Two Dimensional Shock Recognition
and Roe's Scheme

M. J. Baines

Department of Mathematics, University of Reading
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ABSTRACT

This report discusses shock recognition for two dimensional conservation laws and a generalisation of Roe's scheme for one dimensional systems which concentrates on shock capture.

Two Dimensional Shock Recognition and Roe's Scheme

§1. Introduction

In Roe's scheme [1] for shocked flows in one space dimension, the so-called fluctuation

$$\phi = F_R - F_L \tag{1.1}$$

in an interval (L,R), written in terms of the Roe matrix $\tilde{A}(u_R,u_L)$ (see [1]) as

$$F_R - F_L = \tilde{A}(u_R,u_L)\,(u_R - u_L) \tag{1.2}$$

recognises a shock when it matches the jump condition

$$F_R - F_L = S(u_R - u_L) \tag{1.3}$$

i.e. when $u_R - u_L$ is an eigenvector of $\tilde{A}(u_R,u_L)$ and S (the shock speed) is the corresponding eigenvector. In the above L and R denote left and right states of the interval, u is the dependent variable and F the flux function for the hyperbolic system

$$u_t + F_x = 0 \quad . \tag{1.4}$$

By isolating the components of ϕ in the directions of the eigenvectors of $\tilde{A}(u_R,u_L)$ Roe's algorithm recognises each simple wave component of a shock wave exactly (to grid resolution), and is then able to advance values of u in time in accordance with the appropriate upwind direction (given by the sign of the corresponding eigenvalue).

In this report we seek a generalisation of the shock recognition aspect of Roe's scheme in two dimensions. There are two new problems arising from the additional dimension. One is the estimation of the inclination of the shockwave to the grid. The other is the generalisation of the algorithm itself, in particular, the choice of a generalised Roe matrix.

§2. Rotated Fluxes

Consider the two dimensional scalar equation

$$u_t + F_x + G_y = 0 \ , \tag{2.1}$$

where F and G are functions of u only. In order to discuss discontinuities we consider inclinations of the co-ordinate system such that the divergence of (F,G) comes entirely from a one-dimensional jump.

In a new co-ordinate system (n,t) , rotated through an angle θ with respect to (x,y) , let (F,G) become $(\mathcal{F},\mathcal{G})$. Then if the divergence comes entirely from a jump in the n direction we have

$$F_x + G_y = \mathcal{F}_n \qquad \mathcal{G}_t = 0 \ . \tag{2.2}$$

The second of these gives

$$\left(-\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}\right) \left(-\sin\theta F + \cos\theta G\right) = 0 \ , \tag{2.3}$$

which leads to

$$\sin^2\theta F_x - \cos\theta\sin\theta\left(F_y + G_x\right) + \cos^2\theta G_y = 0 \ . \tag{2.4}$$

If we can find a θ for which this is satisfied then from (2.2) and (2.3)

$$\mathcal{F}_n = \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right) \left(\cos\theta F + \sin\theta G\right) \tag{2.5}$$

$$= \cos^2\theta F_x + \cos\theta\sin\theta\left(F_y + G_x\right) + \sin^2\theta G_y \tag{2.6}$$

$$= F_x + G_y \ , \tag{2.7}$$

consistent with (2.2).

§3. Shock Angle

In general there are two distinct angles θ which satisfy (2.4). To understand these angles write

$$F_x = A(u)u_x \qquad G_y = B(u)u_y \ , \tag{3.1}$$

where

$$A(u) = \frac{dF}{du} \qquad B(u) = \frac{dG}{du} \ . \tag{3.2}$$

Then (2.4) becomes

$$\sin^2\theta\ A(u)u_x - \cos\theta\sin\theta\left(A(u)u_y + B(u)ux\right) + \cos^2\theta\ B(u)u_y = 0 \qquad (3.3)$$

or

$$\left(\sin\theta\ A(u) - \cos\theta\ B(u)\right)\ \left(\sin\theta\ u_x - \cos\theta\ u_y\right) = 0\ , \qquad (3.4)$$

which gives two solutions α and β, where

$$\tan\alpha = \frac{B(u)}{A(u)}\ , \qquad \tan\beta = \frac{u_y}{u_x}\ . \qquad (3.5)$$

The first of these is the local flow direction and reflects the fact that the divergence comes from one term only in one-dimensional flow. The second is the direction of the gradient of u, perpendicular to u = constant contour lines.

§ 4. Systems Case

The above results hold for systems of the form

$$u_t + F_x + G_y = 0 \qquad (4.1)$$

where u, F, G are N-vectors, except for the following. In (2.2), $\mathcal{G}_t = 0$ for only one component, in general, say i. $A(u)$ and $B(u)$ in (3.1) are now matrices but (3.4) still holds for the component i leading to

$$\tan\beta_i = \frac{u_y^{(i)}}{u_x^{(i)}}\ , \qquad (4.2)$$

where $u^{(i)}$ is the ith component of u.

$\mathcal{G}_t \neq 0$ for the remaining components, in general, but is given by the left hand side of (2.4). Similarly $\mathcal{F}_n$ is not given by (2.7) but by (2.6).

Any component i leads to an angle given by (4.2): the choice of i is free but may perhaps be made by considering which i leads to the greatest jump as measured by $\mathcal{F}_n^{(i)}$ (suitably normalised).

§5. Discretisation

We discuss now a discretisation of these results (c.f. Baines [2]).

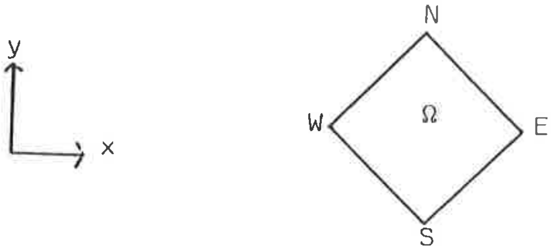

Fig. 1

For a square grid, diagonally orientated, as shown in fig. 1, a discretised form of the (averaged) divergence comes from the integral

$$\frac{1}{d^2}\int_{\Omega}\mathrm{div}(F,G)\,d\Omega$$

$$= \frac{1}{d^2}\oint_{\partial\Omega}(F,G).d\underline{S} \quad , \qquad (5.1)$$

where d is the length of the diagonal of the cell. A discretised form of (5.1) is obtained by using trapezium rule quadrature on the sides EN , NW, WS, SE (fig. 1). The result is the expression

$$\left(F_{EW} + G_{NS}\right)/d \qquad (5.2)$$

where $F_{EW} = F_E - F_W$, $G_{NS} = G_N - G_S$; this is an obvious discretisation of the divergence $F_x + G_y$.

Replacing derivatives by differences in §3 above we see that any discontinuity will occur at an approximate angle β , where

$$\tan\beta = \frac{u_{NS}}{u_{EW}} \qquad (5.3)$$

and $u_{NS} = u_N - u_S$, $u_{EW} = u_E - u_W$. This is for the scalar problem. For a system we expect a discontinuity at an angle β , where β is one of the β_i given by

$$\tan\beta_i = \frac{u_{NS}^{(i)}}{u_{EW}^{(i)}} \quad , \qquad (5.4)$$

the choice of i being possibly that for which the discretised divergence

$$\cos^2\beta F_{EW}^{(i)} + \cos\theta\sin\theta\left(F_{NS}^{(i)} + G_{EW}^{(i)}\right) + \sin^2\theta G_{NS}^{(i)} \tag{5.5}$$

is greatest (after normalisation). In practice the same angle may be expected for each component, since any discontinuity here has a one-dimensional character.

§6. Jump Condition

Consider now the one-dimensional jump itself. From (2.6), (2.7),

$$\mathfrak{F}_n = F_x + G_y = \cos^2\beta\ F_x + \cos\beta\sin\beta(F_y + G_x) + \sin^2\beta\ G_y \tag{6.1}$$

with β given by (4.2). Using (3.1), (6.1) becomes

$$\mathfrak{F}_n = (A(u)\cos\beta + B(u)\sin\beta)\quad(u_x\cos\beta + u_y\sin\beta)$$

$$= (A(u)\cos\beta + B(u)\sin\beta)\frac{\partial u}{\partial n}\ . \tag{6.2}$$

Hence in the scalar case the jump condition for a discontinuity moving in the direction n is

$$[\mathfrak{F}_n] = \{A(u)\cos\beta + B(u)\sin\beta\}\ [u_n] \tag{6.3}$$

and the shock speed is

$$S = A(u)\cos\beta + B(u)\sin\beta\ . \tag{6.4}$$

Convenient discretised forms of (6.3) and (6.4) are

$$\mathfrak{F}_{RL} = \left\{\frac{F_{RL}}{u_{RL}}\cos\beta + \frac{G_{RL}}{u_{RL}}\sin\beta\right\} u_{RL} = F_{RL}\cos\beta + G_{RL}\sin\beta \tag{6.5}$$

and $$S = \frac{F_{RL}}{u_{RL}}\cos\beta + \frac{G_{RL}}{u_{RL}}\sin\beta. \tag{6.6}$$

consistent with approximations (c.f. (6.1))

$$F_{EW} = \mathfrak{F}_{RL}\cos\beta\ ,\ G_{NS} = \mathfrak{F}_{RL}\sin\beta\ . \tag{6.7}$$

In the system case an equation of the form (6.1) holds for each component of $\underline{u}$. In order to discretise (6.2), however, we need a generalisation of the discretisation of (3.1) to the system case.

In one dimension Roe [1] introduces the matrix $\tilde{A}(u_R,u_L)$, (see fig. 2), with the central property

$$\tilde{A}(u_E,u_W)\quad(u_E - u_W) = F_E - F_W\ . \tag{6.8}$$

each eigenvector of which propagates with the speed of the corresponding eigenvalue and, when shocked, is captured exactly. The existence of such matrices has been discussed by Harten [2]. Correspondingly in the two-dimensional case we study the matrix $\tilde{\mathcal{A}}_{RL}$ with the property

$$\mathcal{A}_{RL}\,(u_R - u_L) = \mathcal{F}_R - \mathcal{F}_L \quad . \tag{6.9}$$

Comparison with (6.2) shows that we should take

$$\mathcal{A}_{RL} = A_{RL}\cos\beta + B_{RL}\sin\beta \quad , \tag{6.10}$$

to be the appropriate Roe matrix in this case, where

$$A_{RL}u_{RL} = F_{RL} \quad , \quad B_{RL}u_{RL} = G_{RL} \quad . \tag{6.11}$$

Relations between u_R , u_L and u_E , u_W , u_N , u_S are given in § 7.

7. The Algorithm

We now discuss the algorithm itself, beginning with a summary of Roe's procedure in one dimension.

Fig. 2

Having calculated ϕ from (1.1) (see fig. 2), the Roe algorithm for the scalar case is

$$\begin{aligned} u_R^{n+1} &= u_R^n - \frac{\Delta t}{\Delta x}\phi & \quad (A_{RL} > 0) \\ u_L^{n+1} &= u_L^n - \frac{\Delta t}{\Delta x}\phi & \quad (A_{RL} < 0) \end{aligned} \tag{7.1}$$

where
$$A_{RL} = \frac{F_{RL}}{u_{RL}} \tag{7.2}$$

(c.f. (3.2)) and Δt , Δx are t , x steps $(\Delta x = d)$.

For the system case we calculate first the eigenvectors $\tilde{\underline{e}}_j$ and eigenvalues $\tilde{\lambda}_j$ of the Roe matrix $\tilde{\mathcal{A}}$(see (6.9) , (6.10)). The fluctuation ϕ is then decomposed into components ϕ_j in the directions

$\underset{\sim}{e}_j$. Then for each j we apply (7.1) with ϕ replaced by ϕ_j and A_{RL} replaced by $\tilde{\lambda}_j$ (Δx is replaced by d). The point of this exercise is to ensure that correct upwinding is applied for each simple wave. For the two-dimensional case with the set-up shown in fig. 1, the same procedure can be carried out for the two part-fluctuations

$$\phi_1 = F_E - F_W \, , \qquad \phi_2 = G_N - G_S \, , \tag{7.3}$$

whose sum is ϕ.

In the scalar case a useful property of such a procedure is that there exists a maximum principle.

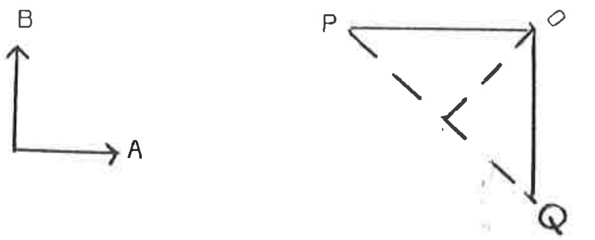

Fig. 3

Thus, if O is a point of the grid and $A_{OP} > 0$, $B_{OQ} > 0$, the algorithm gives

$$u_O^{n+1} = u_O - \frac{\Delta t}{d}\left(F_O - F_P\right) - \frac{\Delta t}{d}\left(G_O - G_Q\right) \tag{7.4}$$

$$= u_O - \frac{\Delta t}{d} A_{OP}\left(u_O - u_P\right) - \frac{\Delta t}{d} B_{OP}\left(u_O - u_Q\right)$$

$$= u_O\left(1 - \nu_{OP} - \nu_{OQ}\right) + \nu_{OP}u_P + \nu_{OQ}u_Q \, , \tag{7.5}$$

where $\nu_{OP} = \frac{\Delta t}{d} A_{OP}$, $\nu_{OQ} = \frac{\Delta t}{d} B_{OQ}$. Provided that $0 \le \nu_{OP}$, $0 \le \nu_{OQ}$, $\nu_{OP} + \nu_{OQ} \le 1$, the coefficients in (7.5) are non-negative and u_O^{n+1} lies between the minimum and maximum of u_O , u_P , u_Q: (n subscripts have been omitted).

For an oblique shock at a small angle to the (A,B) direction, however, such a maximum principle will work against sharp shock resolution and it is a poor design tool.

In the case of systems, if we are to recognise simple waves moving perpendicularly into and out of the shock, we need to apply the algorithm to the components of ϕ along the eigenvectors of the appropriate Roe matrix, which is

$$\mathcal{A}_{RL} = A_{RL} \cos\beta + B_{RL} \sin\beta \quad . \tag{7.6}$$

In the case of the two-dimensional Euler equations the eigenvectors of the matrix

$$\mathcal{A} = A \cos\beta + B \sin\beta \tag{7.7}$$

are

$$\begin{bmatrix} 1 \\ u \pm c\cos\beta \\ v \pm c \sin\beta \\ H \pm c(u\cos\beta + v\sin\beta) \end{bmatrix} \begin{bmatrix} 0 \\ c\sin\beta \\ -c\cos\beta \\ c(u\sin\beta - v\cos\beta) \end{bmatrix} \begin{bmatrix} 1 \\ u \\ v \\ \frac{1}{2}(u^2 + v^2) \end{bmatrix} \tag{7.8}$$

in the usual notation, with corresponding eigenvalues

$$u\cos\beta + v\sin\beta \pm c \quad , \quad u\cos\beta + v\sin\beta \quad \text{(twice)} \quad . \tag{7.9}$$

The eigenvalues and eigenvectors of $\tilde{\mathcal{A}}_{RL}$ are the same as those with u , v , H replaced by $\tilde{u}$, $\tilde{v}$, $\tilde{H}$ where

$$\tilde{f} = \frac{\rho_L^{\frac{1}{2}} f_L + \rho_R^{\frac{1}{2}} f_R}{\rho_L^{\frac{1}{2}} + \rho_R^{\frac{1}{2}}} \quad . \tag{7.10}$$

It remains to show how u_R , u_L are to be calculated from u_E , u_W , u_N and u_S. Assuming that the discontinuity passes through the centre of the cell, linear interpolation gives

$$\left.\begin{aligned} u_R &= \tfrac{1}{2}(1 - \sin \overline{\beta - \tfrac{\pi}{4}})\, u_E + \tfrac{1}{2}(1 + \sin \overline{\beta - \tfrac{\pi}{4}})\, u_N \\ u_L &= \tfrac{1}{2}(1 - \sin \overline{\beta - \tfrac{\pi}{4}})\, u_W + \tfrac{1}{2}(1 + \sin \overline{\beta - \tfrac{\pi}{4}})\, u_S \end{aligned}\right\} \quad \beta \in \left[0, \tfrac{\pi}{2}\right]$$

$$\left.\begin{aligned} u_R &= \tfrac{1}{2}(1 - \sin \overline{\beta - \tfrac{3\pi}{4}})\, u_N + \tfrac{1}{2}(1 + \sin \overline{\beta - \tfrac{3\pi}{4}})\, u_W \\ u_L &= \tfrac{1}{2}(1 - \sin \overline{\beta - \tfrac{3\pi}{4}})\, u_S + \tfrac{1}{2}(1 + \sin \overline{\beta - \tfrac{3\pi}{4}})\, u_E \end{aligned}\right\} \quad \beta \in \left[\tfrac{\pi}{2}, \pi\right] \tag{7.11}$$

and the same expressions with u_R , u_L interchanged when $\beta\epsilon\left[\pi,\frac{3\pi}{2}\right]$, $\beta\epsilon\left[\frac{3\pi}{2},2\pi\right]$, respectively.

Thus, given u_E , u_W , u_N , u_S and β , u_R , u_L may be calculated and hence, using (7.10) and (7.8), (7.9) , the eigenvectors and eigenvalues of $\tilde{\mathcal{A}}$. The two part-fluctuations ϕ_1 , ϕ_2 (see 7.3) are then projected onto these eigenvectors and each component treated as in the scalar one-dimensional case with wavespeed equal to the corresponding eigenvalue.

8. Conclusion

In this discussion we have concentrated on devising a Roe-like scheme which recognises possible shocks in two dimensions. The scheme is first order accurate although it can be made second order accurate using another technique due to Roe [4].

Clearly a scheme which devotes itself to shock capturing will seem cumbersome away from shocks. For this reason some way of deciding more definitely whether a shock is likely to be present is desirable. One possibility is to compare β's from adjacent cells, seeking linear continuity.

Since this report was prepared Davis [5] has described a rotationally biased upwind difference scheme for the Euler equations in which he uses a rotated coordinate system with one-dimensional Van Leer flux splitting to demonstrate that steady oblique shocks can be modelled sharply. He obtains the shock angle by requiring that the shock is normal to the velocity jump.

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REFERENCES

1. Roe, P.L., "Approximate Riemann solvers, parameter vectors, and difference schemes", J. Computational Phys., 43 (1981) p.357.

2. Baines, M.J., "Numerical Algorithms for the scalar wave equation in two and three dimensions", Numerical Analysis Report 1/83, Department of Mathematics, University of Reading.

3. Harten, A., "On the symmetric form of systems of conservation laws with entropy, ICASE report no. 81-34, Oct. 1981.

4. Roe, P.L. and Baines, M.J., "Algorithms for advection and shock problems", Proc. 4th GAMM conference, Paris, 1981.

5. Davis, S.F., "A rotationally biased upwind difference scheme for the Euler equations", ICASE report no. 172179, July 1983.