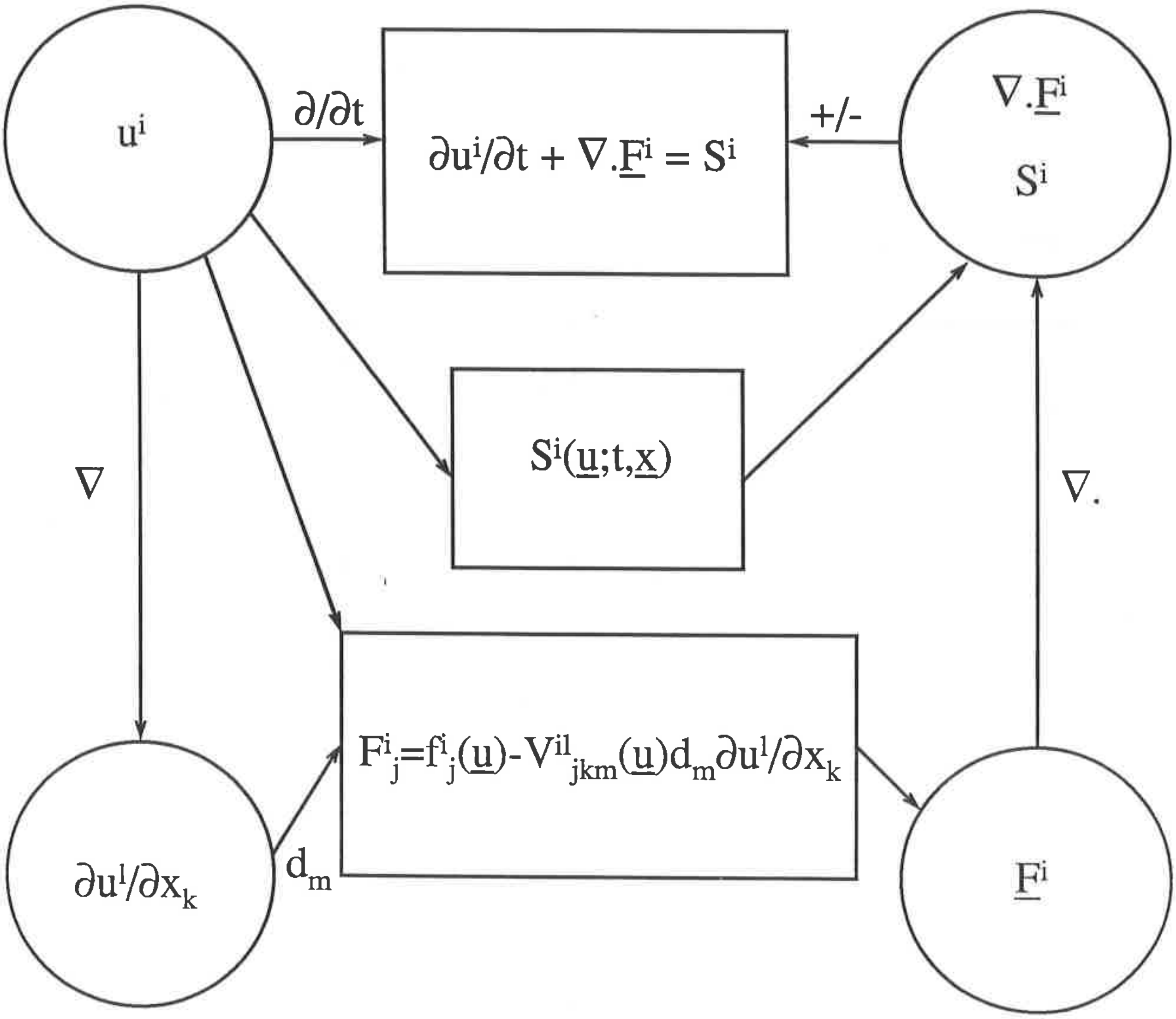

Figure 8

Tonti's Diagram Extended for Nondiffusive Conservation Laws

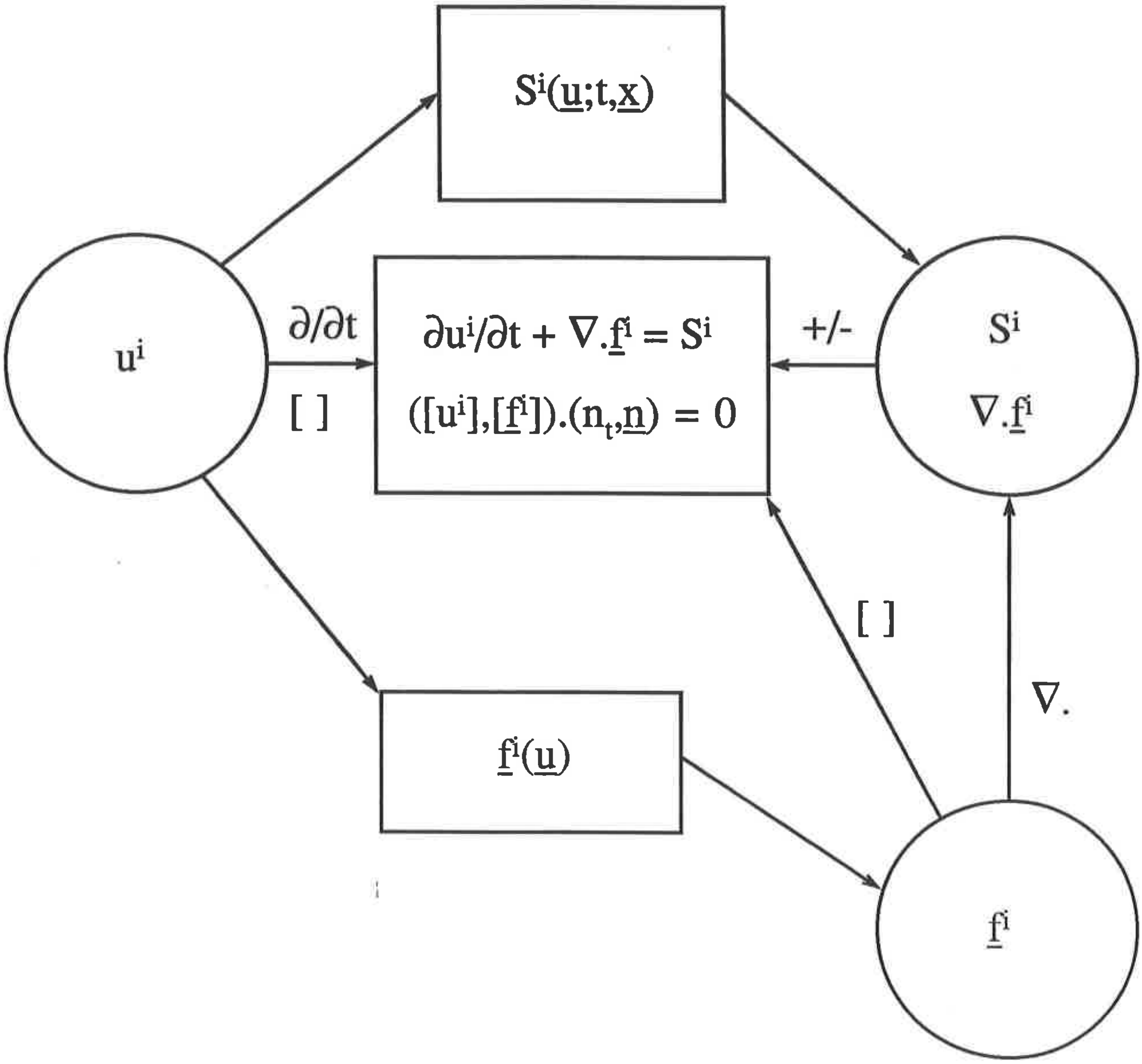

Figure 9

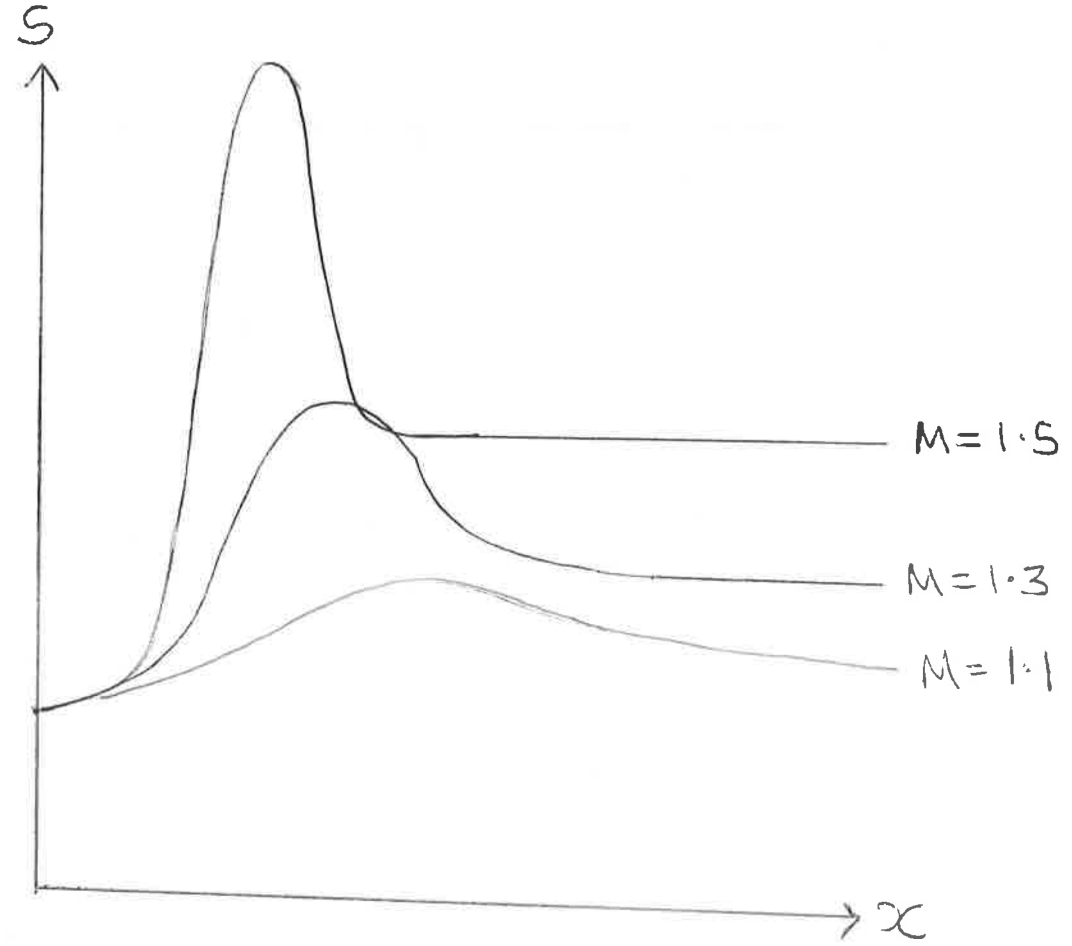

Figure 10