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A CENTRE THEOREM FOR TWO-DIMENSIONAL COMPLEX HOLOMORPHIC SYSTEMS AND ITS GENERALIZATION

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U N I V E R S I T Y   O F   R E A D I N G

A centre theorem for two-dimensional complex holomorphic systems and its generalization

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Abstract

We consider a two-dimensional complex holomorphic system. In particular, we use the centre manifold theory together with the singular point theory of Briot and Bouquet [1] to establish a centre theorem concerning the behaviour of the phase paths of the system in the neighbourhood of an equilibrium point having a single purely imaginary eigenvalue. An extended centre theorem is established for the corresponding N-dimensional complex holomorphic system $(N \geq 3)$.

1 Introduction

We consider the complex dynamical system,

$$z_t = F(z,w) \quad , \qquad (z,w,t) \in D \times I \,, \qquad (1.1)$$
$$w_t = G(z,w)$$

where $I \subseteq \mathbb{R}$ is a connected open interval and $D \subseteq \mathbb{C}^2$ is a simply connected domain. $F, G : D \to \mathbb{C}$ are complex valued functions of the complex variables $(z,w) \in D$. In particular F and G are holomorphic functions of (z,w) throughout D (see Range, ch.1, §1.2, [2]). It should be noted that (1.1) can be written as a C^∞ four-dimensional real autonomous system in an appropriate domain of $\mathbb{R}^4$ (Range, ch.1, Corol. (1.5), [2]). Systems of the type (1.1) arise in telecommunications problems (see, for example, [3], [4], [5]). (insert)
We examine the behaviour of integral paths $(z(t), w(t))$ in the two-dimensional complex phase space (z,w). In particular, we consider the nature of integral paths of (1.1) in the neighbourhood of an equilibrium point which has associated eigenvalues, one of which is purely imaginary whilst the other has non-zero real part. We establish the existence of a family of concentric closed orbits surrounding the equilibrium point, and we conclude that the equilibrium point is a centre, and topologically equivalent to that of the associated linearized system.

2 Local behaviour via centre manifold theory

Without loss of generality, we take $z = w = 0$ to be an equilibrium point of (1.1) in D, which is simple; that is, $\det[J(F,G)] \neq 0$ at $z = w = 0$, where $J(F,G)$ is the Jacobian matrix of $F(z,w)$, $G(z,w)$. This condition ensures that $z = w = 0$ is an isolated equilibrium point. We consider the situation when the associated linearized system is such that one eigenvalue of $J(F,G)|_{(0,0)}$ is purely imaginary, whilst the other has non-zero real part. For simplicity, we consider that the linearized part of (1.1) at $z = w = 0$ has been put into normal form. We may then write (1.1) as,

$$z_t = i\mu z + f(z,w) \quad , \qquad (z,w,t) \in D \times I \qquad (2.1)$$
$$w_t = \lambda w + g(z,w)$$

(insert)

In telecommunications systems, the transmission of high speed digital signals can be affected by atmospheric distortion as a result of multipath interference. Distortion of this type is removed by introducing adaptive equalisers which are tapped delay devices. Since complex–valued data streams are usually transmitted, the control equations for the equaliser are complex–valued and have the form,

$$\phi_t = -\mu_0 \, \mathrm{Im}[e^{i\phi} R_0] ,$$

$$z_{jt} = -\mu_j \, e^{i\phi} R_j \quad , \quad j = 1, ..., N .$$

Here z_j $(j = 1, ..., N)$ are the variable tap weights which are adjusted to remove signal distortion, R_j $(j = 1, ..., N)$ are nonlinear functions of z_j $(j = 1, ..., N)$, μ_j $(j = 1, ..., N)$ are real, positive feedback factors and ϕ in the phase of the carrier signal. In the simplest case, with $N = 2$ and ϕ = constant, we obtain the two–tap adaptive equaliser system, which can be written as

$$z_t = \omega$$

$$\omega_t = \nu_1 + \nu_2 \, z + \nu_3 \, \omega + \gamma_1 \, z^2 + \gamma_2 \, z\omega$$

with z, ω, ν_i, γ_i complex. This system falls into the class of complex dynamical systems given by (1.1), and motivates their study.

where $Re(\lambda) \neq 0$, $\mu \in \mathbb{R}\backslash\{0\}$ and $f(z,w)$, $g(z,w)$ are holomorphic in D, with Taylor series,

$$f(z,w) = \sum_{n=2}^{\infty}\Big(\sum_{\alpha+\beta=n} a_{\alpha\beta} z^{\alpha} w^{\beta}\Big) \quad ,$$

$$g(z,w) = \sum_{n=2}^{\infty}\Big(\sum_{\alpha+\beta=n} b_{\alpha\beta} z^{\alpha} w^{\beta}\Big) \quad , \tag{2.2}$$

convergent in some neighbourhood of $z = w = 0$ ($a_{\alpha\beta}$, $b_{\alpha\beta}$, $\alpha,\beta \in \mathbb{N}$ are Taylor coefficients of f, g at $z = w = 0$, Range, ch.1, §1.6, [2]).

(2.3) Remark

Without loss of generality, we will take $Re(\lambda) < 0$ in (2.1). We simply reverse the sign of t to consider the case when $Re(\lambda) > 0$.

We now apply the centre manifold theory (see, for example, Carr [6], Wiggins [7]) to the equivalent C^{∞} four-dimensional real system to classify the phase space structure of (2.1) in the neighbourhood of $z = w = 0$. We use Theorems (1), (2) together with comments (2.6) of Carr [6] (ch.1 and ch.2, §2.6) to deduce that there exits a real two-dimensional centre manifold in a neighbourhood of $z = w = 0$, described by,

$$W_C = \{(z,w) \in \mathbb{C}^2 \;:\; w = L(z) \;,\; |z| < \delta \;,\; L(0) = 0 \;,\; \mathbf{DL}(0) = 0\} \;, \tag{2.4}$$

for $\delta > 0$ sufficiently small. In (2.4), $L : D_\delta \to \mathbb{C}$, $D_\delta = \{z : |z| < \delta\}$ and writing $z = x + iy$ and $L = u(x,y) + iv(x,y)$, then,

$$\mathbf{DL}(0) = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}\Bigg|_{x = y = 0} \tag{2.5}$$

In addition, the centre manifold theory guarantees that $u(x,y)$ and $v(x,y)$ are C^r functions in some neighbourhood D^r of $x = y = 0$ for each $r \in \mathbb{N}$ (see [6], ch.2, §2.6). However, this does not imply that the complex function $L(z)$ is a holomorphic function of z in any neighbourhood of $z = 0$ ($u(x,y)$, $v(x,y)$ do not necessarily satisfy the Cauchy-Riemann equations in any neighbourhood of $z = 0$).

Theorem 2 of [6] (ch.1, P.4) determines that all phase paths of (2.1) in a neighbourhood of $z = w = 0$ contract exponentially (in t) onto the centre manifold W_c. Hence the nature of the equilibrium point $z = w = 0$ is determined by the dynamics of (2.1) restricted to the centre manifold W_c. The dynamics on the centre manifold are governed by the reduced scalar complex equation,

$$z_t = i\mu z + f(z,L(z)) \ , \quad |z| < \delta \ . \tag{2.6}$$

Thus, to study the behaviour of (2.1) in the neighbourhood of the equilibrium point $z = w = 0$, we need only examine the dynamics of the scalar complex equation (2.6) close to $z = 0$. Clearly $z = 0$ is an isolated equilibrium point of (2.6), with a single imaginary eigenvalue $i\mu$. The behaviour of (2.6) depends crucially upon whether the function $L(z)$ is a holomorphic function of z in a neighbourhood of $z = 0$. As remarked earlier, this is not guaranteed by the centre manifold theory, even when f, g are themselves holomorphic functions of (z,w) in a neighbourhood of $z = w = 0$.

When $L(z)$ is holomorphic in a neighbourhood of $z = 0$, then $f(z, L(z))$ is also holomorphic in a neighbourhood of $z = 0$ (since $f(z,w)$ is holomorphic in a neighbourhood of $z = w = 0$) and the local behaviour of (2.6) can be determined by the theory of scalar complex holomorphic equations (see, for example, Brickman and Thomas [8], Sverdlove [9], Needham and King [10]). In particular (noting that with $L(z)$ holomorphic at $z = 0$, then, $L(z) = \sum_{n=N}^{\infty} \ell_n z^n$ in some neighbourhood of $z = 0$, with $N \geq 2$) we have that the behaviour near $w = z = 0$ is that of a centre, with the concentric family of closed orbits lying on the centre manifold in the neighbourhood of $z = w = 0$ (see [8] or [9], theorem (2.6)). We can summarise this in,

(2.7) Proposition

Let $z = w = 0$ be an equilibrium point of (1.1) at which $J(F,G)$ has a single purely imaginary eigenvalue $i\mu$, whilst the other eigenvalue λ has non-zero real part. Then (1.1) has a complex one-dimensional centre manifold in a neighbourhood of $z = w = 0$ described by the complex function $L(z)$ of the complex variable z, for z sufficiently close to $z = 0$. All phase paths of (1.1) in the neighbourhood of $z = w = 0$ contract exponentially onto the centre manifold as $t \to \infty$ (when $\text{Re}(\lambda) < 0$) or as $t \to -\infty$ (when $\text{Re}(\lambda) > 0$). Moreover, when $L(z)$ is holomorphic in a neighbourhood of $z = 0$, then (1.1) has a centre family in the neighbourhood of $z = w = 0$. This centre family of concentric, closed, periodic orbits lies on the centre manifold.

□

A general centre theorem now follows, provided we can establish that $L(z)$ is holomorphic in a neighbourhood of $z = 0$.

3 A holomorphic centre manifold via Briot-Bouquet theory

We consider first the following singular initial value problem for $\xi : D_{\delta'} \to \mathbb{C}$ (where $D_{\delta'} = \{z : |z| < \delta'\}$) ,

$$[i\mu z + f(z,\xi)] \frac{d\xi}{dz} = \lambda\xi + g(z,\xi) \ , \quad z \in D_{\delta'} \ , \quad \xi(0) = \xi'(0) = 0 \ ,$$

which we will henceforth refer to as IVP. We have,

(3.1) Lemma

$w = L(z)$ is a centre manifold of (2.1) at $z = w = 0$ which is holomorphic in a neighbourhood of $z = 0 \Leftrightarrow \xi = L(z)$ is a solution of IVP which is holomorphic in a neighbourhood of $z = 0$.

proof

$\Rightarrow$ Suppose $w = L(z)$ is a centre manifold of (2.1) at $z = w = 0$ which is holomorphic in $|z| < \delta$. Then by definition, and the Cauchy-Riemann equations,

$$L(0) = L'(0) = 0 \ . \tag{3.2}$$

Now let $|z_0| < \delta$ and put $w_0 = L(z_0)$, with $w_s(t)$, $z_s(t)$ being the integral path of (2.1) satisfying $z_s(0) = z_0$, $w_s(0) = w_0$. Since $w = L(z)$ in an invariant manifold of (2.1), then $w_s(t) = L(z_s(t)) \ \forall \ |t| < \delta''$ such that $|z_s(t)| < \delta$. However $w_s'(t) = L'(z_s(t))z_s'(t)$, $|t| < \delta''$, with in particular $w_s'(0) = L'(z_s(0))z_s'(0)$ which gives, via (2.1),

$$\lambda w_0 + g(z_0, w_0) = L'(z_0)(i\mu z_0 + f(z_0, w_0)) \ . \tag{3.3}$$

Equation (3.3) therefore holds $\forall\ z_0$ with $|z_0| < \delta$. Equations (3.2) and (3.3) establish that $L(z)$ satisfies IVP in $|z| < \delta$, as required.

$\Leftarrow$ Suppose that $\xi = L(z)$ is a solution of IVP which is holomorphic in $|z| < \delta'$ for some $\delta' > 0$. We need to show that the (unique) solution of (2.1) with initial conditions $z(0) = z_0$, $w(0) = L(z_0)$ $(0 < |z_0| < \delta')$ is given by $z_s(t)$, $w_s(t)$, where

$$\begin{aligned} w_s(t) &= L(z_s(t)) , \\ z_{st} &= i\mu z_s + f(z_s, L(z_s)) , \end{aligned} \tag{3.4}$$

for $|t| < \tilde{\delta}$, with $\tilde{\delta}$ such that $|z_s(t)| < \delta'$. Now,

$$z_{st} - i\mu z_s - f(z_s, w_s) = f(z_s, L(z_s)) - f(z_s, w_s) \equiv 0$$

$$\begin{aligned} w_{st} - \lambda w_s - g(z_s, w_s) &= L'(z_s)z_{st} - \lambda L(z_s) - g(z_s, L(z_s)) \\ &= \frac{[\lambda L(z_s) + g(z_s, L(z_s))]}{[i\mu z_s + f(z_s, L(z_s))]} \times [i\mu z_s + f(z_s, L(z_s))] \\ &\quad -[\lambda L(z_s) + g(z_s, L(z_s)) \equiv 0 \end{aligned}$$

via (3.4) and IVP. Thus $(z_s(t), w_s(t))$ as given by (3.4) provides the solution of (2.1) in $|t| < \tilde{\delta}$ subject to initial conditions $(z_0, L(z_0))$ and the result follows.

□

We next establish that IVP has a unique solution holomorphic in a neighbourhood of $z = 0$.

<u>(3.5) Lemma</u>

IVP has a unique solution $w = L(z)$ which is holomorphic in a neighbourhood of $z = 0$.

<u>proof</u>

We introduce $\psi(z)$ by the transformation,

$$w(z) = z\psi(z) \ , \quad |z| < \delta' \ , \tag{3.6}$$

and re-write IVP in terms of $\psi(z)$ and z, which becomes,

$$(\mu i z + f(z, \psi z))(\psi + z\psi_z) = \lambda\psi z + g(z, \psi z) \ , \tag{3.7}$$

$$\psi(0) = 0 \ , \quad |z| < \delta' \ . \tag{3.8}$$

We can write (3.7) as,

$$(1 + p(z,\psi))(\psi + z\psi_z) = -\frac{i\lambda}{\mu}\psi + q(z,\psi) \ , \quad |z| < \delta' \ , \tag{3.9}$$

where now,

$$p(z,\psi) = \frac{1}{i\mu}\sum_{n=2}^{\infty}\Big(\sum_{\alpha+\beta=n} a_{\alpha\beta}\psi^{\beta}\Big)z^{n-1} \ ,$$

$$q(z,\psi) = \frac{1}{i\mu}\sum_{n=2}^{\infty}\Big(\sum_{\alpha+\beta=n} b_{\alpha\beta}\psi^{\beta}\Big)z^{n-1} \ , \tag{3.10}$$

convergent in some neighbourhood of $z = \psi = 0$, and are both therefore holomorphic in that neighbourhood. We can simplify (3.9) to,

$$z\psi_z = -\Big(\frac{i\lambda}{\mu} + 1\Big)\psi - \frac{i}{\mu} b_{20} z + Q(z,\psi) \ , \quad |z| < \delta' \ , \tag{3.11}$$

where,

$$Q(z,\psi) = \Big(q(z,\psi) - \frac{b_{20}}{i\mu} z\Big) - \frac{q(z,\psi)p(z,\psi)}{(1+p(z,\psi))} + \frac{i\lambda}{\mu}\frac{\psi p(z,\psi)}{(1+p(z,\psi))} \ , \tag{3.12}$$

is holomorphic in a neighbourhood of $z = \psi = 0$ and has,

$$Q(z,\psi) = O(\psi^2, z^2) \ \text{ as } |\psi|, \ |z| \to 0 \ . \tag{3.13}$$

We also observe that $(\frac{i\lambda}{\mu} + 1) \notin \mathbb{N}\cup\{0\}$. Equation (3.11) is now in the form of the equation of Briot and Bouquet [1] (see also Sansone and Conti [11], ch.3, §2), and an application of Theorem 1 of [11] (p.115, ch.3) establishes that equation (3.11) has a unique solution $\psi = \Psi(z)$ holomorphic in a neighbourhood of $z = 0$ and satisfying the initial condition $\Psi(0) = 0$. Hence, via the transformation (3.6), IVP has a unique solution $w = L(z)$ holomorphic in a neighbourhood of $z = 0$, with $L(z) = z\Psi(z)$, and $L(0) = L'(0) = 0$, as required.

□

We now have,

(3.14) Proposition

The system (2.1) has a unique centre manifold $w = L(z)$ at $z = w = 0$ which is holomorphic in a neighbourhood of $z = 0$.

proof

Follows from lemma (3.5) using lemma (3.1)

□

Finally we have established the following centre theorem,

(3.15) Theorem

Let $z = w = 0$ be an equilibrium point of (1.1) at which $J[F,G]$ has a single purely imaginary eigenvalue, whilst the other eigenvalue has non-zero real part. Then (1.1) has a unique complex one-dimensional centre manifold at $z = w = 0$ which is holomorphic in a neighbourhood of $z = 0$. This centre manifold contains a centre family of closed, periodic, orbits of (1.1) surrounding $z = w = 0$. All phase paths of (1.1) in the neighbourhood of $z = w = 0$ contract exponentially (in t) into this centre manifold as $t \to \infty$ $(\mathrm{Re}(\lambda) < 0)$ or $t \to -\infty$ $(\mathrm{Re}(\lambda) > 0)$.

proof

Follows directly from propositions (2.7) and (3.14).

□

We can make the following comments concerning theorem (3.15) :

(3.16) Remarks

(i) Theorem (3.15) establishes that the phase space structure of (1.1) and that of its corresponding linearization about $z = w = 0$ are topologically equivalent in a neighbourhood of $z = w = 0$.

(ii) The period of each of the periodic orbits on the centre manifold is $T = {}^{2\pi}/\mu$, and each has zero mean shift about $z = 0$, that is,

$$\int_0^T z_p(t)dt = 0 \ ,$$

for each periodic orbit $z_p(t)$. This follows directly from the theory of scalar holomorphic equations, [8], [9], [10].

(iii) Limit cycles in (1.1) cannot be created at a simple Hopf bifurcation.

We now develop a generalization of the centre theorem to N-dimensional holomorphic systems.

4 N-dimensional holomorphic systems

We generalize the two-dimensional complex system (1.1) to the N-dimensional system ($N \in \mathbb{N}$),

$$\mathbf{u}_t = \mathbf{H}(\mathbf{u}) \ , \quad (\mathbf{u},t) \in D \times I \ , \tag{4.1}$$

where $D \subseteq \mathbb{C}^N$ is a simply connected domain, $\mathbf{u} \in D$ and $\mathbf{H} : D \to \mathbb{C}^N$. In component form we write $\mathbf{u} = (z, w_1, \ldots, w_{N-1})^T$ and $\mathbf{H} = (F, G_1, \ldots, G_{N-1})^T$ with $z, w_i \in \mathbb{C}$ and $F, G_i : D \to \mathbb{C}$ $(i = 1, \ldots, N-1)$ being holomorphic functions of $\mathbf{u}$ in D. Again, (4.1) can be written as a C^∞, 2N-dimensional real autonomous system in a suitable domain of $\mathbb{R}^{2N}$.

We consider the nature of integral paths of (4.1) in the neighbourhood of an equilibrium point which has associated eigenvalues, one of which is purely imaginary whilst the others have non-zero real parts. We establish the existence of a family of concentric closed orbits surrounding the equilibrium point, leading to a generalization of theorem (3.15).

We take $\mathbf{u} = \mathbf{0}$ to be the equilibrium point of (4.1) and assume that the linearized part of (4.1) at $\mathbf{u} = \mathbf{0}$ has been put into normal form. Thus we may write,

$$\begin{aligned} z_t &= i\mu z + f(z, w_1, \ldots, w_{N-1}) \ , \\ w_{it} &= \lambda_i w_i + g_i(z, w_1, \ldots, w_{N-1}) \ , \quad i = 1, \ldots, N-1 \ , \end{aligned} \tag{4.2}$$

where $\mu \in \mathbb{R} \backslash \{0\}$, $\text{Re}(\lambda_i) \neq 0 (i = 1, \ldots, N-1)$ and $f(\mathbf{u})$, $g_i(\mathbf{u}) (i = 1, \ldots, N-1)$ are holomorphic in D with $|f(\mathbf{u})|, |g_i(\mathbf{u})| = O(|\mathbf{u}|^2)$ as $|\mathbf{u}| \to 0$. Thus, in a neighbourhood of $\mathbf{u} = 0$, f and g_i $(i = 1, \ldots, N-1)$ have Taylor series,

$$f(\mathbf{u}) = \sum_{n=2}^{\infty} \Big(\sum_{p_1+p_2+\dots+p_N=n} a_{p_1p_2\dots p_N} z^{p_1} w_1^{p_2} \dots w_{N-1}^{p_N} \Big) ,$$
$$g_i(\mathbf{u}) = \sum_{n=2}^{\infty} \Big(\sum_{p_1+p_2+\dots+p_N=n} b^i_{p_1p_2\dots p_N} z^{p_1} w_1^{p_2} \dots w_{N-1}^{p_N} \Big) . \tag{4.3}$$

We can again apply centre manifold theory to the equivalent C^∞, 2N-dimensional real system to classify the behaviour of (4.2) in phase space in a neighbourhood of $\mathbf{u} = \mathbf{0}$. We require the extended versions of theorems (1), (2) and comment (2.6) in [6] (as extended to systems for which $\mathrm{Re}(\lambda_i)$ may be positive or negative, and reviewed by Wiggins, [7], ch.2, §2.1c). These results establish the existence of a real two-dimensional centre manifold in a neighbourhood of $\mathbf{u} = \mathbf{0}$, described by,

$$W_c = \{\mathbf{u} \in \mathbb{C}^N : w_i = L_i(z) \ , \quad |z| < \delta \ , \quad L_i(0) = 0 \ ,$$
$$\mathbf{DL}_i(0) = \mathbf{0} \ , \ i = 1, \dots , N-1\} \tag{4.4}$$

for some $\delta > 0$. In (4.4) $L_i : D_\delta \to \mathbb{C}$ and with $L_i = u_i + iv_i$, then the definition of $\mathbf{DL}_i$ follows (2.5). The functions $u_i(x,y)$, $v_i(x,y)$ $(i = 1, \dots , N-1)$ are C^r functions in some neighbourhood D_r of $x = y = 0$ for each $r \in \mathbb{N}$. However, as before, this does not guarantee that the functions $L_i(z)$ are holomorphic in any neighbourhood of $z = 0$.

The phase paths in the neighbourhood of $\mathbf{u} = \mathbf{0}$ contract onto the centre manifold either as $t \to \infty$ or as $t \to -\infty$ and the nature of the equilibrium point $\mathbf{u} = \mathbf{0}$ is determined by the dynamics of (4.1) restricted to the centre manifold W_c. The dynamics on the centre manifold are governed by the reduced complex scalar equation,

$$z_t = i\mu z + f(z, L_1(z), \dots , L_{N-1}(z)) \ , \quad |z| < \delta \ . \tag{4.5}$$

Classification of (4.5) in $|z| < \delta$ then determines the nature of the equilibrium point $\mathbf{u} = \mathbf{0}$ of (4.1). $z = 0$ is an isolated equilibrium point of (4.5) with a single imaginary eigenvalue $i\mu$. To establish the centre theorem for (4.1) we again show that $L_i(z)$ $(i = 1, \dots , N-1)$ are holomorphic functions of z in some neighbourhood of $z = 0$, after which the result follows from (4.5) and the theory of [8], [9], [10], as in section 2.

We introduce the initial value problem,

$$[i\mu z + f(z,\xi_1, \dots , \xi_{N-1})]\, \xi_{iz} = \lambda_i\xi_i + g_i(z,\xi_1, \dots , \xi_{N-1}) \ , \quad |z| < \delta'$$

$$i = 1, \dots , N - 1 \ , \quad \text{with,}$$

$$\xi_i(0) = \xi_{iz}(0) = 0 \ , \quad i = 1, \dots , N - 1 \ ,$$

which we shall henceforth refer to as IVPN. Corresponding to lemma (3.1), it is readily established that $w_i = L_i(z)$ $(i = 1, \dots , N - 1)$ in a centre manifold of (4.2) at $\mathbf{u} = 0$ which is holomorphic in a neighbourhood of $z = 0$ if and only if $\xi_i = L_i(z)$ $(i = 1, \dots , N - 1)$ in a solution of IVPN which is holomorphic in a neighbourhood of $z = 0$. We study IVPN using the Briot-Bouquet theory for systems (see [11], ch.3, compliments 5, [12]). First we introduce the transformation,

$$\xi_i(z) = z\psi_i(z) \ , \tag{4.6}$$

after which IVPN becomes,

$$[1 + \bar{f}(z, z\psi_1, \dots, z\psi_{N-1})]\,[\psi_i + z\psi_i'] = -\frac{i\lambda_i}{\mu}\psi_i + \bar{g}_i(z, z\psi_1, \dots, z\psi_{N-1}) \ , \quad |z| < \delta' \ , \tag{4.7}$$

$$\psi_i(0) = 0 \ , \quad i = 1, \dots , N - 1 \ . \tag{4.8}$$

Here $\bar{f} = \dfrac{1}{i\mu z} f$, $\bar{g}_i = \dfrac{1}{i\mu z} g_i$ are holomorphic functions of $z, \psi_1, \dots , \psi_{N-1}$ in a neighbourhood of $z = \psi_1 = \dots = \psi_{N-1} = 0$. A further rearrangement leads to,

$$\begin{aligned} z\psi_i' &= \Big[\frac{-i\lambda_i}{\mu}\psi_i + \bar{g}_i(z, z\psi_1, \dots, z\psi_{N-1})\Big] \\ &\quad \times [1 + R(z, z\psi_1, \dots, z\psi_{N-1})] - \psi_i \ , \quad i = 1, \dots, N-1 \ , \end{aligned} \tag{4.9}$$

with,

$$R(z, z\psi_1, \dots, z\psi_{N-1}) = \frac{-\bar{f}(z, z\psi_1 \ , \dots, z\psi_{N-1})}{(1 + \bar{f}(z, z\psi_1, \dots, z\psi_{N-1}))} \ . \tag{4.10}$$

We observe that,

$$\bar{g}_i = \frac{b^i_{20....0}}{i\mu} z + 0(z^2, \psi_1^2, ..., \psi_{N-1}^2) ,$$

$$R = \frac{a_{20....0}}{i\mu} z + 0(z^2, \psi_1^2, ..., \psi_{N-1}^2) , \tag{4.11}$$

as $|z|$, $|\psi_1|$, ... , $|\psi_{N-1}| \to 0$. Finally (4.9) becomes,

$$z\psi_i' = -\sigma_i \psi_i - i\frac{b^i_{20....0}}{\mu} z + \chi_i(z, \psi_1, ..., \psi_{N-1}) , \quad i = 1, ..., N-1 , \tag{4.12a}$$

with,

$$\sigma_i = \frac{i\lambda_i}{\mu} + 1 \notin \mathbb{N} \cup \{0\} , \quad i = 1, ..., N-1 , \tag{4.12b}$$

and, $\chi_i(z, \psi_1, ... , \psi_{N-1})$ is holomorphic in a neighbourhood of $z = \psi_1 = ... = \psi_{N-1} = 0$ with,

$$\chi_i = 0(z^2, \psi_1^2, ..., \psi_{N-1}^2) \text{ as } |z|, |\psi_1|, ... |\psi_{N-1}| \to 0 . \tag{4.13}$$

Equations (4.12a) subject to initial conditions

$$\psi_i(0) = 0 , \quad i = 1, ... , N - 1 , \tag{4.14}$$

are equivalent to IVPN. The equations (4.12) are now in the standard form for application of the Briot-Bouquet theory ([11], [12]), which establishes that provided none of the σ_i is a non-negative integer, then equations (4.12a) have a unique solution $\psi_i = \Psi_i(z)$ $(i = 1,2, ... , N - 1)$ which satisfies conditions (4.14) and is holomorphic in a neighbourhood of $z = 0$. Since $\mathrm{Re}(\lambda_i) \neq 0 \ \forall \ i = 1, ... , N - 1$, then, via (4.12b), $\sigma_i \notin \mathbb{N} \cup \{0\} \ \forall \ i = 1, ... , N - 1$ and so the Briot-Bouquet theorem holds. We conclude, via transformation (4.6), that IVPN has a unique solution $\xi_i = z\Psi_i(z)$ $i = 1, ... , N - 1$ which is holomorphic in a neighbourhood of $z = 0$, from which we deduce that (4.1) has a unique one dimensional complex centre manifold at $\mathbf{u} = \mathbf{0}$, $w_i = z\Psi_i(z)$ $(i = 1, ... , N - 1)$ which is holomorphic in a neighbourhood of $z = 0$. We therefore have established the following generalization of theorem (3.15), (4.15) Theorem

Let $\mathbf{u} = \mathbf{0}$ be an equilibrium point of (4.1) at which $J[H]$ has a single purely imaginary eigenvalue, whilst the other eigenvalues all have non-zero real parts, then (4.1) has a unique complex one dimensional centre manifold at $\mathbf{u} = \mathbf{0}$ which is holomorphic in a neighbourhood of $z = 0$. This centre manifold contains a centre family of closed periodic orbits of (4.1) surrounding $\mathbf{u} = \mathbf{0}$. All phase paths of (4.1) in the neighbourhood of $\mathbf{u} = \mathbf{0}$ contract onto this centre manifold as $t \to \infty$ or $t \to -\infty$.

□

We note finally that remarks (3.16) also apply to theorem (4.15).

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