



School of Mathematical, Physical and Computational Sciences

**Mathematical modelling of oil slick dynamics
on a rough sea surface.**

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Abstract

This thesis investigates the dynamics of oil slicks formed on water surfaces after, for example, accidental spills under the influence of surface-wave motion. The spreading of oil in the marine environment is a complex phenomenon influenced by a variety of natural dispersive factors such as winds, currents and water waves with various applications in the environmental studies.

The main result of the previous studies is that surface waves have a profound influence on the behaviour of oil films ultimately leading to topological changes such as oil layer rupture and the following emulsification.

The emulsification process is by itself a very complex phenomenon, involving dynamics on multiple length scales, which is usually very difficult to analyse on a rigorous basis. Therefore, the conceptual framework is currently focused on using empirical approaches with a large number of various phenomenological assumptions.

The idea of this study is to introduce a more rigorous approach, reducing the number of phenomenological assumptions as much as possible, to understand the oil slick behaviour in the presence of wave motion at the intermediate stages of spreading, before the emulsification processes become important, on the basis of a macroscopic model. The results are expected to form a

basis to obtain distributions pre-conditioned by the wave motion which can be used then to inform the future or existing empirical models providing more accurate account of the overall oil layer dynamics.

The particular focus in this study is on the behaviour of oil layers under the action of travelling wave perturbations in both open and semi-confined geometries, including cases with a continuous oil leakage source.

Mathematically, the idea is to use a simplified thin-film lubrication model, which is a useful mathematical approximation used in many studies and applications involving small ratios of geometric characteristic parameters, instead of treating the full Navier-Stokes system, which is known to be rather difficult. The model then will be solved numerically using a moving-mesh approach to study how wave motion interacts with the oil layer.

It will be demonstrated that wave motion in the form of travelling waves can significantly facilitate spreading in open conditions, while in semi-confined domains the same wave-induced motion can lead to localization of the oil layer, even under very modest requirements for the wave amplitudes.

The results will provide new insights into how surface-wave dynamics and geometric constraints interact to control oil spreading and confinement under realistic marine conditions.

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Declaration

I declare that the work presented in this thesis is my own original research and that all sources of information have been properly acknowledged and referenced. I also declare that some parts of this thesis come from my published work and are cited properly, mainly in Chapters 4, 5 and 6.

Hanan Hozan

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Chapter 1

Introduction

The dynamics of oil spills on the sea surface represents a significant phenomenon with practical implications [Lehr and Simecek-Beatty, 2000, Council, 2003, Cordes et al., 2016, Fingas, 2011b, Lukyanov et al., 2024]. Early theoretical studies, in particular in 60th and 70th of the last century, supplemented by laboratory experiments, have concentrated on the simplified scenarios of oil spreading. These research works primarily examined the dynamics of liquid films on nearly flat water-oil interfaces, that is essentially on the surface of calm seas, to delineate the fundamental physical mechanisms and driving 'forces' involved and to identify the various stages of oil spreading [Shewmaker et al., 1954, Fay, 1969, Fay, 1971, Hoult, 1972a, Foda and Cox, 1980, Phillips, 1997].

It has been determined in laboratory settings that the oil spreading can be categorised into three consecutive, distinct regimes based on the dominant forces involved. Specifically, these include a brief, initial gravity-inertial phase (lasting a few minutes), followed by a gravity-viscous spreading

regime (lasting several hours), and culminating in the viscous-surface tension regime [Fay, 1971, Hoult, 1972a].

Oil spreading in real-world environmental conditions is much more complicated than in lab settings because it is affected by many natural factors. These include water waves, wind, ocean currents, turbulence, oil evaporation, dispersion, emulsification caused by water movement, biodegradation, and how oil interacts with the shoreline.

Particularly important is the role of surface tension (capillary effects and the dynamic contact angle contributions) [Lukyanov and Likhtman, 2016], which predominates in the final spreading stage under laboratory conditions. This factor is crucial as it directly governs the emulsification process at the micro-scale.

Therefore, in practical efforts to forecast the overall environmental impact, factors such as rough sea conditions have historically been incorporated empirically into models. This approach emphasizes simplified and empirical methodologies, yet ensures the inclusion of all pertinent natural factors at realistic scales [Murray, 1972, Venkatesh et al., 1979, Raj and Griffiths, 1979, Aravamudan et al., 1981, Huang, 1983, Lehr et al., 1984a, Lehr et al., 1984b, on Modeling of Oil Spills, 1996].

Particularly, recent research efforts have been concentrated on understanding the emulsification processes and the large-scale behaviour of water-oil emulsions formed as a result. This includes focusing on the large-scale dynamics and monitoring of the distribution of oil micro-droplets [Papadimitrakakis et al., 2005, Li et al., 2011, Skartlien et al., 2013, Soloviev et al., 2016, Geng et al., 2016b, Nissanka and Yapa, 2016, Nissanka and Yapa, 2017, Li et al.,

2017a, Nissanka and Yapa, 2018, Fingas and Brown, 2018, Garcia-Pineda et al., 2020, Zhao et al., 2020, Sundhar and Rajan, 2023].

One of the recent achievements was the progress made towards understanding the dynamics and distribution of oil droplets [Nissanka and Yapa, 2016, Nissanka and Yapa, 2017, Li et al., 2017a, Nissanka and Yapa, 2018]. Once the oil slick is broken into small areas and then into droplets by the surface breaking waves, the droplets in the form of emulsions become entrained to the water layers by the turbulent motion.

The dispersion process is complicated by interactions between the droplets, basically by their further breakup and coalescence [Chen et al., 2009a, Chen et al., 2009b]. This leads to the formation of characteristic droplet distributions which ultimately determine the fate and further evolution of the entire environmental system. The interesting and still open question, as far as the rigorous consideration is concerned, is then what kind of processes take place before the emulsification has gained momentum, as they should precondition and strongly influence the final stages of the oil dispersion in the ocean water systems. They definitely deserve more quantitative consideration, which can be delivered as we will demonstrate in this thesis.

The goal of this study is to provide a rigorous and quantitative analysis of the oil spreading on water surface, while at the same time taking into account more realistic factors. This will be achieved by using the thin film approximation of the system of the Navier Stokes equations as the mathematical model, and by exploring several scenarios relevant to real sea conditions using very effective moving mesh numerical techniques.

Specifically, in this thesis we explore the influence of oil discharge and water-

wave motion on the spreading of oil slicks and their potential for the breakup of oil slicks by surface water waves at the initial stages of oil-film spreading, essentially at the onset of the gravity-viscous regime.

It is well-established that factors such as winds, currents, and water turbulence significantly impact the spreading rate [Zhao et al., 2020]. Notably, the spreading rates predicted by the original Fay-Hoult theory [Fay, 1969, Hoult, 1972a], which assumes calm sea conditions, are markedly lower than those typically observed in real-world conditions [Zhao et al., 2020].

In this thesis I will demonstrate that, under relatively modest sea surface conditions, a continuous oil slick can rapidly disintegrate into surface fractions. This may happen, depending on the nature of the wave motion, either significantly before the emulsification process begins to exert a noticeable effect, namely at the very start of the gravity-viscous regime, or simultaneously with it, for example when the breaking waves predominate in the surface wave motion.

The depletion process appears to significantly influence the further evolution of the oil slick. The formation of void areas is expected to considerably enhance emulsification processes along substantially longer boundaries due to the impingement of breaking waves on the oil slicks [Li et al., 2017a, Lukyanov et al., 2024]. Furthermore, it will be shown that the spreading rate is affected by wave motion even in the absence of depletion [Lukyanov et al., 2024].

In the subsequent sections, specifically in Chapter 2, we will introduce a one-dimensional mathematical model based on the thin film approximation. The framework is Eulerian and is related to velocity-based frames by a flux term, giving an ALE (Arbitrary Lagrangian- Eulerian) description.

If the velocity is the fluid velocity a Lagrangian description is obtained. This model will eventually be analysed analytically and numerically to investigate various dynamics of the oil slick. The numerical method and its benchmarking procedure will be explained in detail in Chapter 3.

In Chapter 4, we will concentrate on the influence of the travelling surface wave perturbations on the dynamics of oil slicks. It will be demonstrated that there are different regimes of spreading. The one-dimensional model will also be generalised to a radially symmetric domain.

Chapter 5 will be devoted to numerical simulations of the oil spreading on calm water surface in the presence of the oil discharge.

Chapter 6 will be dedicated to the combined effects of different surface wave modes and the external flux in semi-confined geometries.

In Chapter 7, we will conclude and briefly discuss further developments of the approach developed in the thesis.

The final Chapter 8 is dedicated to some extra material elaborating on the Stokes drift effect, which could be useful in the interpretation of the results.

Chapter 2

Mathematical models and approximations

Consider an oil layer with density ρ and dynamic viscosity μ , spreading on the wavy surface of the sea, as depicted in Fig. 2.1 in one dimension.

In this study, we assume that the oil is Newtonian, although it is noted that various crude oil compositions could exhibit non-Newtonian behaviour.

The positions of the oil-water and oil-gas interfaces are denoted by

$$x_3 = B(x_1, x_2, t)$$

and

$$x_3 = B(x_1, x_2, t) + h(x_1, x_2, t)$$

respectively, where h represents the thickness of the oil layer.

The mathematical framework is the well-known thin-film lubrication approximation, the details of its applications in various situations can be found in

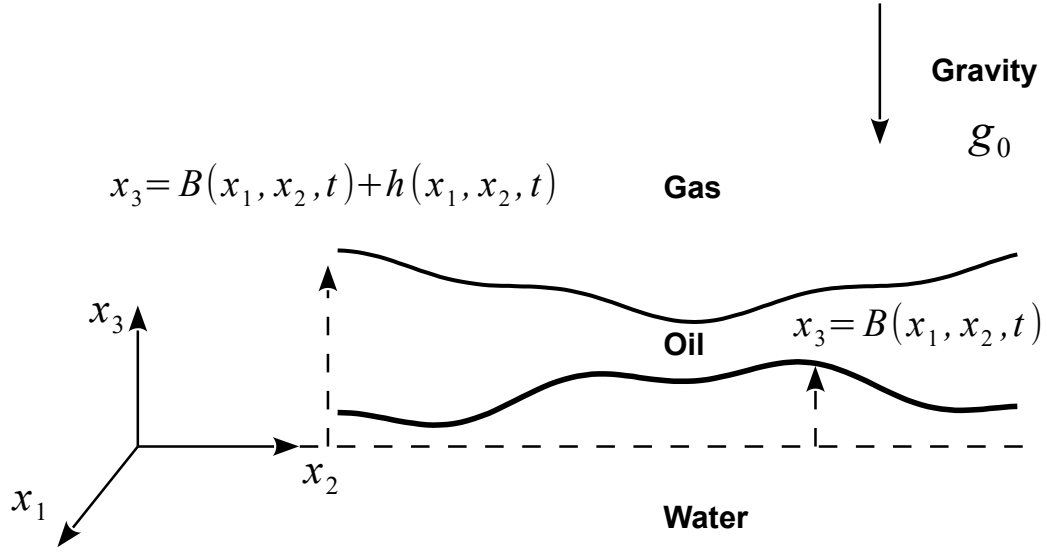


Figure 2.1: Illustration of the oil layer on the water surface problem geometry.

text books and reviews [Reynolds, 1886, Dowson, 1979, Oron et al., 1997, Shikhmurzaev, 2007, Craster and Matar, 2009]. For example, this approximation has been widely used in the analysis of coating processes, in the dynamics of evaporative thin films and in the analysis of hydraulic jumps in decontamination flows [Oron et al., 1997, Shikhmurzaev, 2007, Benilov, 2014, Calver et al., 2022]. There are many examples of the application of the methodology in various engineering applications, but here we will focus on details pertinent to the specific spreading regimes under consideration.

The idea of the approach [Reynolds, 1886] is that the model distinguishes between two length scales: L in the $x_{1,2}$ -directions, which might represent the diameter of the oil slick or the wavelength of surface disturbances, and

$h(\text{mm})$	$L(\text{m})$	ε	$U(\text{m/s})$	α_g	Re	St
1	50	$2 \cdot 10^{-5}$	1	$2 \cdot 10^{-5}$	$2 \cdot 10^{-3}$	$2 \cdot 10^{-3}$
1	50	$2 \cdot 10^{-5}$	0.1	$2 \cdot 10^{-4}$	$2 \cdot 10^{-4}$	$2 \cdot 10^{-4}$
1	10	10^{-4}	1	10^{-4}	10^{-2}	10^{-2}
10	50	$2 \cdot 10^{-4}$	1	0.02	0.2	0.2

Table 2.1: Oil layer parameters and the associated non-dimensional quantities at $\rho_{oil} = 8.7 \cdot 10^2 \text{ kg/m}^3$ and $\mu_{oil} = 8.7 \cdot 10^{-3} \text{ Pa} \cdot \text{s}$.

H in the x_3 -direction, corresponding to the thickness of the oil layer.

The ratio of these scales, defined as $\frac{H}{L} = \varepsilon$, is considered as a small parameter, $\varepsilon \ll 1$. This assumption is based on the typical ranges of oil-slick characteristic lengths, with $1 \text{ m} \leq L \leq 100 \text{ m}$ covering either the size of the oil slick or the scale of perturbations, and $10^{-4} \text{ m} \leq H \leq 10^{-2} \text{ m}$ representing the thickness of the oil layer.

In our analysis we employ several simplifying assumptions for clarity and focus. Firstly, we assume the presence of long water waves only, where the characteristic wavelength L significantly exceeds the thickness of the oil layer. In this context, surface tension effects can be disregarded, as we will demonstrate later, as L vastly surpasses the capillary length

$$L \gg L_c = \sqrt{\frac{\gamma}{\rho g_0}},$$

which is approximately 2 mm or less for most liquids at room temperature [Lukyanov et al., 2024]. Here, γ represents the oil-gas coefficient of surface tension, and g_0 denotes the acceleration due to gravity.

Our second assumption is inherently linked to the first one. Within the long-wavelength regime we assume that the oil layer exerts no retro-active effect

on water movement. This assumption of negligible feedback resonates with similar assumptions made in the suspension theory on the particles following the liquid flow [Guazzelli and Morris, 2012]. The assumption is reasonable when considering the spreading process over a non-stationary water interface. In terms of non-dimensional parameters, the statement is equivalent to small Stokes numbers (reduced Reynolds number), $St = \frac{\rho U H^2}{\mu L} \ll 1$, defined assuming that the characteristic size of the object carried by the liquid motion with the characteristic velocity U is H .

As we will see later, this definition is also equivalent to the Reynolds number in the lubrication theory. As one can see in Table 2.1, the condition is always fulfilled in the range of parameters of our analysis in the study.

The governing equations in the thin film approximation are obtained by using the small parameter ε and averaging the Navier-Stokes equations over the oil layer, that is by introducing liquid fluxes

$$q_{1,2} = \int_B^{h+B} v_{1,2} dx_3,$$

where $v_{1,2}$ are the components of the horizontal velocity.

2.1 The system of Navier-Stokes equations and non-dimensional parameters

The Navier-Stokes equations in the incompressible tensor form in the case of Newtonian fluids are given by

$$\rho \left(\frac{\partial v_j}{\partial t} + v_k \frac{\partial v_j}{\partial x_k} \right) = -\frac{\partial p}{\partial x_j} + \mu \frac{\partial^2 v_j}{\partial x_k \partial x_k} + F_j, \quad (2.1)$$

$$\frac{\partial v_k}{\partial x_k} = 0, \quad (2.2)$$

where i, j are indices from 1 to 3, p is pressure and $F_j = -\rho g_0 \delta_{j3}$ is the gravity body-force according to the system arrangement, as is shown in Fig. 2.1.

The system is brought into non-dimensional form by introduction of the reduced variables

$$\tilde{x}_{1,2} = x_{1,2}/L, \quad \tilde{x}_3 = x_3/H, \quad (2.3)$$

$$\tilde{v}_{1,2} = v_{1,2}/U, \quad \tilde{v}_3 = v_3/\varepsilon U, \quad \tilde{t} = t/t_0 \quad (2.4)$$

where U is the characteristic velocity and $t_0 = L/U$ is the characteristic time scale associated with the longitudinal length scale L .

The pressure in the system is assumed to be dominated by viscous contributions, so it is normalised by the characteristic pressure $p_0 = \frac{\mu U}{\varepsilon H} = \frac{\mu U L}{H^2}$ in the oil layer generated by viscous forces on the basis of the transverse length scale H , $\tilde{p} = p/p_0$ following [Shikhmurzaev, 2007, Craster and Matar, 2009].

In non-dimensional form the governing system of equations becomes

$$\begin{aligned} \text{Re} \left(\frac{\partial v_1}{\partial t} + v_k \frac{\partial v_1}{\partial x_k} \right) &= - \frac{\partial p}{\partial x_1} + \varepsilon^2 \frac{\partial^2 v_1}{\partial x_1^2} + \varepsilon^2 \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2}, \\ \text{Re} \left(\frac{\partial v_2}{\partial t} + v_k \frac{\partial v_2}{\partial x_k} \right) &= - \frac{\partial p}{\partial x_2} + \varepsilon^2 \frac{\partial^2 v_2}{\partial x_1^2} + \varepsilon^2 \frac{\partial^2 v_2}{\partial x_2^2} + \frac{\partial^2 v_2}{\partial x_3^2}, \\ \varepsilon^2 \text{Re} \left(\frac{\partial v_3}{\partial t} + v_k \frac{\partial v_3}{\partial x_k} \right) &= - \frac{\partial p}{\partial x_3} + \varepsilon^4 \frac{\partial^2 v_3}{\partial x_1^2} + \varepsilon^4 \frac{\partial^2 v_3}{\partial x_2^2} + \varepsilon^2 \frac{\partial^2 v_3}{\partial x_3^2} - \alpha_g, \end{aligned} \quad (2.5)$$

$$\frac{\partial v_k}{\partial x_k} = 0. \quad (2.6)$$

In the system (2.5)-(2.6), we retain for simplicity the notation used for dimensional quantities to non-dimensional variables.

The system contains two non-dimensional parameters, the Reynolds number Re , which is customarily used in the thin film approximation by making use of the layer thickness length-scale (reduced Reynolds number) $Re = \varepsilon \frac{\rho U H}{\mu}$ [Craster and Matar, 2009], and parameter $\alpha_g = \varepsilon \frac{\rho g_0 H^2}{\mu U}$ characterizing the strength of the gravity field in comparison to viscous stress.

2.1.1 Navier-Stokes equation boundary conditions

Boundary conditions for the system of the Navier-Stokes equations in the problem formulation are set on the oil-gas free surface and on the oil-water interface [Shikhmurzaev, 2007, Craster and Matar, 2009].

2.1.1.1 Boundary conditions on the oil-gas interface

The boundary conditions to (2.1) - (2.2) on the oil-gas interface, that is at the location $x_3 = h(x_1, x_2, t) + B(x_1, x_2, t)$, in dimensional form are negligible stress tangential to the surface area and the normal stress balance, also known as the capillary boundary condition, that is

$$n_j (\delta_{il} - n_i n_l) \sigma_{ij}^{oil} = 0 \quad (2.7)$$

and

$$n_i n_j (\sigma_{ij}^{oil} - \sigma_{ij}^{gas}) = \gamma \nabla \cdot \mathbf{n}, \quad (2.8)$$

where σ_{ij} are components of the stress tensor in the approximation of a Newtonian liquid, n_j are the components of the normal vector to the interfacial surface, δ_{ij} is the Kronecker-Delta tensor and γ is surface tension of the gas-oil interface. Later on, for simplicity, we omit notations relevant for different media in the problem and assume that the stress tensor components are given for the oil. The divergence $\nabla \cdot \mathbf{n}$ of the normal vector represents the curvature of the oil-gas interface.

The stress tensor in the gas following the assumption of the negligible tangential stress components is simply $\sigma_{ij}^{gas} = -p_a \delta_{ij}$, as it should be in the case of inviscid media, where p_a is the characteristic value of the atmospheric gas pressure.

In non-dimensional form, if we normalise the stress by the characteristic value p_0 ,

$$n_j (\delta_{il} - n_i n_l) \sigma_{ij} = 0 \quad (2.9)$$

and

$$n_i n_j (\sigma_{ij} + p_a \delta_{ij}) = \frac{\varepsilon^2}{Ca} \nabla \cdot \mathbf{n}, \quad (2.10)$$

Here, p_a is the normalised external gas pressure, keeping the same notation for simplicity of the nomenclature, and $Ca = \frac{\mu U}{\gamma}$ is the capillary number.

2.1.1.2 Boundary conditions on the oil-water interface

On the oil-water interface, the retro-active effects of the oil layer on the water movement are neglected assuming that the characteristic wavelength is always larger than a few meters. This condition provides a lower limit on the critical, maximal wavelength when retro-active effects could possibly contribute due to the non-linear coupling of short-gravity waves with the wavelength of the order of a couple of meters or less and the capillary waves [Alpers and Hühnerfuss, 1989, Behroozi et al., 2007]. As the retroactive effects from the oil layer on the water motion have been disregarded in the approximation, only one dynamic boundary condition is necessary, that is the continuity of velocity, which is applied here.

Basically, on the oil-water interface $x_3 = B(x_1, x_2, t)$, the no-slip condition is assumed in tangential to the interface direction, such that the oil motion is driven by the water-wave velocity V_i ,

$$(\delta_{il} - n_i n_l) v_i = (\delta_{il} - n_i n_l) V_i(x_1, x_2, x_3, t). \quad (2.11)$$

2.2 The thin film approximation

To obtain a simplified system of governing equations we make use of the small parameter of the problem $\varepsilon \ll 1$ by neglecting all terms proportional to ε^2 and ε^4 . In this way, it is possible to preserve the inertial terms in the limit of thin films, where the pressure is created by viscous forces, so that if necessary the model can be simplified to the regime of small Reynolds numbers or equally easily generalised into the finite Reynolds number case. The system of equations (2.5)-(2.6) becomes

$$\begin{aligned} \text{Re} \left(\frac{\partial v_1}{\partial t} + v_\ell \frac{\partial v_1}{\partial x_\ell} \right) &= - \frac{\partial p}{\partial x_1} + \frac{\partial^2 v_1}{\partial x_3^2}, \\ \text{Re} \left(\frac{\partial v_2}{\partial t} + v_\ell \frac{\partial v_2}{\partial x_\ell} \right) &= - \frac{\partial p}{\partial x_2} + \frac{\partial^2 v_2}{\partial x_3^2}, \\ 0 &= - \frac{\partial p}{\partial x_3} - \alpha_g, \end{aligned} \tag{2.12}$$

$$\frac{\partial v_k}{\partial x_k} = 0. \tag{2.13}$$

From the last equation in (2.12) the pressure can be obtained explicitly as

$$p = -\alpha_g x_3 + \phi(x_1, x_2), \tag{2.14}$$

where ϕ is a function of x_1 and x_2 only, which is to be determined from boundary conditions.

2.2.1 Boundary conditions in the thin film approximation

Boundary conditions for the simplified system of the Navier-Stokes equations on the oil-gas free surface and on the oil-water interface in the thin-film approximation are obtained by simplifying the boundary conditions to the full system of the Navier-Stokes equations given by (2.9) - (2.11), details relevant to various applications of the thin-film lubrication approximation can be also found in [Shikhmurzaev, 2007, Craster and Matar, 2009].

2.2.1.1 Boundary conditions on the oil-gas interface

The orientation of a free surface is characterised by the normal vectors. The normal vectors to the oil-gas interface at $x_3 = h(x_1, x_2, t) + B(x_1, x_2, t)$ are given by

$$n_j = \pm \frac{\frac{\partial s}{\partial x_j}}{\sqrt{\left(\frac{\partial s}{\partial x_1}\right)^2 + \left(\frac{\partial s}{\partial x_2}\right)^2 + \left(\frac{\partial s}{\partial x_3}\right)^2}}$$

where $s = x_3 - h - B$.

In non-dimensional form, neglecting contributions of the order of ε^2 , it is obvious that

$$\sqrt{n_k n_k} = \sqrt{\varepsilon^2 \left(\frac{\partial(h+B)}{\partial x_1}\right)^2 + \varepsilon^2 \left(\frac{\partial(h+B)}{\partial x_2}\right)^2 + 1} \approx 1.$$

Therefore, using the approximate form of the normalising factor,

$$n_1 = \varepsilon \frac{\partial(h+B)}{\partial x_1},$$

$$n_2 = \varepsilon \frac{\partial(h+B)}{\partial x_2}$$

and

$$n_3 = -1,$$

where the normal vector pointing downwards in the liquid domain is selected. The boundary conditions to (2.1) - (2.2) on the oil-gas interface at the location given by $x_3 = h(x_1, x_2, t) + B(x_1, x_2, t)$, as we have already stated, in dimensional form are negligible stress tangential to the surface area and the normal stress balance, also known as the capillary boundary condition [Shikhmurzaev, 2007, Craster and Matar, 2009], that is

$$n_j (\delta_{il} - n_i n_l) \sigma_{ij} = 0 \quad (2.15)$$

and

$$n_i n_j (\sigma_{ij} + p_a \delta_{ij}) = \gamma \nabla \cdot \mathbf{n}, \quad (2.16)$$

where γ is surface tension of the gas-oil interface.

Changing to non-dimensional variables, if we normalise stress by the characteristic value p_0 , the stress tensor components for a Newtonian fluid in non-dimensional form are given by

$$\sigma_{12} = \varepsilon^2 \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right), \quad \sigma_{13} = \varepsilon \left(\frac{\partial v_1}{\partial x_3} + \varepsilon^2 \frac{\partial v_3}{\partial x_1} \right), \quad \sigma_{23} = \varepsilon \left(\frac{\partial v_2}{\partial x_3} + \varepsilon^2 \frac{\partial v_3}{\partial x_2} \right),$$

and

$$\sigma_{11} = -p + 2\varepsilon^2 \frac{\partial v_1}{\partial x_1}, \quad \sigma_{22} = -p + 2\varepsilon^2 \frac{\partial v_2}{\partial x_2}, \quad \sigma_{33} = -p + 2\varepsilon^2 \frac{\partial v_3}{\partial x_3}.$$

The curvature in the thin film approximation

$$\nabla \cdot \mathbf{n} = \varepsilon \left(\frac{\partial^2(h+B)}{\partial x_1^2} + \frac{\partial^2(h+B)}{\partial x_2^2} \right).$$

Equations (2.15) in the leading order of ε are reduced to two conditions

$$\sigma_{13} \approx \varepsilon \left. \frac{\partial v_1}{\partial x_3} \right|_{x_3=h+B} = 0$$

and

$$\sigma_{23} \approx \varepsilon \left. \frac{\partial v_2}{\partial x_3} \right|_{x_3=h+B} = 0.$$

Using $n_1, n_2 = O(\varepsilon)$ and $n_3 = -1$, the leading contribution in the normal stress balance is $\sigma_{33} + p_a$, while the curvature term is taken from the thin-film approximation written above.

The normal stress boundary condition (2.16) keeping leading order terms is down to

$$-p + 2\varepsilon^2 \frac{\partial v_3}{\partial x_3} + p_a = \frac{\varepsilon^3}{Ca} \left(\frac{\partial^2(h+B)}{\partial x_1^2} + \frac{\partial^2(h+B)}{\partial x_2^2} \right).$$

Here, p_a is the external gas pressure and $Ca = \frac{\mu U}{\gamma}$ is the capillary number. The parameter α_g does not appear directly in the normal stress boundary condition. It enters through the vertical momentum equation (2.12)

$$0 = -\frac{\partial p}{\partial x_3} - \alpha_g.$$

Integrating this equation with $p|_{x_3=h+B} = p_a$ gives

$$p = p_a + \alpha_g(h + B - x_3),$$

If the characteristic ratio of the hydrostatic pressure to the viscous stress, that is the parameter

$$\alpha_g = \frac{\rho g_0 \varepsilon H^2}{\mu U} \gg \varepsilon^2$$

and

$$\alpha_g = \frac{\rho g_0 \varepsilon H^2}{\mu U} \gg \varepsilon^3 Ca^{-1}.$$

then viscous and surface tension contributions to the pressure at the free surface can be neglected, so that $p|_{x_3=h+B} = p_a$. Note, the last condition is equivalent to

$$L \gg L_c = \sqrt{\frac{\gamma}{\rho g_0}},$$

that is the longitudinal length scale is much larger than the capillary length $L_c \approx 2$ mm. This condition is easily satisfied in the long-wave approximation. The first condition is also well satisfied as one can observe from Table 2.1.

As a result, in the thin film approximation, neglecting surface tension effects, conditions (2.15) and (2.16) are reduced to

$$\left. \frac{\partial v_1}{\partial x_3} \right|_{x_3=h+B} = 0, \quad \left. \frac{\partial v_2}{\partial x_3} \right|_{x_3=h+B} = 0 \quad (2.17)$$

and

$$p|_{x_3=h+B} = p_a \quad (2.18)$$

respectively.

2.2.1.2 Boundary conditions on the oil-water interface

As we have already stated, on the oil-water interface, the retro-active effects of the oil layer on the water movement are neglected, so that, as a result of the assumption, only one dynamic boundary condition is necessary, that is the continuity of velocity, which is applied in our approach.

Basically, on the oil-water interface $x_3 = B(x_1, x_2, t)$, the no-slip condition is assumed in tangential to the interface direction, such that the oil motion is driven by the water-wave velocity V_i ,

$$(\delta_{il} - n_i n_l) v_i = (\delta_{il} - n_i n_l) V_i(x_1, x_2, x_3, t). \quad (2.19)$$

In the thin film approximation, the last condition is equivalent to

$$v_i = V_i(x_1, x_2, t) \quad i = 1, 2. \quad (2.20)$$

2.2.1.3 Kinematic boundary conditions and the conservation of mass

The motions of both interfaces are subject to kinematic boundary conditions, which are

$$v_3|_{x_3=B} = \frac{\partial B}{\partial t} + v_1 \frac{\partial B}{\partial x_1} + v_2 \frac{\partial B}{\partial x_2}. \quad (2.21)$$

and

$$v_3|_{x_3=h+B} = \quad (2.22)$$

$$\frac{\partial(h+B)}{\partial t} + v_1 \frac{\partial(h+B)}{\partial x_1} + v_2 \frac{\partial(h+B)}{\partial x_2}.$$

Integrating the incompressibility condition (2.13) from $x_3 = B$ to $x_3 = h+B$, and applying the Leibniz integral rule together with the kinematic conditions (2.21) and (2.22), gives

$$\frac{\partial h}{\partial t} + \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} = 0, \quad (2.23)$$

where

$$q_{1,2} = \int_B^{h+B} v_{1,2} dx_3,$$

are the liquid fluxes in the oil layer.

The result is the key in formulating a closed form of the governing equation in the model.

2.2.2 Averaging of the governing equations using the Karman-Pohlhausen ansatz

The way to deal with the remaining governing equations (2.12) is to use the so-called classical Karman-Pohlhausen ansatz [Pohlhausen, 1921], see also a review [Craster and Matar, 2009], for the velocity distribution in the oil layer

$$v_{1,2} = V_{1,2} - \frac{3}{2} \frac{q_{1,2} - V_{1,2}h}{h^3} \left\{ x_3^2 - 2(h+B)x_3 + B(B+2h) \right\}, \quad (2.24)$$

which satisfies the boundary conditions on the oil-water and the oil-gas interfaces.

Apart from respecting the boundary conditions, the velocity distribution given by the ansatz mimics the distributions commonly obtained in rectilinear viscous flows, in particular in the well-known Hagen–Poiseuille rectilinear flow characterised by a parabolic velocity profile distribution.

The distribution is, of course, self-consistent, as it is not difficult to show that

$$\int_B^{h+B} v_{1,2} dx_3 = q_{1,2}.$$

Before turning our attention to the averaging procedure, the distribution of pressure using the boundary condition (2.18) is needed.

Explicitly, from (2.14)

$$p = p_a + \alpha_g(h + B - x_3).$$

Integrating the first two equations in (2.12) with respect to x_3 from B to $h+B$ by using the explicit expression for pressure, one finally gets a closed system of the governing equations in the thin film approximation to determine q_1 , q_2 and h as functions of time t and spatial variables x_1 and x_2

$$\frac{\partial h}{\partial t} + \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} = 0, \quad (2.25)$$

$$\begin{aligned} & \frac{\partial q_1}{\partial t} + \frac{\partial}{\partial x_1} \left(\frac{6}{5} \frac{q_1^2}{h} - \frac{2}{5} (q_1 V_1) + \frac{1}{5} V_1^2 h \right) + \\ & \frac{\partial}{\partial x_2} \left(\frac{6}{5} \frac{q_1 q_2}{h} - \frac{1}{5} (q_1 V_2 + q_2 V_1) + \frac{1}{5} V_1 V_2 h \right) = \\ & -h \frac{\alpha_g}{Re} \frac{\partial(h+B)}{\partial x_1} - \frac{3}{Re} \frac{q_1 - V_1 h}{h^2} \end{aligned} \quad (2.26)$$

and

$$\begin{aligned} \frac{\partial q_2}{\partial t} + \frac{\partial}{\partial x_1} \left(\frac{6}{5} \frac{q_1 q_2}{h} - \frac{1}{5} (q_1 V_2 + q_2 V_1) + \frac{1}{5} V_1 V_2 h \right) + \\ \frac{\partial}{\partial x_2} \left(\frac{6}{5} \frac{q_2^2}{h} - \frac{2}{5} (q_2 V_2) + \frac{1}{5} V_2^2 h \right) = \\ -h \frac{\alpha_g}{Re} \frac{\partial(h+B)}{\partial x_2} - \frac{3}{Re} \frac{q_2 - V_2 h}{h^2}. \end{aligned} \quad (2.27)$$

The obtained system of governing equations resembles various versions of the lubrication thin film models used in other applications [Craster and Matar, 2009], but it has essential specific features, as the mathematical model was tuned for the problem in hand.

Our further considerations will be based on several simplifying assumptions in order to obtain observable results. First of all, we will concentrate on the one-dimensional problem formulation. Secondly, as one can readily observe in Table 2.1, the Reynolds number $Re \ll 1$, therefore we consider the limit of small Reynolds numbers.

2.3 The model in the limit of small Reynolds numbers

In the limit of small Reynolds number $Re \ll 1$ the system of equations (2.25)-(2.27) is reduced to a single non-linear advection-diffusion equation.

$$\frac{\partial h}{\partial t} + \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} = \frac{\partial h}{\partial t} + \nabla \cdot \mathbf{q} = 0, \quad (2.28)$$

$$q_i = -\frac{\alpha_g}{3} \frac{\partial(h+B)}{\partial x_i} h^3 + V_i h, \quad i = 1, 2.$$

As the oil slick is finite in the horizontal dimensions, we assume that there is a smooth boundary Γ of the two-dimensional domain in $\mathbb{R}^2$ with elements (x_1, x_2) , where

$$h|_{\Gamma} = 0.$$

As the boundary is physically free to move, one needs to investigate additional conditions regulating mathematically the boundary motion in the model.

A well-known, particular example of the moving boundary-value problem to a partial differential equation of a similar type is the so-called Stefan problem, where the boundary motion between two different phases of matter, such as water and ice, is regulated by the Stefan condition [Gupta, 2003]. The Stefan condition results from the conservation of thermal energy due to the latent heat required to achieve the phase transition. In a similar way, the additional condition in our case can be obtained from the conservation of mass at the moving boundary. Basically, in the reference frame moving with the boundary, the total mass flux through the boundary is assumed to be zero.

Mathematically, this can be stated as

$$\mathbf{q}' \cdot \mathbf{n}_{\Gamma} = \mathbf{q} \cdot \mathbf{n}_{\Gamma} - h(\mathbf{u} \cdot \mathbf{n}_{\Gamma}) = 0,$$

where $\mathbf{q}'$ is the flux density in the moving frame of reference and $\mathbf{n}_{\Gamma}$ is the normal vector to the boundary Γ in the (x_1, x_2) -plane.

The motion of the boundary in the normal direction is described by means

of the flux at the boundary, that is the normal component of the boundary velocity $\mathbf{u}$

$$\mathbf{u} \cdot \mathbf{n}_\Gamma = \mathbf{q} \cdot \mathbf{n}_\Gamma h^{-1}. \quad (2.29)$$

As one can see, the capillary effects, in particular the capillary pressure associated with surface curvature near the contact-line region where h vanishes, have been neglected in (2.29). This is a subtle point. Such a simplification can be possibly justified as in the problem, we study, the contact-line region is assumed to be in fast motion such that the characteristic capillary number $Ca = \frac{\mu U}{\gamma} \approx 0.5$ at $U = 1$ m/s, $\gamma \approx 20$ mN/m [Shaw, 2003] and $\mu \approx 10^{-2}$ Pa·s is close to $Ca = 1$, when viscous stresses prevail. Note, the capillary effects in dynamic wetting usually dominate at rather small values $Ca \ll 1$ [Schwartz, 1989, Shikhmurzaev, 2007, Lukyanov and Likhtman, 2016]. We also note, that in the case of creeping flow conditions over a calm sea surface area, such an assumption may be inaccurate as we know that the contact line effects result in fingering instabilities in spreading over solids [Troian et al., 1989]. In a way, the condition at the moving boundary, we set, is a compromise to obtain observable results but taking into account many uncertainties about the current state of the art of the moving-contact line problem over easily deformable liquid surfaces.

2.4 One-dimensional problem formulations

To analyse the effects of the external forcing due to the wave motion, we consider a one-dimensional problem posed on a compact manifold $x_l \leq x \leq$

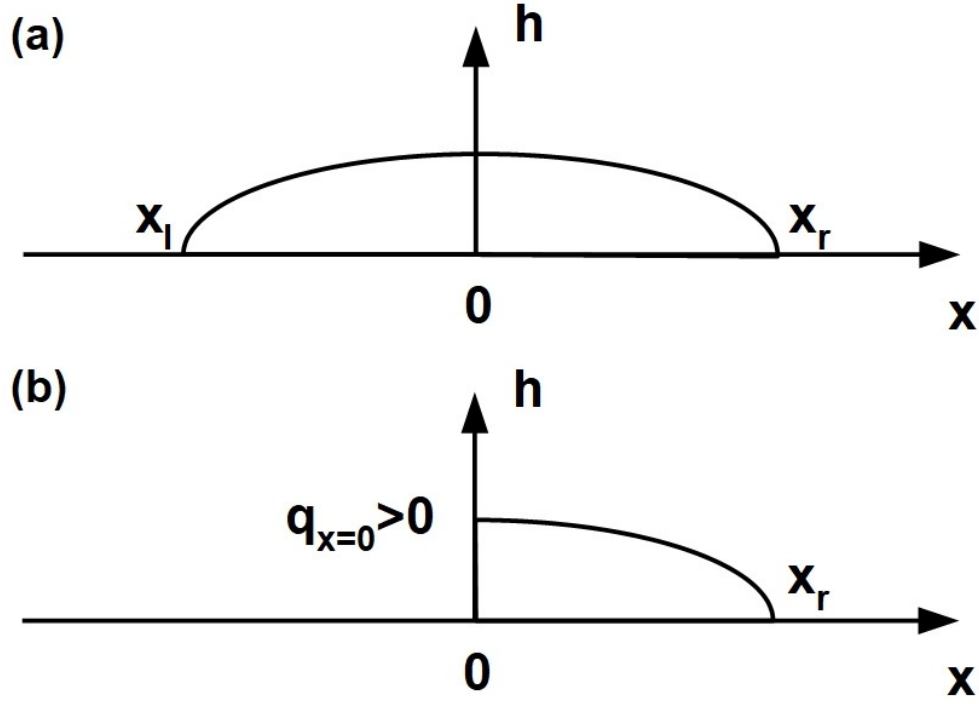


Figure 2.2: Illustration of the one-dimensional problem formulations.

x_r governed by

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad (2.30)$$

$$q = -\frac{\alpha_g}{3} \frac{\partial(h+B)}{\partial x} h^3 + Vh$$

with various boundary conditions.

We shall be interested in two basic problem formulations and geometries as shown in Fig. 2.2.

- In the first problem formulation, demonstrated in panel (a) of Figure 2.2, both ends are moving boundaries with the boundary conditions

$$h(x_l, t) = h(x_r, t) = 0. \quad (2.31)$$

The domain boundaries are moving at the rate defined by the flux value in a stationary reference frame.

$$\frac{dx_l}{dt} = \frac{q}{h} \Big|_{x=x_l} = -\frac{\alpha_g}{3} \frac{\partial(h+B)}{\partial x} h^2 + V, \quad (2.32)$$

and

$$\frac{dx_r}{dt} = \frac{q}{h} \Big|_{x=x_r} = -\frac{\alpha_g}{3} \frac{\partial(h+B)}{\partial x} h^2 + V. \quad (2.33)$$

corresponding to the boundary of a two-dimensional oil slick. Although $h(x_l, t) = h(x_r, t) = 0$ at the boundaries, the ratio q/h does not vanish. Near the edge the flux q behaves as h^3 times the gradient of h , and the gradient of h becomes singular in such a way that the ratio q/h approaches a finite non-zero value. Therefore, equations (2.32) and (2.33) correctly give the finite speed of the moving boundary.

- The second moving boundary-value problem, shown in panel (b) of Figure 2.2, is to simulate a situation when there is some external source of the oil coming in at one end of the domain. In this case one boundary at $x_l = 0$ is fixed, that is stationary, where one boundary condition is required to single out a particular solution. In our model the flux is given at this end, so that the total flux into the domain may not be zero and the total amount of the oil not conserved.

By continuity

$$q(0, t) = q_0(t). \quad (2.34)$$

Here $q_0(t)$ is a given function of time. At the other end we have, as in

the previous case, the boundary conditions

$$h(x_r) = 0,$$

and

$$\frac{dx_r}{dt} = \frac{q}{h} \Big|_{x=x_r} = -\frac{\alpha_g}{3} \frac{\partial(h+B)}{\partial x} h^2 + V. \quad (2.35)$$

2.4.1 Conservation of mass

As we have seen in the formulation of the boundary condition at the moving ends, the key feature is the conservation of mass or the mass balance. Intuitively, in the absence of external fluxes into the domain, the mass must be conserved.

Indeed, the rate of change of the total mass M by the Leibniz integral rule is

$$\frac{dM}{dt} = \frac{d}{dt} \int_{x_l}^{x_r} h \, dx = \frac{dx_r}{dt} h(x_r, t) - \frac{dx_l}{dt} h(x_l, t) + \int_{x_l}^{x_r} \frac{\partial h}{\partial t} \, dx,$$

which is a one-dimensional equivalent of the general results given by the Reynolds Transport Theorem.

Using (2.30), the last term in the right-hand side can be written as

$$\int_{x_l}^{x_r} \frac{\partial h}{\partial t} \, dx = - \int_{x_l}^{x_r} \frac{\partial q}{\partial x} \, dx = q(x_l, t) - q(x_r, t).$$

Combining the results, one gets

$$\frac{dM}{dt} = \frac{dx_r}{dt} h(x_r, t) - \frac{dx_l}{dt} h(x_l, t) + q(x_l, t) - q(x_r, t).$$

Therefore, in the first problem by the boundary conditions (2.32) - (2.33) the total rate of change

$$\frac{dM}{dt} = 0.$$

In the second problem formulation, as the end $x_l = 0$ is fixed,

$$\frac{dM}{dt} = q(x_l, t) = q_0(t).$$

2.4.2 Water waves

In the model, the water wave motion is assumed to be given. It serves as the driving force of the oil layer dynamics, neglecting the backlash.

As the main model for the water wave perturbations, we will choose the water waves in the approximation of the deep water in the linear regime.

The water-oil interface perturbations functions B and V , are taken as harmonics of the deep-water waves [Lighthill, 1978] in the linear approximation

$$B(x, t) = \sum_j B_0^j \sin(k_j x - \omega_j t), \quad (2.36)$$

$$V(x, t) = \sum_j B_0^j \varepsilon \omega_j \sin(k_j x - \omega_j t).$$

Here B_0^j is the amplitude, ω_j is the frequency and k_j is the wave number ($k = 2\pi/\lambda$, where λ is the wavelength). In non-dimensional form, the dispersion relation is

$$\omega_j^2 = \varepsilon^{-1} Fr^{-2} k_j, \quad (2.37)$$

where $Fr^{-2} = \frac{g_0 H}{U^2}$ is the Froude number.

These follow from the deep water irrotational water wave theory, where it has been established that, in dimensional form, the surface elevation η , the flow potential Φ and the dispersion equation are, in two-dimensional setting, as follows

$$\eta(x, t) = A \sin(kx - \omega t), \quad (2.38)$$

$$\Phi(x, z, t) = -Ag_0\omega^{-1} \cos(kx - \omega t) \exp(kz), \quad (2.39)$$

and

$$\omega^2 = g_0 k, \quad (2.40)$$

where A is wave amplitude, and ω and k are the wave angular frequency and the wave number respectively. Here z is the vertical Cartesian coordinate (i.e. $z = x_3$).

In the thin-film scaling, we identify $B(x, t)$ with the non-dimensional surface elevation η and $V(x, t)$ with the non-dimensional horizontal velocity at the interface obtained from $\partial\Phi/\partial x$.

In the thin film approximation, provided that $A \ll 2\pi k^{-1}$, $\exp(kz) \approx 1$, and the result (2.36) follows.

To understand the nature of the approximation made and its sensitivity to the water wave parameters, we consider some non-linear solutions available for the surface wave motion. This will be instructive and informative to consider the second-order Stokes waves on arbitrary depth [Whitham, 1974].

In this case, one has

$$\eta(x, t) = A \left\{ \cos(kx - \omega t) + kA \frac{3 - P_s^2}{4P_s^2} \cos(2kx - 2\omega t) \right\} + O((kA)^3) \quad (2.41)$$

with the dispersion relation

$$\omega^2 = g_0 k P_s + O((kA)^2), \quad (2.42)$$

where $P_s = \tanh(kH_w)$ and H_w is the mean depth of the water layer.

This is the standard second-order Stokes dispersion relation for water of finite depth [Whitham, 1974]. As one can see, there are dependencies on the ratio of the water layer depth and the wave length of the wave and the ratio of the wave amplitude to the wave length. Those non-linear contributions are essential when parameters kA and kH_w come close to one.

On the other hand, in the limit

$$kH_w \rightarrow \infty,$$

one has

$$\lim_{kH_w \rightarrow \infty} P_s \rightarrow 1.$$

If we then take the second limit $kA \rightarrow 0$, then the deep water relationships (2.41) and (2.42) are recovered.

2.5 Diffusion-dominated model

If $B = 0$ and $V = 0$ the governing equation (2.30) is reduced to a non-linear diffusion equation of the porous medium equation (PME) type,

$$\frac{\partial h}{\partial t} = A_g \frac{\partial}{\partial x} \left(h^3 \frac{\partial h}{\partial x} \right) \quad (2.43)$$

where $A_g = \alpha_g/3$, and α_g is the non-dimensional gravity parameter introduced in Section 2.1. As in the general case the PDE is augmented with two sets of the boundary conditions mimicking the evolution of an oil slick, either by itself or with some discharge of oil coming into the layer, that is either

$$h(x_l, t) = h(x_r, t) = 0 \quad (2.44)$$

or

$$q(x_l, t) = q_0(t), \quad h(x_r, t) = 0. \quad (2.45)$$

To understand the properties of the model, consider first some particular solutions available. Namely, in the first case the problem is known to have self-similar solutions [Aronson, 1986], subject to $h = 0$ at the moving boundaries, of the form

$$h(x, t) = \frac{A_s}{t^{\frac{1}{n(3n+2)}}} \left(C - \left(\frac{x}{t^{\frac{1}{3n+2}}} \right)^2 \right)^{1/3}, \quad (2.46)$$

here n denotes the spatial dimension of the problem, $A_s = \left(\frac{3}{2A_g(3n+2)} \right)^{1/3}$ and C is a parameter, which is defined by the total initial amount of the oil. In

our thesis, we take $n = 1$, corresponding to the one-dimensional case. The solution (2.46) represents an isolated oil slick, similar to that shown in panel (a) of Figure 2.2. The slick is spreading out and flattening due to conservation of mass, and is closely resembling the spreading processes observed on the surface of calm water [Fay, 1969, Fay, 1971, Hoult, 1972a]. Therefore, the model seems to be at least qualitatively adequate to describe dynamics of oil slicks. We also note, that the self-similar solutions to the non-linear diffusion equation are attractors [Aronson, 1986], so that an arbitrary initial profile satisfying boundary conditions will ultimately evolve to one of them.

To understand the model quantitatively, we briefly compare it with observations before elaborating on the details. Parameter n in (2.46) is an integer defined by the problem dimension, which is $n = 1$ for one-dimensional spreading and $n = 2$ in the case of two-dimensional axisymmetric spreading [Lukyanov et al., 2024].

If we consider the axisymmetric 2D case, where $n = 2$, the algebraic form (2.46) suggests that the boundary (radius r) of the domain is moving according to

$$r \propto t^\alpha, \quad \alpha = 1/(3n + 2) = 1/8,$$

which is well in the range of the exponents found in droplet spreading experiments [Lelah and Marmur, 1981] over solid surfaces at different conditions

$$r \propto t^\alpha, \quad 1/10 \leq \alpha \leq 1/6.$$

This rate is still slower than that found in the spreading experiments over calm waters in an axisymmetric case with the exponent $\alpha = 1/4$ [Hoult,

1972a]. In a one-dimensional case, $n = 1$, the situation is similar. The self-similar solution implies $\alpha = 1/5$, while the viscous spreading rate has an exponent $\alpha = 3/8$ [Hoult, 1972a].

Such a trend should be expected, as the liquid substrate provides a very effective lubrication layer for the spreading liquid. Therefore, the model we utilise is not expected to very accurately describe spreading over a steady interface or in the limit of very small amplitudes $B_0 \ll 1$, though it is still expected to provide very good qualitative guidance.

In the next section, we consider the mathematical derivation of self-similar solutions to a class of the non-linear diffusion problems in detail, which will be used to analyse the profile behaviour, even in the case of external source present. The latter will be the subject of our analysis in Chapter 5.

2.5.1 Self-similar solutions of the non-linear diffusion equation

The foundation of self-similarity is in the invariance of problem formulations, that is in the invariance of the governing equation with the set of boundary conditions, to some transformation of variables [Barenblatt, 1996].

The boundary value problem involving the non-linear diffusion equation (2.43) with the boundary condition (2.44), set on the moving boundary, is invariant under the transformation of variables $t' = t/\lambda$, $x' = x/\lambda^\beta$ and $h' = h\lambda^{-\alpha}$, for any positive constant $\lambda > 0$.

Hence the combinations of dependent and independent variables x/t^β and ht^α are also invariants, which suggest new dependent and independent variables

$w = x t^{-\beta}$ and $\phi(w) = h t^\alpha$.

These properties allow some important particular solutions of the non-linear diffusion equation on a compact support with moving boundaries, the so-called Barenblatt solutions of the moving boundary-value problem, similar to the example considered in the previous section [Aronson, 1986].

In the short derivation which follows, we consider the more general case of the PME,

$$\frac{\partial h}{\partial t} = A \frac{\partial}{\partial x} \left(h^m \frac{\partial h}{\partial x} \right) \quad (2.47)$$

for a general m .

Invariance of the governing equation directly implies that the parameters of the transformation must satisfy the condition

$$m\alpha + 2\beta = 1. \quad (2.48)$$

By changing to the new (self-similar) variables $w = x t^{-\beta}$ and $\phi = h t^\alpha$, the main governing non-linear diffusion equation a partial differential equation, can be transformed into the ordinary differential equation (ODE) in the self-similar variables w and ϕ ,

$$\beta w \frac{d\phi}{dw} + \alpha \phi + A \frac{d}{dw} \left(\phi^m \frac{d\phi}{dw} \right) = 0. \quad (2.49)$$

To obtain the self-similar solutions requires a set of boundary conditions which depend on the particular problem formulation. On a general note, one may notice that equation (2.49) is quite difficult to treat analytically, so that later on we consider some specific examples and use numerical solutions.

2.5.2 Self-similarity when the amount of oil is conserved

We are looking for a symmetric solution such that $h(x) = h(-x)$ and, of course, $x_l(t) = -x_r(t)$, and due to conservation of the oil

$$\int_{x_l}^{x_r} h(x, t) dx = 2 \int_0^{\hat{w}} t^{-\alpha+\beta} \phi(w) dw = 2I_0, \quad (2.50)$$

where $\hat{w} = x_r t^{-\beta}$. Here $\hat{w}$ is a constant (independent of time).

For the left-hand side to be independent of time, the power of t must be zero:

$$-\alpha + \beta = 0.$$

Then one can immediately deduce from (2.48) and (2.50) that

$$\alpha = \beta = \frac{1}{m+2}.$$

It follows that equation (2.49) can be simplified to

$$\frac{1}{m+2} \frac{d}{dw} (w\phi) + A \frac{d}{dw} \left(\phi^m \frac{d\phi}{dw} \right) = 0, \quad (2.51)$$

which is complemented with two boundary conditions

$$\phi(w)|_{w=\hat{w}} = 0, \quad (2.52)$$

and

$$\left. \frac{d\phi}{dw} \right|_{w=0} = 0. \quad (2.53)$$

Integrating equation (2.51) with respect to w once, we obtain

$$\frac{1}{m+2} w\phi + A \phi^m \frac{d\phi}{dw} = C_1, \quad (2.54)$$

where C_1 is a constant.

From the boundary condition (2.53), $C_1 = 0$, and integrating (2.54),

$$\frac{1}{2(m+2)} w^2 + \frac{A}{m} \phi^m = C_2,$$

where C_2 is a constant, which from the boundary condition (2.52) is

$$C_2 = \frac{\hat{w}^2}{2(m+2)}.$$

Consequently, we obtain

$$\phi(w) = \left(\frac{m}{2A(m+2)} \right)^{1/m} (\hat{w}^2 - w^2)^{1/m}. \quad (2.55)$$

Note that at the moving boundary $w = \hat{w}$, the derivative $\frac{d\phi}{dw}$ becomes infinite, producing a vertical tangent (90° slope) in the plot of $\phi(w)$. This follows directly from the form $\phi(w) \sim (\hat{w} - w)^{1/m}$ near $w = \hat{w}$. For $m = 3$ (the case shown in Figure 2.3), the slope behaves as $(\hat{w} - w)^{-2/3}$, which tends to $-\infty$ as $w \rightarrow \hat{w}^-$. This infinite gradient at the edge is a characteristic feature of the porous-medium equation when $m > 1$ and is the mathematical origin of the waiting-time phenomenon discussed in detail in Section 2.5.4.

To find the value of the constant $\hat{w}$, one needs to use conservation of mass (2.50). Substituting the explicit form of the self-similar solution (2.55) into this equation yields

$$\int_0^{\hat{w}} (\hat{w}^2 - w^2)^{1/m} dw = \left(\frac{2A(m+2)}{m} \right)^{1/m} I_0.$$

By using the trigonometric substitution $w = \hat{w} \sin \theta$, we obtain

$$(\hat{w})^{\frac{2+m}{m}} \int_0^{\frac{\pi}{2}} (\cos \theta)^{\frac{2+m}{m}} d\theta = \left(\frac{2A(m+2)}{m} \right)^{1/m} I_0.$$

The trigonometric substitution $w = \hat{w} \sin \theta$ is chosen because it transforms the integral into a standard form involving only powers of $\cos \theta$, with the integration limits changing from $w = 0$ to $w = \hat{w}$ into $\theta = 0$ to $\theta = \pi/2$. The integral on the left hand side can be computed via the formula (3.62.1) in [Zwillinger and Jeffrey, 2007]. That is

$$\int_0^{\frac{\pi}{2}} (\cos \theta)^{\frac{2+m}{m}} d\theta = 2^{\frac{2}{m}} \left(\frac{\left(\Gamma\left(\frac{1+m}{m}\right)\right)^2}{\Gamma\left(\frac{2(1+m)}{m}\right)} \right),$$

where Γ denotes the Gamma function. Hence

$$\hat{w} = \left[\left(\frac{A(2+m)}{2m} \right)^{1/m} \left(\frac{\Gamma\left(\frac{2(1+m)}{m}\right)}{\left(\Gamma\left(\frac{1+m}{m}\right)\right)^2} \right) I_0 \right]^{\frac{m}{2+m}}. \quad (2.56)$$

The solution (2.55) can be written in the original variables as

$$h = \frac{1}{t^{\frac{1}{2+m}}} \left(\frac{m}{2A(m+2)} \right)^{1/m} \left(\hat{w}^2 - \left(\frac{x}{t^{\frac{1}{2+m}}} \right)^2 \right)^{1/m}. \quad (2.57)$$

One can immediately see that the obtained solution (2.57) is equivalent to

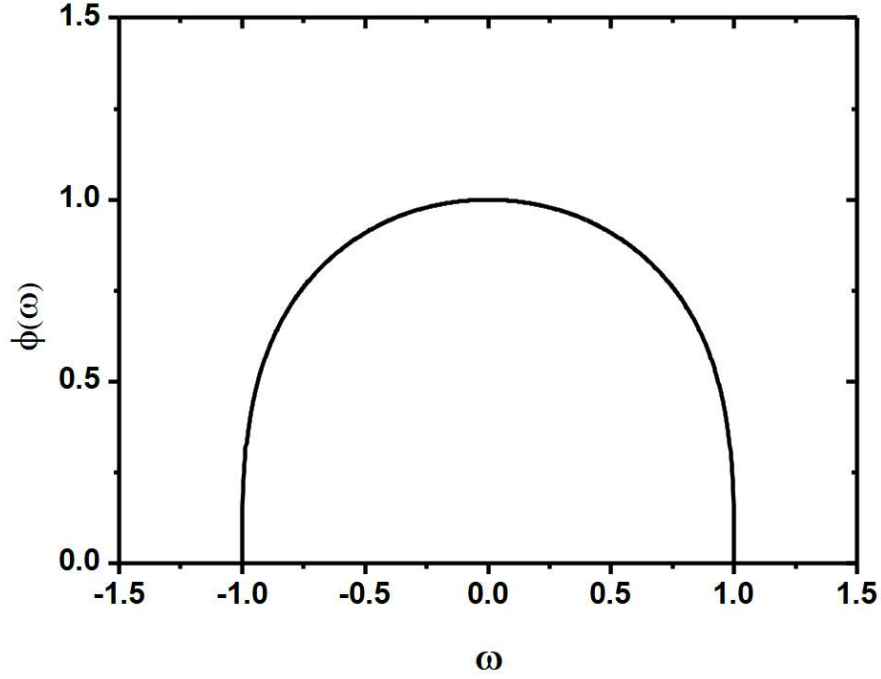


Figure 2.3: Self-similar solution (2.55), $\phi(w)$ as a function of the invariant variable w at $m = 3$ and $\hat{w} = 1$.

(2.46) in the particular case $m = 3$ at $n = 1$.

2.5.3 Self-similarity when the amount of oil is changing

It follows that when the amount of oil is changing

$$\frac{dM}{dt} = \frac{d}{dt} \int_0^{x_r} h dx = t^{\beta-\alpha-1}(\beta - \alpha) \int_0^{\hat{w}} \phi dw = q_0 t^\gamma. \quad (2.58)$$

The right-hand side $q_0 t^\gamma$ represents the prescribed time-dependent influx rate at the fixed boundary $x = 0$, as given by the boundary condition (2.45) [Huppert, 1982, Lister, 1992]. This relation (2.58) will be used in Chapter 5 to

derive and analyse the self-similar solutions.

Then we require for the self-similarity that

$$\alpha = \beta - \gamma - 1. \quad (2.59)$$

Then, using (2.48), this leads to

$$\alpha = -\frac{2\gamma + 1}{m + 2}, \quad \beta = \frac{1 + m(\gamma + 1)}{m + 2}.$$

Equation (2.58) subject to the constraint (2.59) directly implies that

$$(\beta - \alpha) \int_0^{\hat{w}} \phi dw = (\gamma + 1) \int_0^{\hat{w}} \phi dw = q_0,$$

which can be used to determine the problem parameter $\hat{w}$. Equation (2.58) also implies that for a self-similar solution to exist, the parameter $\gamma > -1$ if the external flux is a source, that is $q_0 > 0$. As can be seen from (2.59), depending on the exponent γ of the rate of change of the external flux, there could be different regimes of spreading, which can be identified independently of the source strength q_0 .

The self-similar profile should satisfy (2.49) with

$$\phi(\hat{w}) = 0, \quad (2.60)$$

and

$$\phi^m \frac{\partial \phi}{\partial w} \Big|_{w=0} = -\frac{q_0}{A}. \quad (2.61)$$

The problem (2.49), (2.60) and (2.61) will be analysed in Chapter 5 in detail.

2.5.4 Self-similar solutions and waiting times

The solution (2.55) is illustrated in Fig. 2.3 when $m = 3$, the coefficient of non-linearity in the mathematical model. One can observe from (2.57) that the gradient of the profile is singular at the moving boundary $x_r = \hat{w} t^{1/5}$, that is

$$\frac{dh}{dx} \propto \frac{1}{\left(\hat{w}^2 - \left(\frac{x}{t^{1/5}}\right)^2\right)^{2/3}}$$

At the same time, the speed is finite,

$$\frac{dx_r}{dt} = \frac{q}{h} \propto h^2 \frac{dh}{dx},$$

so that the singularity in the gradient is compensated by $h(x_r) = 0$.

This directly implies that if the initial data does not have singular gradients at the boundaries, then due to the boundary condition $h(x_r) = 0$ the velocity must be vanishing

$$\frac{dx_r}{dt} = 0,$$

until some moment of time when the gradient at the boundary coincides with that of the self-similar solution.

This in turn implies that there should be some waiting time until the end point starts moving, that is when the gradient of the solution becomes singular.

One needs to note that this effect with application to our situation may be artificial, because the capillary effects, which have been ignored at the boundary in our model, would always produce some finite boundary motion

over a liquid surface.

2.6 Summary

In summary, we have developed a thin film lubrication model, which can be used to analyse oil layer behaviour on the dynamic water surfaces, that is in the presence of the surface water waves.

Preliminary analysis of the model in the absence of water-wave motion has demonstrated that the model is qualitatively consistent with observations of oil spreading. Two representative setups have been selected for the further studies in the thesis shown in Fig. 2.2. While some self-similar solutions are available in the absence of the water-wave motion, in general the problem can only be solved (and then approximately) using numerical techniques. In the next chapter, the numerical method is discussed.

Chapter 3

Numerical method: basics and verification

In this chapter, we introduce a numerical technique to approximate the solution of the governing model (2.30), which includes a moving boundary and the method is designed to follow the physics.

In Chapter 2, we studied a simplified case of the model with no external forces, which led a nonlinear diffusion equation of the porous medium type, often called the lubrication equation in the thin-film approximation. For that case, we obtained an exact analytical solution (2.46), known as the Barenblatt (self-similar) solution. In this chapter, we use this exact solution as a benchmark to verify the accuracy of the numerical method. We consider the role of the non-linearity, by omitting external forces and setting $B = 0$ and $V = 0$:

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{3.1}$$

$$q(x, t) = -A_g \frac{\partial h}{\partial x} h^3,$$

where q is the flux at $m = 3$ and $A_g = \frac{\alpha_g}{3}$, where $\alpha_g = \varepsilon \rho g_0 H^2 / \mu U$ is the non-dimensional gravity parameter introduced in Section 2.1 . The goal is to solve this equation numerically and compare the result with the known Barenblatt solution.

There are several ways to approximately solve differential equations (in particular non-linear diffusion equations) numerically, including finite difference, finite volume, and finite element methods. In this thesis, we apply a conservation-based finite difference method to solve a non-linear diffusion equation on a moving domain effectively, and simultaneously manage moving boundaries. The approach adapts the domain dynamically to follow the physics. It builds on established adaptive meshing techniques from prior works [Russell, 1992, Huang and Russell, 2001, Baines et al., 2005, Baines et al., 2006, Baines et al., 2011, Budd et al., 2009, Lee et al., 2015, Baines and Sarahs, 2018, Baines and Christou, 2021].

3.1 Background on moving-mesh methods

Moving mesh methods are techniques used to solve partial differential equations (PDEs) when the solution varies rapidly in space over time, for example, when the solutions have sharp gradients, moving boundaries, or wave fronts. These methods adjust the domain dynamically in the areas where the solution is changing the most, or where it contains moving boundaries. This allows the description to be more accurate and efficient than using a fixed

grid [Huang and Russell, 2010].

In many cases, it is especially useful to use a mesh that also moves with the solution, for example in Lagrangian methods. This also helps keep high resolution in the important parts of the domain. The method we use starts with a basic grid (either uniformly spaced or not) and moves the points around as the solution evolves without changing the total number of points or how they are connected.

First we give a brief survey of adaptive mesh techniques.

3.1.1 Location-based methods

Location-based moving mesh strategies rely on a coordinate transformation between a fixed computational domain Ω_c and the evolving physical domain Ω_p . This transformation is described by a time-dependent mapping $x(\xi, t)$, where ξ is a coordinate in the computational domain and t is time. The transformation allows mesh points to follow important features of the solution in the physical space.

Following [Cao et al., 2003], consider a one-dimensional spatial domain where the adaptive mapping $x(\xi, t)$ defines the mapping from a computational domain Ω_c to a physical domain Ω_p . X is a function of both ξ (a coordinate in the computational domain) and t (time). This function defines how each point ξ in the computational domain maps to a point x in the physical domain at a given time t . If the mesh on Ω_c is uniform, meaning that the spacing between points (grid points) is equal throughout the domain, then the Jacobian derivative $\frac{\partial \xi}{\partial x}$ measures the local density of the mesh points in

the physical domain Ω_p . One way to control the mesh is to select a monitor function $m(x) > 0$ which is proportional to the mesh density at any given time and guides how the mesh is distributed based on solution features, i.e.

$$\frac{\partial \xi}{\partial x} = c m(x) \quad (3.2)$$

where c is a constant. This may come from an equidistribution principle or a physical rule applied to the monitor function $m(x)$ [Lee et al., 2015].

An example of an adaptive approach is the Moving Mesh Partial Differential Equations (MMPDEs) method based on mass balance. This method constitutes a widely used class, especially for one-dimensional problems. A secondary PDE is derived from a monitor principle to define a moving coordinate ξ associated with the physical coordinate x [Budd et al., 1996, Huang and Russell, 1994] and many others. The reviews by Huang and Russell [Huang and Russell, 2001] or Huang, Russell or Budd give a good survey of MMPDE methods. In addition, the survey presented in [Budd et al., 1996, Cao et al., 2003, Huang and Russell, 2010] summarizes several location-based mesh adaptation strategies, including classical contributions by Winslow [Winslow, 1966], Thompson et al. [Thompson et al., 1985], Brackbill and Saltzman [Brackbill and Saltzman, 1982], Dvinsky [Dvinsky, 1991], Brackbill [Brackbill, 1991], Knupp [Knupp, 1996], and Knupp and Robidoux [Knupp and Robidoux, 1999], who proposed a range of elliptic PDEs and variational formulations for generating adaptive mappings between computational and physical domains.

3.1.2 Velocity-based moving-mesh methods

We now turn to velocity-based moving-mesh methods, which are the numerical approach applied in this thesis. These methods employ a velocity to relocate nodes, automatically generating boundary velocities, ideal for problems with moving boundaries. Unlike location-based methods, which require a mapping between a computational space (Ω_c) and a physical space (Ω_p), velocity-based methods work directly in the physical domain.

In velocity-based methods, mesh movement is defined using a moving coordinate system that specifies the mesh velocity explicitly. This type of system is typically Lagrangian, in which the spatial coordinates themselves evolve as functions of time. These evolving coordinates represent a transformation from a fixed (Eulerian) system. For example, the classical Lagrangian coordinate system tracks individual fluid particles or mesh points. Each particle is identified by its fixed reference position ξ (a constant). The physical position $x = x(\xi, t)$ of the particle evolves according to the velocity $v(x, t)$ using

$$\left. \frac{dx}{dt} \right|_{\xi} = v(x, t).$$

3.1.2.1 Arbitrary Lagrangian-Eulerian (ALE) methods

The goal of the Arbitrary Lagrangian-Eulerian (ALE) method of Hirt, Amsden and Cook [Hirt et al., 1974] is to combine the features of the Eulerian and Lagrangian methods. The main idea is that the velocity does not have to be restricted to either the Eulerian or Lagrangian, instead it can be chosen arbitrarily to accommodate external boundaries and also to improve accuracy

and robustness of the solution process through given boundary conditions.

3.1.2.2 The Geometric Conservation Law (GCL)

In their 2002 paper [Cao et al., 2002], Cao, Huang, and Russell introduced the Geometric Conservation Law (GCL) to MMPDE. Their approach is based on a variational principle that leads to a minimisation problem, used to compute the mesh velocities. This is a method that aims to preserve geometric properties, particularly the volume of each element, ensuring that, for example, a moving fluid does not gain or lose mass solely due to mesh motion. This is done by requiring that the integral of a suitable monitor function remains constant over each mesh interval. The advantages of this method are that the level of mesh adaptation can be easily controlled, and there is a theoretical guarantee that the coordinate transformation exists and remains nonsingular.

3.1.2.3 The Conservation Method

The conservation method was developed by Baines, Hubbard, and Jimack [Baines et al., 2005, Baines et al., 2006, Baines et al., 2011]. The main idea of this method is to ensure that the local mass in a cell remains constant over time. Its implementation follows a similar approach to the GCL method, but the difference is in how the mesh velocities are determined. Instead of reformulating the GCL as a minimization problem during discretization, as in Cao et al [Cao et al., 2002], this method uses the conservative (integral) form of the PDE to compute the discrete mesh velocities directly. The method works by solving the PDE together with the Eulerian form of the conservation equa-

tion to compute a velocity, which is then used to move the points of the mesh in a Lagrangian manner.

The solution of the PDE is then recovered algebraically from the Lagrangian (integral) form of the same conservation law. In Baines et al. [Baines et al., 2005, Baines et al., 2006, Baines et al., 2011] the approach is demonstrated for mass-conserving problems, where the global mass remains constant over time, for example the porous medium equation and a fourth-order non-linear diffusion equation (in these problems the conserved quantity is local mass) before generalising to problems that do not conserve mass such as those with source terms. For these non mass-conserving problems the theory is generalised using the idea of relative mass, which represents the fraction of the total mass associated with each element patch [Lee, 2011, Baines and Christou, 2021]

3.2 A finite difference velocity-based moving-mesh method based on conservation

In this section, we explore how a velocity-based moving mesh method can be used to numerically solve the special case of the nonlinear diffusion equation (3.1), which in our study models the spread of an oil slick across a calm surface of water.

We implement a conservation-based finite difference approach [Lee et al., 2015], which was originally based on earlier finite element versions of the velocity-based moving mesh method developed by Baines, Hubbard, and

Jimack in [Baines et al., 2005, Baines et al., 2011].

In §3.2.1, we address the mass-conserving oil-slick problems by constructing a mesh that preserves the mass between nodes using velocities derived from local conservation laws.

We first apply this method to situations where the total mass in the domain is conserved. When mass changes due to boundary influx, the method is extended in §3.4 by preserving normalised partial masses.

The Barenblatt self-similar solution (2.46), derived in Chapter 2, is used here as a benchmark for assessing accuracy. This well-understood solution makes it ideal for validating our method in the case of non-linear diffusion and calm water.

We also investigate the case where we start with a different initial profile. We can often observe a “waiting time” [Lee, 2011, Partridge, 2013] where the solution at the boundary stays stationary for a certain time until the internal profile develops into the self-similar shape locally at the boundary. The conservation moving mesh method is able to capture this transition effectively.

3.2.1 Mass conserving problems

In this section we give details of a moving mesh strategy for solving the non-linear diffusion equation (3.1) under mass-conserving boundary conditions. Specifically, we first assume zero net flux across the domain boundaries, which means that no material enters or leaves the domain. Under these conditions,

the total mass

$$M(t) = \int_{\Omega(t)} h(x, t) dx,$$

of the solution remains constant over time, which can be expressed as:

$$\frac{d}{dt} \int_{\Omega(t)} h(x, t) dx = - \int_{\partial\Omega(t)} \mathbf{q} \cdot \mathbf{n} dS, \quad (3.3)$$

where $\mathbf{q} = -Ah^m \nabla h$ is the diffusive flux. Before describing the details of the numerical method, we recall the aspects of the theory that we use.

The mesh adapts dynamically to the evolving solution by preserving the partial masses between mesh nodes. This strategy follows the physics and improves numerical accuracy, especially near sharp gradients and moving fronts. Conserving the integral of the solution (local mass) is particularly advantageous in regions where the derivative of h is large because the diffusive flux $q \propto h^m \nabla h$ is highly sensitive to small errors in h when gradients are steep. In the PME this flux has the nonlinear form $q \propto h^m \nabla h$ with $m > 1$, so that even a tiny error in the local value of h produces a large error in the flux near the front where h is small.

By exactly conserving local mass, the method automatically concentrates mesh points where they are most needed, greatly reducing numerical diffusion and preserving the correct speed and shape of the moving front.

In this thesis we consider two types of boundary conditions. The first one is Dirichlet conditions, in which the solution vanishes at both ends of a domain. The second has Neumann boundary on one end and a Dirichlet condition on the other (which is relevant to simulation of the oil-slick problem).

3.2.2 The PME with Dirichlet boundary conditions

We first consider (3.1) over a fixed one-dimensional domain $x_l \leq x \leq x_r$ with homogeneous Dirichlet boundary conditions. The solution $h(x, t)$ evolves within this spatial interval and is prescribed to vanish at the domain boundaries, that is

$$h(x_l, t) = 0, \quad h(x_r, t) = 0, \quad t > 0. \quad (3.4)$$

The solution starts from a given initial condition on h at $t = t_0$, that is

$$h(x, t_0) = h_0(x). \quad (3.5)$$

The porous medium equation (PME) has been widely studied, and many of its analytical properties are well established [Aronson, 1986, Barenblatt, 1996, Vázquez, 2007]. In particular the solution may observe a “waiting time” [Lee, 2011, Partridge, 2013]).

Here, two key features are especially important: the presence of a moving boundary and the conservation of the approximate total mass.

As we have shown in Chapter 2, section (2.4.1), the total mass M of the solution remains constant over time. This is expressed as:

$$\frac{dM}{dt} = \frac{d}{dt} \left(\int_{x_l}^{x_r} h(x, t) dx \right) = 0, \quad (3.6)$$

3.2.3 The basic discretisation

Given that the total mass is conserved, it is consistent to divide the partial domain in such a way that the mass up to a moving point remains constant over time.

To state this property in another way, assume that the cumulative mass from $x = x_l$ up to node x_j is

$$m_j(t) = \int_{x_l(t)}^{x_j(t)} h(x, t) dx, \quad j = 0, \dots, N, \quad (3.7)$$

where each partial mass m_j is constant in time.

By subtraction of adjacent partial masses the local mass in an interval (x_{j-1}, x_j) is defined by

$$m_j - m_{j-1} = \int_{x_l}^{x_j(t)} h(x, t) dx - \int_{x_l}^{x_{j-1}(t)} h(x, t) dx = \int_{x_{j-1}(t)}^{x_j(t)} h(x, t) dx,$$

for $j = 1, \dots, N$, which also remains constant in time. Equation (3.7) is consistent with (3.6), since $m_N = \sum_j (m_j - m_{j-1}) = M$ is the total mass. In other words, the local interval masses $m_j - m_{j-1}$, summed across the domain equals the total mass. This guarantees that the moving mesh method preserves both local and global mass conservation.

3.2.4 Computation

In the numerical method, for a given mesh x_j^n and solution h_j^n , we first compute the mesh velocity $v_j(t)$ of the point X_j then step forward in time using an ODE for j to find the update mesh, and thirdly update the moved

solution h_j . This procedure is detailed in the following subsections and the subsequent algorithm .

Determining the mesh velocity

To obtain the formula for the mesh velocity, we differentiate the partial mass equation (3.7) with respect to time using the Leibniz's Integral Rule and apply the boundary conditions $h(x_l(t), t) = 0$ and $q(x_l(t), t) = 0$, finding

$$0 = \int_{x_l(t)}^{x_j(t)} \frac{\partial h}{\partial t} dx + h_j(t)v_j(t), \quad j = 1, \dots, N-1. \quad (3.8)$$

Substituting for $\frac{\partial h}{\partial t}$ using equation (3.1)

$$0 = -q(x_j(t), t) + h_j(t)v_j(t),$$

where $q = -Ah^3 \frac{\partial h}{\partial x}$. Rearranging the equation to solve it for the mesh velocity $v_j(t)$ and since $h_j(t) > 0$ in the interior, the interior nodes move in a way that

$$v_j(t) = \frac{q(x_j(t), t)}{h_j(t)}.$$

Substituting the flux expression for q , we get the final form

$$v_j(t) = -Ah^2(x_j(t), t) \frac{\partial h}{\partial x} \Big|_{x_j(t)} = \frac{A}{3} \frac{\partial h^3}{\partial x} \Big|_{x_j(t)} \quad (3.9)$$

which holds for $j = 1, \dots, N-1$. In the numerical method, the spatial derivative in (3.9) is approximated at the interior nodes by the central finite difference.

The velocities $v_0(t)$ and $v_N(t)$ at the boundary nodes $j = 0$ and $j = N$ are also given by (3.9) and are calculated by one-sided finite difference approximations based on neighbouring interior nodes.

Updating the nodal positions

To advance the mesh in time, we define time levels as $t^n = n\Delta t$ for $n = 0, 1, 2, \dots$, where $\Delta t > 0$ is the time step.

The position of the j -th mesh node at time t^n is denoted by $x_j^n \approx x_j(t^n)$.

The updated positions of the mesh nodes at the next time level, x_j^{n+1} , are calculated using the mesh velocity v_j^n through a time-stepping scheme. Specifically, the positions are updated using the explicit first-order Euler time-stepping scheme for ODE's, given by:

$$x_j^{n+1} = x_j^n + v_j^n \Delta t, \quad \text{for } j = 0, \dots, N. \quad (3.10)$$

where v_j^n is the mesh velocity at node j and time level t_n . This approach allows us update the mesh dynamically, ensuring that node positions evolve according to the local velocity field.

Recovering the solution

After we update the mesh positions at the new time, we need to recover the solution h_j^{n+1} on this new mesh. This is done by using an incremental form

of (3.7)

$$m_{j+1} - m_{j-1} = \int_{x_{j-1}^{n+1}}^{x_{j+1}^{n+1}} h(x, t_{n+1}) dx = \int_{x_{j-1}^0}^{x_{j+1}^0} h(x, t_0) dx, \quad j = 1, \dots, N-1. \quad (3.11)$$

given that each m_j is a constant determined by the initial conditions. We approximate the integrals in equation (3.11) using a simple quadrature rule. This approach allows us to recover h_j^{n+1} for $j = 1, \dots, N-1$ on the new mesh, as follows. By using a simple mid-point approximation for equation (3.11), we get:

$$(x_{j+1}^{n+1} - x_{j-1}^{n+1})h_j^{n+1} = (x_{j+1}^0 - x_{j-1}^0)h_j^0,$$

at all interior nodes, which implies that the updated solution can be expressed as:

$$h_j^{n+1} = \frac{x_{j+1}^0 - x_{j-1}^0}{x_{j+1}^{n+1} - x_{j-1}^{n+1}} h_j^0, \quad \text{for } j = 1, \dots, N-1. \quad (3.12)$$

The solution at the boundaries, h_0^{n+1} and h_N^{n+1} , is defined by the boundary conditions (3.4) at $x = 0$ or by extrapolation at $x = x_r$. In our problem, the partial mass represents the amount of oil up to the point j . As the mesh points move with the spreading oil, the total amount of oil up to this point remains constant (if no oil is added or removed from the region).

3.2.5 Algorithm

Choose initial node positions x_j^0 and initial solution h_j^0 , $j = 0, \dots, N$ at time $t^0 \geq 0$:

1. Calculate the mesh velocity v_j^n using a discrete form of equation (3.9).
For the boundary velocities, apply a one-sided finite difference scheme.
2. Update the mesh positions x_j^{n+1} using a time-stepping method (3.10).
3. Update the solution h_j^{n+1} using equation (3.12). The boundary solutions, h_0^{n+1} and h_N^{n+1} , are determined by the boundary conditions (3.4).

3.3 Numerical results

In this section, we present numerical experiments using the velocity-based moving mesh finite difference method to solve (3.1) with the boundary condition (3.4). Our simulations focus on examining the solution behaviour under different initial profiles to check how well the method conserves mass and follows the moving front, comparing results against the theoretical Barenblatt self-similar solution to quantify convergence and highlight phenomena such as waiting times.

Our simulation starts at time $t_0 = 1$, and the spatial domain spans $x \in [-4.9, 4.9]$ discretized into equally spaced nodes with $N = 160$ mesh nodes. The time step Δt was chosen to keep the numerical scheme stable, with

$$\Delta t = O\left(\frac{1}{N^2}\right) \quad (\text{i.e., } \Delta t \propto (\Delta x)^2).$$

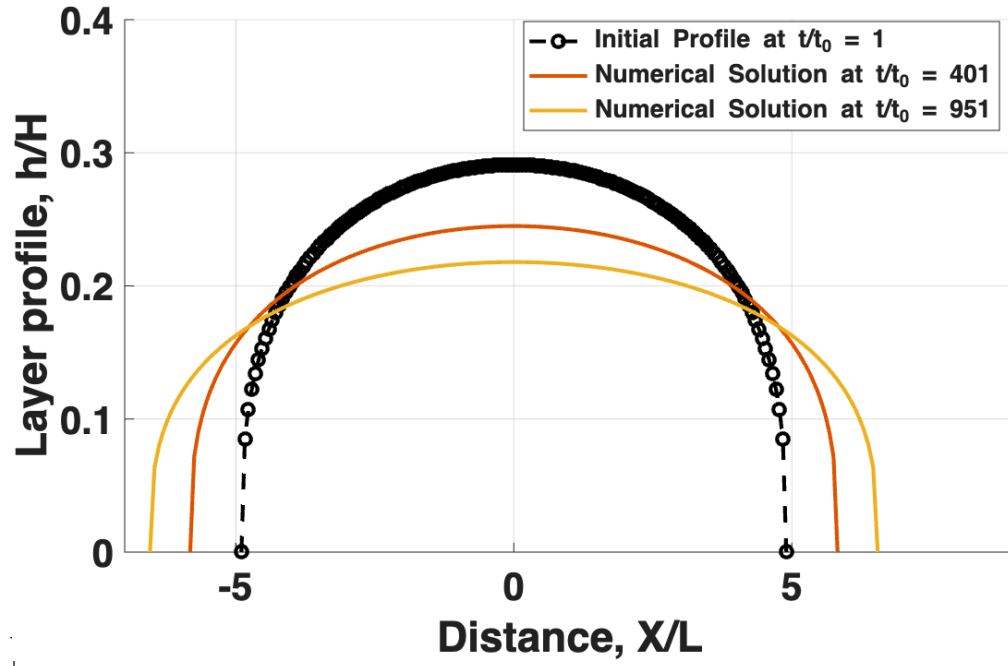


Figure 3.1: Evolution of the numerical solution starting from the self-similar initial profile (2.46). The final time corresponds to $t/t_0 = 951$.

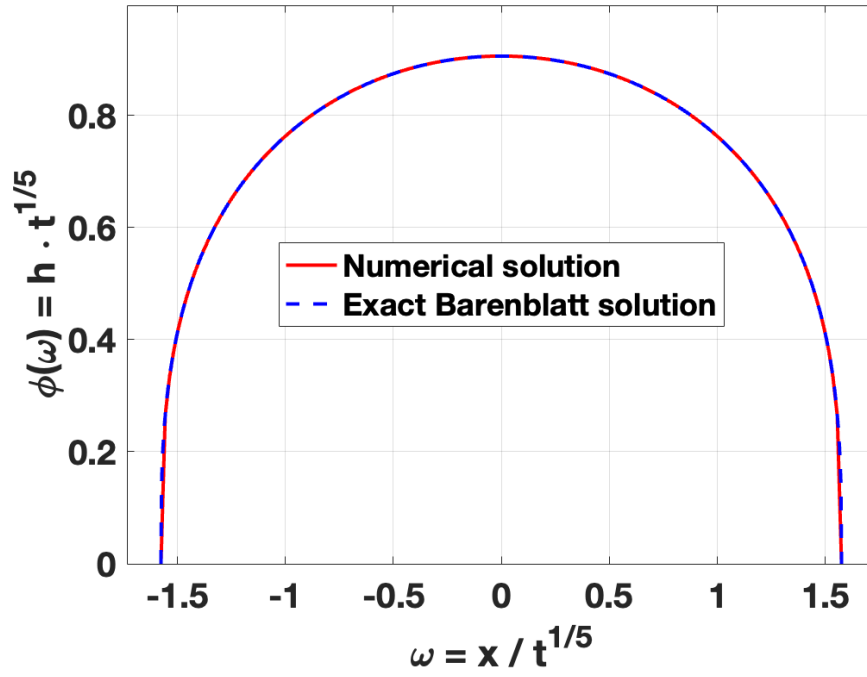


Figure 3.2: Comparison of the numerical solution of (3.1) (solid line), transformed into self-similar variables, with the exact Barenblatt solution (2.46) (dashed line) at the final time $t/t_0 = 951$.

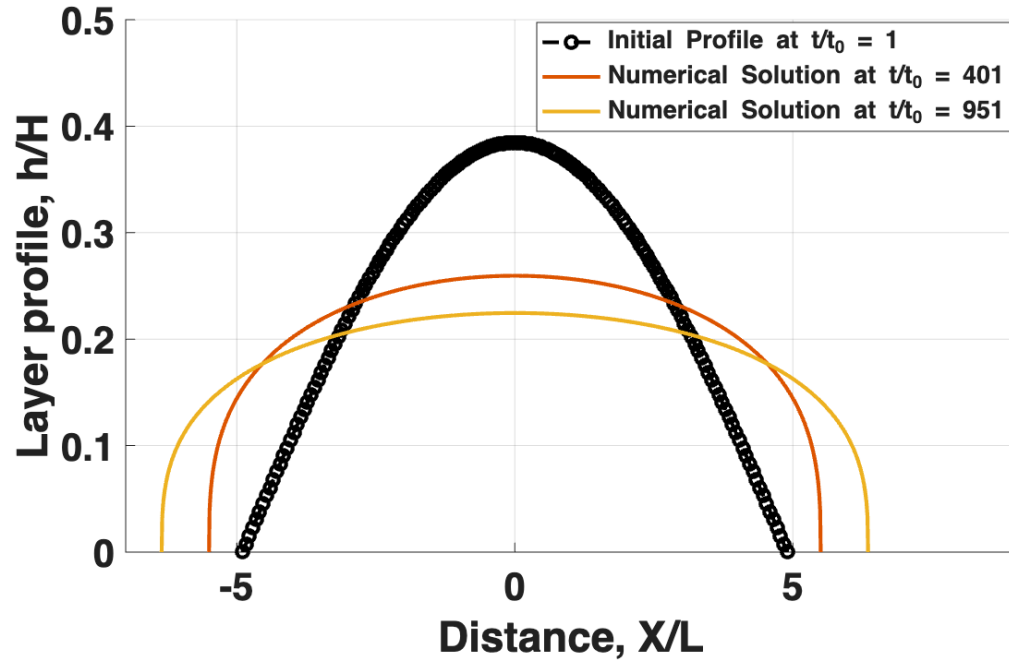


Figure 3.3: Evolution of the numerical solution starting from different initial profile $h(x, t_0) = \cos\left(\frac{\pi}{2}x\right)$. The final time corresponds to $t/t_0 = 951$.

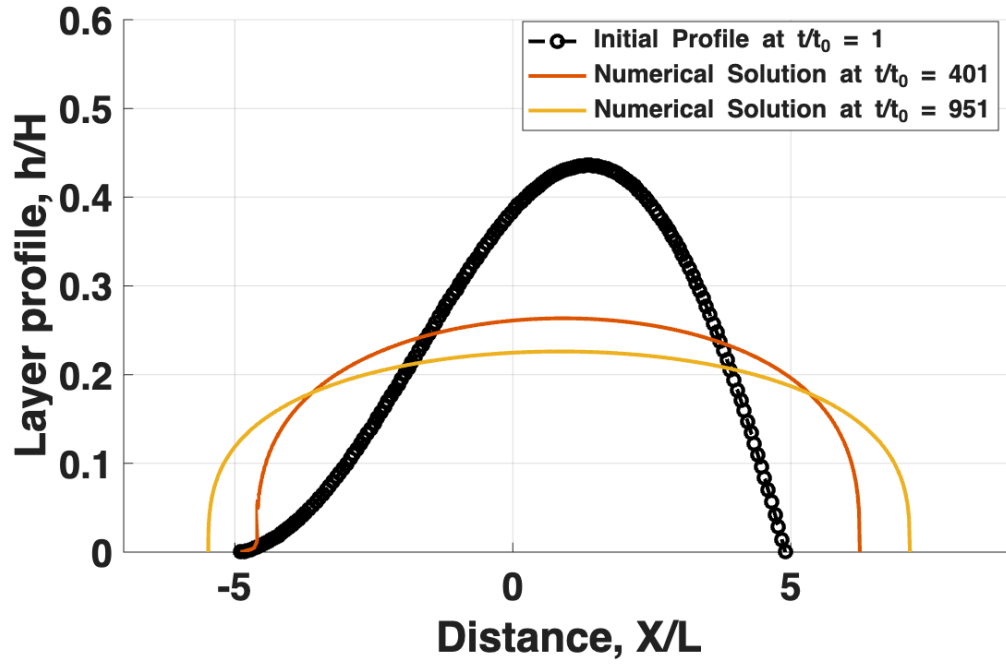
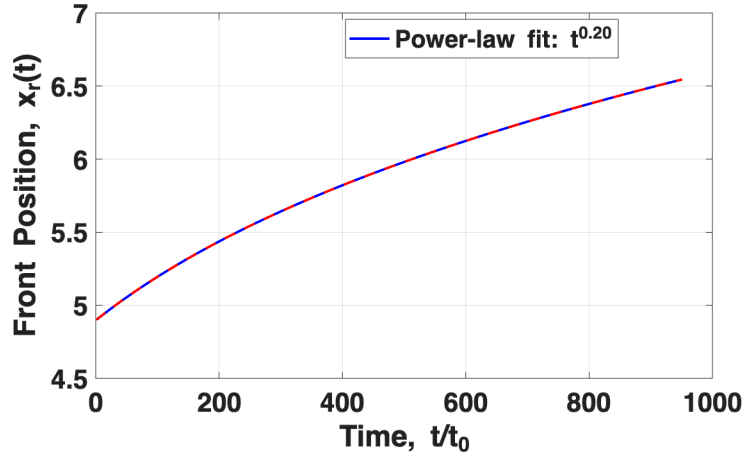
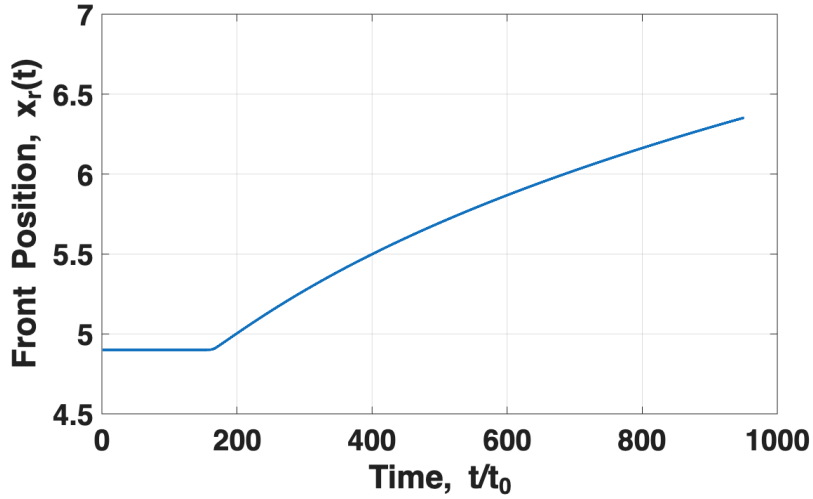


Figure 3.4: Evolution of the numerical solution starting from an asymmetric initial profile $h(x, t_0) = \cos\left(\frac{\pi}{2}x\right) \cdot (1 + 0.9x)$. The final time corresponds to $t/t_0 = 951$.



(a) Evolution of the numerically front position $x_r(t)$ over time, starting from the self-similar initial profile (2.46). The dashed line represents a power-law fit $x_r(t) \sim t^\alpha$, with $\alpha = 0.2$, corresponding to the theoretical scaling $x_r(t) \sim t^{1/5}$.



(b) Evolution of the numerically front position $x_r(t)$ over time using initial profile $h(x, t_0) = \cos(\frac{\pi}{2}x)$.

Figure 3.5: Numerical evolution of the moving front position $x_r(t)$ for two different initial profiles at $t/t_0 = 951$.

Starting with the Barenblatt self-similar initial profile (2.46), Figure 3.1

shows solution snapshots which times. We observe that the layer thickness profile $h(x, t)$ spreads symmetrically with finite propagation speed, a characteristic of the PME for $(m > 1)$ [Aronson, 1986, Vázquez, 2007]. This finite propagation speed produces sharp fronts where the solution height h drops suddenly to zero and the spatial derivative $\partial h / \partial x$ becomes non-differentiable. The final solution at $t/t_0 = 951$ (red curve) aligns closely with the theoretical Barenblatt form, confirming the rapid asymptotic convergence toward self similarity [Vázquez, 2007]. We confirm the close match by transforming the numerical results into self-similar variables. We plot the adjusted height $\phi(w) = h(x, t) t^{1/5}$ against $w = x/t^{1/5}$. As seen in Figure 3.2, the numerical solution at the final time $t/t_0 = 951$ follows the exact Barenblatt curve from (2.55). This confirms that the method preserves the self-similar structure and is effective for simulating oil layer spreading. In addition to the self-similar initial profile, we also conducted simulations using a different initial condition which is not self-similar, specifically $\cos\left(\frac{\pi}{2}x\right)$, to explore the effect of varying initial profiles on the evolution of the solution.

Figure 3.3 illustrates this scenario. The final numerical solution at $t/t_0 = 951$ (yellow curve) continues to evolve toward the Barenblatt self-similar form.

This demonstrate that, regardless of the initial non-self-similar profile, the solution evolves toward the Barenblatt form as $t \rightarrow \infty$, although delays such as waiting times may occur. This kind of delay in front movement, often called 'waiting time,' has been discussed previously in earlier studies [Stojsavljeic, 2007] [Lacey et al., 1982].

In our previous simulations, we have used symmetric initial conditions to enable direct comparison with the exact Barenblatt solution. It is important

to verify the method's performance on asymmetric initial profiles. This allows us to demonstrate the capability of the numerical method in capturing some properties of the PME, such as mass conservation and the convergence to the self-similar solution [Vázquez, 1992].

Figure 3.4 shows that the final numerical solution gradually symmetrises and approaches the self-similar form over time. Moreover, we investigate the evolution of the front position $x_r(t)$ under different initial profiles. As we see in Figure 3.5, the choice of the initial profile has a significant effect on the front's movement over time. In Figure 3.5a, with the self-similar initial profile is applied, the front begins to move immediately from its starting position, following a smooth, monotonically increasing trajectory as time progresses. As seen in the expanding supports of Figure 3.1, the front follows the $t^{1/5}(x_r \propto t^{1/5})$ scaling detailed in Figure 3.5a. In contrast, Figure 3.5b shows the behaviour when the initial condition is given by $\cos\left(\frac{\pi}{2}x\right)$. Here, we observe a clear waiting time at the start of the simulation, the front remains stationary for a while before it begins to propagate. This delay is visible in Figure 3.5b, where the front remains stationary initially before it begins to spread. This waiting time occurs because the initial gradient at the boundary, $\left.\frac{\partial h}{\partial x}\right|_{x=x_r(t)}$, is finite in the cosine case compared to the infinite self-similar gradient, insufficient to generate immediate flux [Lacey et al., 1982, Vázquez, 2007].

Convergence

To assess the accuracy of our method, we conduct a convergence study based on the relative ℓ_2 -norm error. In this case, we have an exact solution, which allows us to quantify the error between the numerical solution and the exact solution. Specifically, we analyze how the error decreases as we refine the spatial grid and adjust the time step accordingly. The spatial domain is discretized using N grid points, where we vary N across different refinement levels: $N = [10, 20, 40, 80, 160, 320]$. For each refinement level N , we calculate the numerical solution $h(x, t)$ at the mesh points x_i , and compare it to the exact self-similar solution $\bar{h}(x, t)$.

The relative ℓ_2 -norm error between the numerical solution $h_i(T)$ and the exact solution $\bar{h}_i(T)$ at the final time T is computed as:

$$E_N(h) = \sqrt{\frac{\sum_{i=0}^N \left(\bar{h}_i(T) - h_i(T) \right)^2}{\sum_{i=0}^N \left(\bar{h}_i(T) \right)^2}}.$$

As N increases, the grid resolution becomes finer, and we expect the error to decrease accordingly. Table 3.1 confirms this expectation, the error decreases consistently as the number of mesh points increases.

N	Error
10	6.54×10^{-3}
20	1.38×10^{-3}
40	2.39×10^{-4}
60	3.77×10^{-5}
80	5.72×10^{-6}
320	9.73×10^{-7}

Table 3.1: Error values for different refinement levels N .

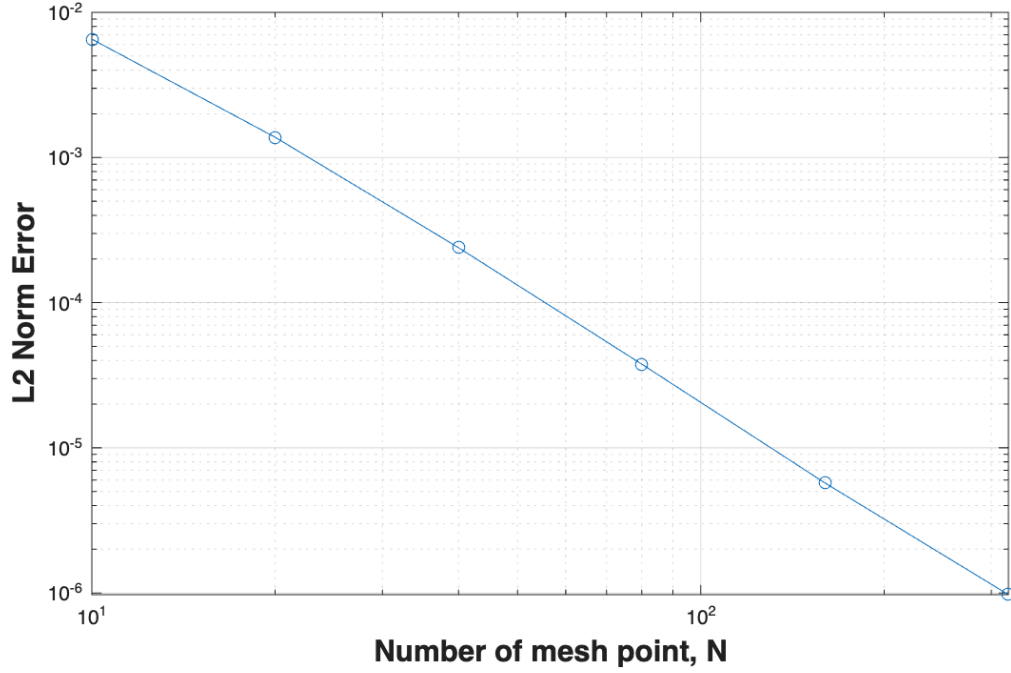


Figure 3.6: Plot of the error for different refinement levels N

To better understand how accurate the method is, we also estimate the order of convergence. This gives us a sense of how quickly the solution improves when the grid is refined.

We assume the total error behaves like a combination of spatial and temporal terms:

$$E_N = A(\Delta x)^q + B(\Delta t)^p,$$

where q and p are the convergence rates in space and time, respectively, and A, B are constants. Since we are only accounting for the spatial errors (the time step Δt is chosen to scale with Δx^2), the temporal error term becomes negligible. Therefore, when the spatial grid size is halved ($\Delta x \rightarrow \Delta x/2$), we have

$$E_N \approx A(\Delta x)^q, \quad E_{2N} \approx A(\Delta x/2)^q.$$

To eliminate the constant A , we divide the two expressions:

$$\frac{E_{2N}}{E_N} \approx \left(\frac{1}{2}\right)^q.$$

Taking the base-2 logarithm of both sides yields

$$\log_2 \left(\frac{E_{2N}}{E_N} \right) = q \log_2 \left(\frac{1}{2} \right) = -q.$$

Hence, the spatial order of convergence is given by

$$q = -\log_2 \left(\frac{E_{2N}}{E_N} \right).$$

We find that $q \approx 2$, which is consistent with the theoretical second-order accuracy in space. This agreement between theoretical and observed convergence validates the method under mesh refinement.

3.3.1 PME with zero-flux Neumann boundary condition

In addition to Dirichlet boundaries, we consider a configuration where a zero-flux Neumann boundary condition is imposed at one end of the domain. This setup reflects a semi-confined domain and enables to verify that the method can preserve mass without depending on full domain. Specifically, we solve the (3.1) on a domain $x \in [0, x_r(t)]$ with:

$$h(x_r, t) = 0, \tag{3.13}$$

at the moving boundary $x = x_r(t)$. The Neumann condition enforces zero gradient at $x = 0$ (i.e. $\left. \frac{\partial h}{\partial x} \right|_{x=0} = 0$), which implies that the flux

$$q(0, t) = -Ah^3 \frac{\partial h}{\partial x},$$

is zero at that boundary. As in the Dirichlet case, the total mass of the solution is conserved because there is no net flux through the domain:

$$\frac{d}{dt} \int_0^{x_r(t)} h(x, t) dx = 0.$$

Due to the boundary conditions, $q(0, t) = 0$ and $h(x_r(t), t) = 0$, which implies that both boundary fluxes vanish. We adopt the same moving mesh strategy introduced in Section 3.2.2, where the mesh nodes move in such a way that local partial masses remain conserved:

$$c_j = \int_0^{x_j(t)} h(x, t) dx, \quad j = 0, \dots, N.$$

This means that the mesh velocity and the update algorithm will have the same expressions with the only difference being the treatment of the boundary conditions.

Using the same derivation as in equation (3.9), the interior node velocity remains:

$$v_j(t) = -\frac{A}{3} \frac{\partial h^3}{\partial x} \Big|_{x_j(t)}.$$

$$v_j^n = -\frac{A}{3} \left(\frac{(h_{j+1}^3)^n - (h_{j-1}^3)^n}{x_{j+1}^n - x_{j-1}^n} \right), \quad j = 1, \dots, N-1.$$

At the left boundary $x_0 = 0$, the zero Neumann condition $\partial h / \partial x = 0$ implies zero flux and hence zero velocity:

$$v_0(t) = 0.$$

The velocity at the moving front $x_N(t) = x_r(t)$ is estimated using a one-sided finite difference, as in the Dirichlet case.

Moreover, the update of mesh points and recovery of the solution is done exactly as in the Dirichlet case (see equations (3.10) and (3.12)). The only change is that the left boundary value h_0^{n+1} is determined using a one-sided approximation can be approximated as:

$$h_0^{n+1} \approx h_1^{n+1}.$$

as $\frac{\partial h}{\partial x} = 0$ at $x = 0$, and the solution at the moving boundary is $h_N^{n+1} = 0$ from the boundary condition $h(x_r, t) = 0$.

3.4 Non-mass-conserving problems with boundary influx

Now, we consider the same one-dimensional diffusion equation (3.1) with domain $x \in [0, x_r(t)]$ where a mass is added through an external influx imposed at the left boundary. Specifically, we are setting a Neumann boundary

condition at the left end of the domain $x = 0$ to prescribe the influx rate,

$$q(0, t) = q_0(t) = -Ah^3 \frac{\partial h}{\partial x} \Big|_{x=0}, \quad (3.14)$$

where $q_0(t) > 0$ is a function of time only representing incoming flux. At the moving right-hand boundary $x_r(t)$ we assume free boundary condition where the thickness h vanishes and the net flux is zero

$$h(x_r, t) = 0. \quad (3.15)$$

Since flux enters the domain through the left boundary $x = 0$, the total amount of oil in the domain

$$M(t) = \int_0^{x_r(t)} h(x, t) dx, \quad (3.16)$$

is not conserved and increases over time. By applying the divergence theorem and the boundary conditions, we get

$$\frac{dM}{dt} = q(0, t) \neq 0, \quad (3.17)$$

as we have proved it in Chapter 2. In such case, the methods that rely on constant total or partial masses (as in previous sections) are no longer applicable, since the influx violates the zero boundary flux assumption. Instead, we handle this non-conservative scenario numerically by introducing a relative depth variable, this method preserves relative mass instead of total mass, allowing the mesh to adapt more accurately to the incoming flux at

the boundary. This approach has been discussed in detail in [Lee, 2011]. The relative depth variable introduce as

$$\bar{h}(x, t) = \frac{h(x, t)}{M(t)},$$

for which the relative mass,

$$\bar{M}(t) = \int_0^{x_r} \bar{h}(x, t) dx = \frac{1}{M} \int_0^{x_r} h(x, t) dx = 1,$$

is conserved. Differentiating $\bar{M}(t)$ with respect to t , we get

$$\frac{d\bar{M}}{dt} = \frac{d}{dt} \int_0^{x_r} \bar{h} dx = 0 \quad (3.18)$$

Relative mass

We now generalize the conservation argument to a subdomain $(0, X(t)) \subset [0, x_r]$, where $X(t)$ is an interior moving point of the domain and the boundary conditions at $X(t)$ are no longer $h = 0$. The goal is to define mesh movement such that the relative partial mass remains constant over time.

Let $\bar{m}(X(t), t)$ denote the mass fraction in the segment from 0 to $X(t)$:

$$\bar{m}(X(t), t) = \int_0^{X(t)} \bar{h}(x, t) dx = \frac{1}{M(t)} \int_0^{X(t)} h(x, t) dx. \quad (3.19)$$

Since we require that $\bar{m}(X(t), t)$ remains constant in time, we impose

$$\bar{m}(X_j(t), t) = \bar{m}_j, \quad j = 0, \dots, N,$$

where $\bar{m}_j$ is a prescribed constant mass fraction. Equivalently,

$$\frac{d}{dt}\bar{m}(X_j(t), t) = 0.$$

Thus, each mesh node $X_j(t)$ moves so that the relative mass between 0 and $X_j(t)$ remains constant over time.

Note that at the fixed left boundary $X = 0$, the relative mass is zero:

$$\bar{m}(0, t) = \frac{1}{M(t)} \int_0^0 h(x, t) dx = 0.$$

At the moving boundary $X = x_r(t)$, the relative mass is

$$\bar{m}(x_r(t), t) = \frac{1}{M(t)} \int_0^{x_r(t)} h(x, t) dx = 1.$$

Now, we differentiate (3.19) with respect to t using Leibniz's rule, and set the derivative to zero for constancy. We obtain

$$0 = \frac{d\bar{m}}{dt} = \frac{dX}{dt} \bar{h}(X, t) + \int_0^{X(t)} \frac{\partial \bar{h}}{\partial t} dx. \quad (3.20)$$

Since

$$\bar{h}(x, t) = \frac{h(x, t)}{M(t)},$$

we have

$$\frac{\partial \bar{h}}{\partial t} = \frac{1}{M(t)} \frac{\partial h}{\partial t} - \frac{h(x, t)}{M^2(t)} \frac{dM}{dt}.$$

Using the conservation equation (3.1) and the mass balance (3.17), we obtain

$$\frac{\partial \bar{h}}{\partial t} = -\frac{1}{M(t)} \frac{\partial q}{\partial x} - \frac{q_0}{M(t)} \bar{h}(x, t). \quad (3.21)$$

Since

$$\int_0^{X(t)} \frac{\partial q}{\partial x} dx = q_X - q_0, \quad q_X = q(X(t), t),$$

and

$$\int_0^{X(t)} \bar{h}(x, t) dx = \bar{m}_X,$$

we get

$$0 = \frac{dX}{dt} \bar{h}(X, t) - \frac{q_X - q_0}{M(t)} - \frac{q_0}{M(t)} \bar{m}_X.$$

Substituting (3.21) inside the integral in the Leibniz formula (3.20), we obtain

$$0 = \frac{dX}{dt} \bar{h}(X, t) - \frac{1}{M(t)} \int_0^{X(t)} \frac{\partial q}{\partial x} dx - \frac{q_0}{M(t)} \int_0^{X(t)} \bar{h}(x, t) dx$$

Rearrange it yields the mesh velocity for interior nodes:

$$\frac{dX}{dt} \bar{h}(X, t) = \frac{q_0(\bar{m}_X - 1) + q_X}{M(t)}, \quad (3.22)$$

or equivalently, in terms of h since $h = M(t)\bar{h}$:

$$\frac{dX}{dt} h(X, t) = q_0(\bar{m}_X - 1) + q_X. \quad (3.23)$$

Thus, equation (3.23) may also be written as

$$\frac{dX}{dt} h(X, t) = q_X + q_0 \left(\frac{m_X}{M(t)} - 1 \right). \quad (3.24)$$

where

$$m_X = \int_0^{X(t)} h(x, t) dx,$$

is the absolute partial mass. These expressions hold for interior nodes $j = 1, \dots, N - 1$.

For boundaries:

In the limit $X \rightarrow 0$, $\bar{m}_X = 0$, $q_X = q_0$, so

$$v_0(t) = 0, \quad (3.25)$$

Therefore, the mesh point at the origin does not move, which is consistent with our assumption that the origin remains stationary.

In the limit $X \rightarrow x_r(t)$, $\bar{m}_X = 1$, recovering

$$v_N(t) = \frac{q_{x_r}}{h} = -Ah^2 \frac{\partial h}{\partial x} \Big|_{x_r(t)}, \quad (3.26)$$

where $q(x(t), t) = -Ah^3 h_x$ aligning with the free boundary motion.

Discretisation of the domain

To implement a finite difference method according to this theory, we first discretize time as $t_n = n\Delta t$, $n = 0, 1, 2, \dots$. These time points will be our snapshots in time for the solution process.

Next, at each t_n , we discretize the spatial domain $[0, x_r(t)]$ using nodal points denoted by $x_j^n = x_j(t_n)$, where $j = 0, \dots, N$, with

$$0 = x_0^n < x_1^n < \dots < x_N^n = x_r(t_n).$$

The solution and velocity are approximated as

$$h_j^n \approx h(x_j(t_n), t_n), \quad v_j^n \approx v(x_j(t_n), t_n).$$

Determining the mesh velocity

The mesh velocities for interior nodes $j = 1, \dots, N - 1$ are discretized from Equation (3.23), incorporating the influx and flux terms. Boundary velocities follow Equations (3.25) and (3.26).

Updating nodal positions through time-stepping

The updated mesh, denoted as x_j^{n+1} , is derived from v_j^n using a time-stepping scheme, such as the Euler time-stepping scheme. There are other time steps scheme available but the Euler time stepping scheme appears to be sufficient for demonstration purposes.

$$x_j^{n+1} = x_j^n + \Delta t v_j^n, \quad j = 0, \dots, N \quad (3.27)$$

where v_j^n is the velocity at the point x_j , Δt is the time step and x_j^{n+1} is the position of x_j at the next time step.

Recovering the solution

After determining the updated mesh, we approximate the updated solution h_j^{n+1} . We equate (3.19) between the limits x_{j+1} and x_{j-1} , at time levels t^{n+1}

and initial t^0 , giving

$$\frac{1}{M(t^{n+1})} \int_{x_{j-1}^{n+1}}^{x_{j+1}^{n+1}} h(x, t^{n+1}) dx = \frac{1}{M(t^0)} \int_{x_{j-1}^0}^{x_{j+1}^0} h(x, t^0) dx \quad (3.28)$$

for $j = 1, \dots, N-1$. To evaluate the integrals in (3.28), we use a midpoint quadrature approximation, we get

$$\frac{1}{M(t^{n+1})} (x_{j+1}^{n+1} - x_{j-1}^{n+1}) \cdot h(x_j^{n+1}) = \frac{1}{M(t^0)} (x_{j+1}^0 - x_{j-1}^0) \cdot h(x_j^0)$$

Therefore, the updated solution can be determined by

$$h(x_j^{n+1}) = \frac{M(t^{n+1})}{M(t^0)} \frac{x_{j+1}^0 - x_{j-1}^0}{x_{j+1}^{n+1} - x_{j-1}^{n+1}} \cdot h(x_j^0), \quad (3.29)$$

for $j = 1, \dots, N-1$. It is important to note that at any given time level, the discrete relative partial mass

$$\frac{1}{M(t_{n+1})} (x_{j+1}^{n+1} - x_{j-1}^{n+1}) h_j^{n+1},$$

is preserved and is equal to the corresponding initial relative partial mass

$$\frac{1}{M(t_0)} (x_{j+1}^0 - x_{j-1}^0) h_j^0.$$

The solution at the boundary h_N is determined by the boundary condition (3.15). For h_0^{n+1} at the influx boundary, approximate as $h_0^{n+1} \approx h_1^{n+1}$ to maintain flux consistency.

The full algorithm

Given a mesh x_j^n , solution h_j^n , and total mass M_j^n , ($j = 0, \dots, N$) at time t^n ,

for $n > 0$.

1. Compute $\dot{M}^n$ from the discrete form of (3.16).
2. Compute the mesh velocity v_j^n from a discrete form of (3.22) or (3.23).
The velocity at the boundary(at the origin) $v_0^n = 0$ and the moving boundary v_N^n we compute it from (3.26).
3. Update the mesh positions using (3.27), then compute the updated total mass M^{n+1} from (3.17) using the forward difference approximation for the time derivative, we get

$$M^{n+1} = M^n + \Delta t \cdot q(0, t^n).$$

4. Compute the updated solution h_j^{n+1} from (3.29)

We check the mass-balance consistency of the numerical scheme under non-mass-conserving conditions by considering a constant influx $q_0(t) = q_0$, where q_0 is a fixed positive constant. In this case, the total mass is expected to increase linearly in time.

This result comes from integrating (3.17) with respect to time from t_0 to t , we get

$$M(t) = M(t_0) + q_0(t - t_0) \tag{3.30}$$

which represents the linear growth of mass due to the continuous influx at the left boundary where $M(t_0)$ is the initial mass.

We perform the numerical simulation on the domain $[0, 1]$, starting from the initial time $t_0 = 0$ until the final time $t = 300$. The initial profile $h(x, t_0)$

was the self-similar solution (2.46) of the mass-conserving case. Although this profile is not a self-similar solution of the influx problem, it provides a convenient initial condition for checking the implementation of the non-mass conserving formulation.

Figure 3.7 shows that the computed total mass matches the analytical prediction (3.30), as expected by construction of the method. As theory predicts (3.30), the total mass should grow linearly over time and that is exactly what we see.

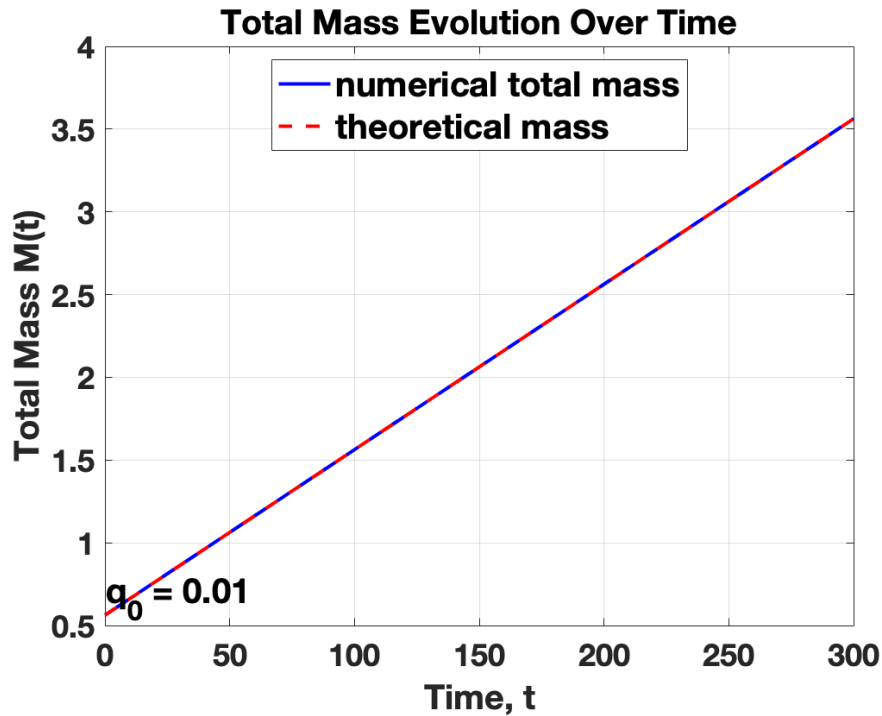


Figure 3.7: A comparison between numerical and theoretical evolution of the total mass (3.30) over time. The blue solid line represents the numerically computed total mass, while the red dashed line corresponds to the theoretical mass. The influx rate is set at $q_0 = 0.01$ and we take $N = 100$. The initial profile is self similar solution (2.46) at $A = \frac{\alpha q}{3} = 1$.

In next section, we focus on verifying the method via convergence tests, while

in Chapter 5 we present how profiles evolve under different influx scenarios and investigate the dynamics under external forcing.

3.5 Error analysis and convergence

In this section, we analyse the errors behaviour by comparing numerical solutions at different levels of spatial discretization to a reference solution. Since we do not have an exact solution for the problem with external influx, we use a highly refined numerical solution obtained with $N = 460$ nodes as the reference, denoted by h_{ref} . We then calculate the relative error for different values of N , specifically $N = 70$, $N = 90$, $N = 110$, $N = 130$, and $N = 150$, over the time interval $t \in [0, 1]$. By comparing these results to the reference solution, we can observe how the error decreases as the number of nodes increases, which provides insight into the convergence behaviour of our method.

To calculate the error for each N , we use the ℓ_2 norm defined as

$$E_N(h) = \sqrt{\frac{\sum_{i=0}^N \left(h_i^{(N)}(T) - h_{\text{ref}}(x_i^{(N)}, T) \right)^2}{\sum_{i=0}^N \left(h_{\text{ref}}(x_i^{(N)}, T) \right)^2}}$$

where $h_i^{(N)}(T)$ denotes the numerical solution computed with N mesh points at the final time T , and $h_{\text{ref}}(x_i^{(N)}, T)$ denotes the reference solution.

Table 3.2 summarizes the relative errors for two cases with different influx. In both cases, the error decreases as N increases, confirming the convergence of the method.

N	$E_N(h)$ for $q_0 = 0.01$
70	1.6746×10^{-4}
90	8.3913×10^{-5}
110	5.5544×10^{-5}
130	4.1244×10^{-5}
150	3.1902×10^{-5}

N	$E_N(h)$ for $q_0 = 0.3$
70	5.8630×10^{-3}
90	3.6111×10^{-3}
110	2.4233×10^{-3}
130	1.7198×10^{-3}
150	1.2693×10^{-3}

Table 3.2: Relative ℓ_2 -norm errors $E_N(h)$ for different numbers of nodes N , with reference solution h_{ref} obtained at $N = 460$. Results are shown for an influx parameter $q_0 = 0.01$ (left table) and $q_0 = 0.3$ (right table) using a time step $\Delta t = 1 \times 10^{-6}$ over the time interval $t \in [0, 1]$.

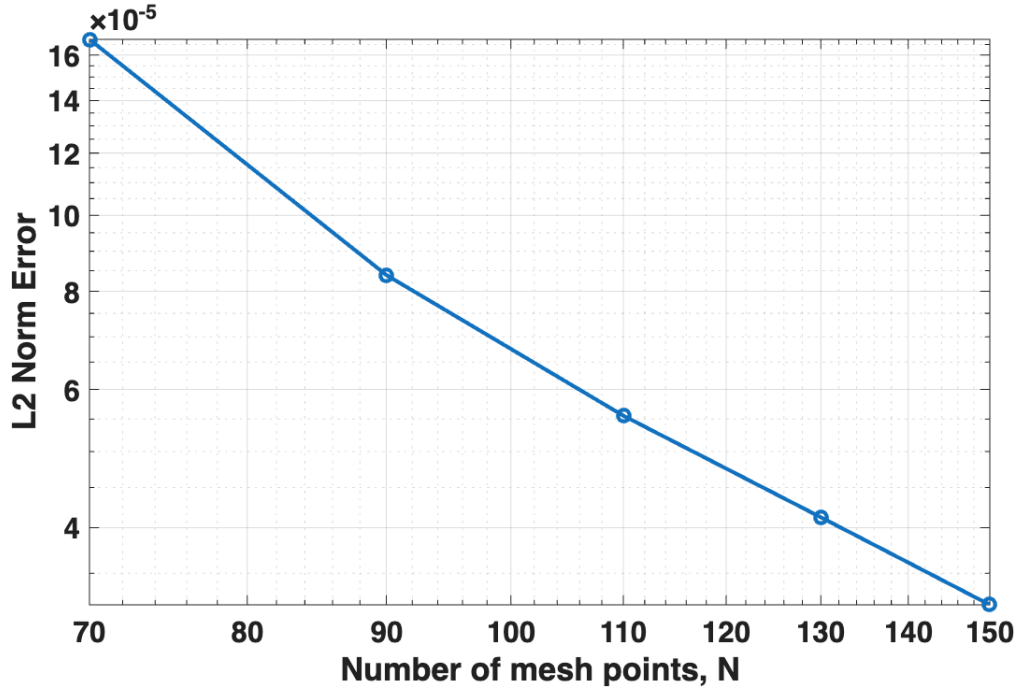


Figure 3.8: Plot of the ℓ_2 -norm error against the total number of mesh points N for $q_0 = 0.01$.

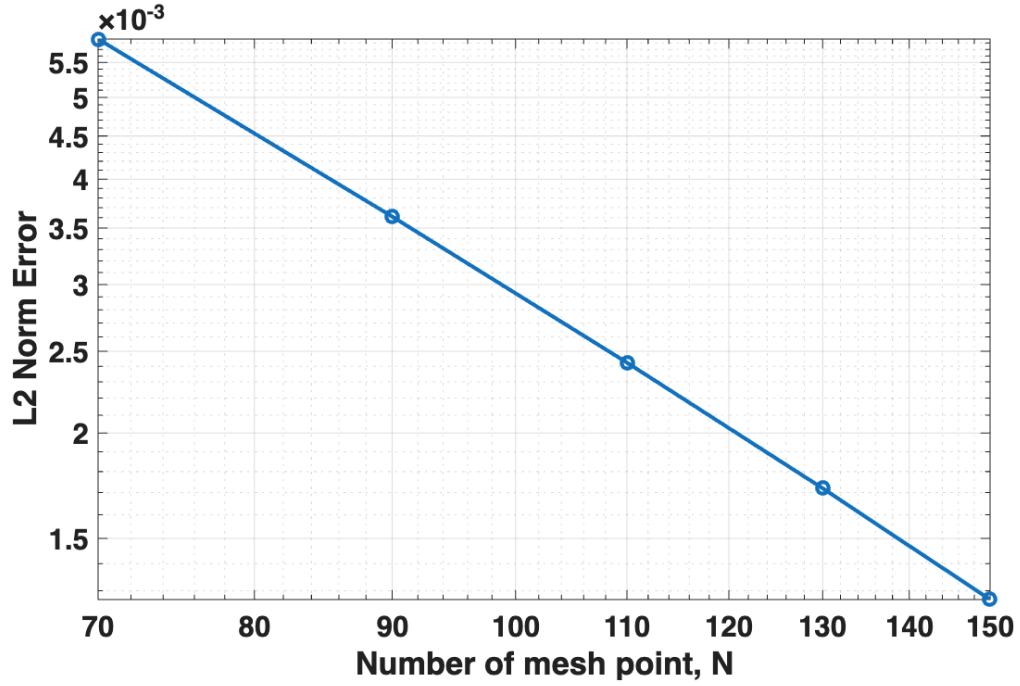


Figure 3.9: Plot of the ℓ_2 -norm error against the total number of mesh points N for $q_0 = 0.3$.

To further assess the accuracy of the method, we first need to understand the order of approximation of the moving-mesh numerical method. Accordingly, we have conducted a set of numerical simulations, as summarized in Tables (3.3). Our goal was to determine the order of convergence, which is an important metric that measures how the error in the numerical solution decreases as the mesh is refined. The order of convergence was calculated using a standard method, which is defined in

$$E_N \propto N^{-p}$$

Here, E_N represents the numerical error corresponding to a mesh with N nodes, and p is the order of convergence. We compute the order of conver-

gence using

$$p \approx \frac{\log\left(\frac{E_{N_1}}{E_{N_2}}\right)}{\log\left(\frac{N_2}{N_1}\right)}.$$

This allows us to estimate how the error decreases as the mesh is refined.

It can be observed that the order of convergence is consistently above one and mostly close to second order, although not always exactly.

We know from the numerical analysis that the central difference scheme is of the second order provided that the mesh is uniform, meaning the error decreases quadratically as the mesh is refined [LeVeque, 2007]. At the same time, we have established theoretically that the same numerical scheme may show lower order of convergence on non-uniform meshes.

We speculate that in the moving-mesh method, one may always expect some distortion of the initially uniform mesh, especially in the presence of an external flux, such as the influx condition at the first point $x = 0$. These distortions can introduce non-uniform spacing as the mesh dynamically adapts to the evolving solution, which in turn may contribute to the observed fluctuations in the measured order of convergence.

To understand the surprising result that the method mostly retains its accuracy despite significant mesh non-uniformity, we have investigated the non-uniformity of the mesh using a special norm.

We used the following scaled measure:

$$\|\delta_1 - \delta_2\| = \frac{1}{N-1} \sum_{i=1}^{N-1} \frac{|\delta_{1,i} - \delta_{2,i}|}{\delta_{1,i}^2},$$

where $\delta_{1,i} = x_{i+1}^{(1)} - x_i^{(1)}$ and $\delta_{2,i} = x_{i+1}^{(2)} - x_i^{(2)}$ are the spacings between

consecutive nodes in the initial and current meshes, respectively. This quantity is not a standard relative error. Instead, it measures the mesh spacing between consecutive mesh sizes, normalized by the square of the mesh size, providing a detailed measure of non-uniformity across the mesh. The result of our analysis in Table 3.3 indicates that the difference of the two adjacent mesh sizes, which is formally linear in terms of the mesh size, scales up and down as a square of the mesh size. This result may help explain the observed behaviour of the numerical method, which remains mostly of the second order.

q_0	T	Norm	Order of convergence p
0.01	1	[0.020305, 0.020080, 0.019934, 0.019831, 0.019755]	2.15
0.05	1	[0.082942, 0.083094, 0.083184, 0.083242, 0.083284]	1.80
0.1	1	[0.141713, 0.142153, 0.142421, 0.142601, 0.142731]	1.91
0.3	1	[0.262953, 0.263776, 0.2642764, 0.264611, 0.264850]	2.00

Table 3.3: Estimated convergence order.

We observed that the relative mass-based moving mesh method performs well when the influx q_0 is not too large compared to the diffusion parameter α_g .

To understand its limitations, we analyse the mesh velocity formula in (3.23):

$$\frac{dX}{dt} h = q_0(\bar{m}_X - 1) + q_X.$$

As we see there are two competing terms in this equation. A negative influx driven term $q_0(\bar{m}_X - 1)$, which is negative when $\bar{m}_X < 1$ for interior nodes

$j = 1, \dots, N - 1$ since $\bar{m}_0 = 0$ and $\bar{m}_N = 1$, and a diffusion driven term

$$q_X = -Ah^2 \frac{\partial h}{\partial x},$$

where $A = \frac{\alpha_g}{3}$ (since $m = 3$). The negative term $q_0(\bar{m}_X - 1)$ appears because the continuous influx increases the total mass $M(t)$. In the relative mass method, each interior node must always carry a fixed fraction m_X of the total mass. When diffusion is weak, the added oil piles up near the inlet $x = 0$ instead of spreading. To maintain the prescribed mass fractions, some interior mesh points are forced to move backward (negative velocity). This is a mesh effect, not a physical backflow of oil.

When the diffusion coefficient A is sufficiently large, the term q_X dominates, and the velocity remains positive, ensuring smooth mesh evolution. However, when A is small or q_0 is large, the negative term $q_0(\bar{m}_X - 1)$ dominates, and we get a negative velocity that causes some of the interior mesh points to move backward. This can lead to tangled nodes or a loss of numerical accuracy and make the method unstable. This analysis is consistent with our numerical observations for $q_0 = 0.01$, $q_0 = 0.1$ or 0.3 and $\alpha_g/m = \frac{3}{3} = 1$, all mesh velocities remain positive. However, when $\alpha_g/m = \frac{0.02}{3}$, negative velocities appear. To avoid these problems, we introduce another method that takes a different approach.

3.5.1 Mass balance

Now, we present a direct numerical method for solving (3.1) with boundary conditions (3.14) and (3.15). In this method we do not enforce the total mass

conservation because of the influx at the fixed boundary $x = 0$. Instead, we apply a local mass balance approach that treats the first mesh interval separately. At the inflow boundary $x = 0$, we apply a direct mass balance that accounts for the external influx $q_0(t)$, while enforcing local mass conservation in all interior cells. We divide the domain into N mesh cells defined by the intervals $[x_j(t), x_{j+1}(t)]$, where $j = 0, 1, \dots, N - 1$. Then we define the local mass in the j -cell as

$$m_j(t)(\text{local mass}) = \int_{x_j(t)}^{x_{j+1}(t)} h(x, t) dx. \quad (3.31)$$

We evolve the mesh such that the local mass in each interior cell $j = 1, \dots, N - 1$ remains constant over time; i.e.,

$$\frac{d}{dt}m_j(t) = 0.$$

This means that the net flux into and out of each cell is balanced by the motion of the mesh. We enforce this condition by computing mesh point velocities that preserve the local mass in each cell.

The first cell, corresponding to the interval $[x_0(t), x_1(t)]$, is the only cell where local mass is not conserved. Instead, the net mass in this cell increases over time because of external influx $q_0(t)$ at the fixed boundary $x = 0$. Note that this does not mean all the incoming mass stays inside the first cell; mass continuously flows out through the moving interface at $x_1(t)$ into the neighbouring cells.

We model this by applying a direct mass balance:

$$\frac{d}{dt}m_0(t) = q_0(t). \quad (3.32)$$

To determine the mesh velocity, we assume that each interior node moves with the local fluid velocity. Since the flux satisfies

$$q(x, t) = h(x, t) v(x, t),$$

the mesh velocity is defined by

$$v(x, t) = \frac{q(x, t)}{h(x, t)}.$$

Accordingly, the velocity of each mesh point $x_j(t)$, for $j = 1, \dots, N$, is given by

$$\frac{dx_j}{dt} = v(x_j(t), t), \quad \text{where} \quad v_j^n = -\frac{q_j^n}{h_{j-1}^n} \quad (3.33)$$

The node at the inflow boundary is fixed at $x_0 = 0$, so we set

$$\frac{dx_0}{dt} = 0.$$

We update the mesh points x_j^{n+1} using the same forward Euler scheme as in the relative method, given by (3.27). After updating the mesh, we recover the solution using the local mass integral definition of (3.31). By applying

the midpoint rule to approximate the integral, we obtain:

$$h_j^{n+1} = \frac{m_j^{n+1}}{x_{j+1}^{n+1} - x_j^{n+1}}, \quad j = 0, \dots, N-1, \quad (3.34)$$

where the mass m_0^{n+1} is defined over the interval $[x_0, x_1]$ and is updated using the boundary influx q_0 so h_0^{n+1} is recovered as

$$h_0^{n+1} = \frac{m_0^{n+1}}{x_1^{n+1} - x_0^{n+1}},$$

, and $h_N^{n+1} = 0$ from the boundary condition (3.15).

The full algorithm

Given a mesh x_j^n , solution h_j^n for $j = 0, \dots, N$, and local masses m_j^n for ($j = 0, \dots, N-1$) at time t^n , for $n \geq 0$.

1. Compute the interior fluxes q_j^n at each mesh point from the discrete form of (3.1). Fluxes at the boundaries determine from (3.14) and (3.15).
2. Compute the mesh velocity v_j^n from a discrete form of (3.33). The velocity at the left boundary $x = 0$ is $v_0^n = 0$.
3. Update the mesh positions x_j^{n+1} by the explicit Euler scheme (3.27).
4. Update the local mass in the first cell m_0^{n+1} using the discrete form of (3.32)

$$m_0^{n+1} = m_0^n + \Delta t \cdot q_0^n.$$

5. Compute the updated solution h_j^{n+1} using (3.34), with $h_N^{n+1} = 0$ from (3.15).

Error analysis and convergence

Next we evaluate the accuracy of the direct method. Our approach follows the same procedure used in the relative method discussed earlier. Because we do not have exact solution for the problem with external influx, we use again a highly refined numerical solution obtained with $N = 640$ as a reference denoted h_{ref} . We then compute the numerical solutions for different sized meshes $N = 10, 20, 40, 80, 160, 320$ over the time $t \in [0, 1]$, and compare them to the reference. For each mesh, the time step (Δt) was chosen proportionally to $(\Delta x)^2$ to satisfy a stability condition appropriate for explicit Euler time stepping used in this method. As a measure of the numerical error, we compute the relative ℓ_2 -norm of the difference between the computed solution and the reference at final time $T = 1$:

$$E_N(h) = \sqrt{\frac{\sum_{i=0}^N \left(h_i^{(N)}(T) - h_{\text{ref}}(x_i^{(N)}, T) \right)^2}{\sum_{i=0}^N \left(h_{\text{ref}}(x_i^{(N)}, T) \right)^2}}$$

Here $h_{N,i}$ represents the value of the numerical solution at node i and $h_{\text{ref},i}$ represents the corresponding value of the reference solution. We can observe how the error decreases as the number of nodes increases. We examine the hypothesis that the numerical error in the solution behaves like $E_N(h) \sim N^{-p}$ for large N , where p is the expected order of convergence. We estimate the order of convergence using

$$p \approx \log_2 \left(\frac{E_N}{E_{2N}} \right).$$

Our numerical results confirm that $p \approx 2$, indicating second-order convergence in space, where the error decreases approximately quadratically with respect to the mesh size. This is what we expected because we used a second-order method for space and chose a small enough time step to make time errors very small.

N	$E_N(h)$
10	4.5830×10^{-2}
20	1.3494×10^{-2}
40	2.5004×10^{-3}
80	4.4319×10^{-4}
160	7.6494×10^{-5}
320	1.2014×10^{-5}

Table 3.4: Relative ℓ_2 errors $E_N(h)$ for different numbers of nodes N , with reference solution h_{ref} obtained at $N = 640$. Result are shown for an influx parameter $q_0 = 0.3$.

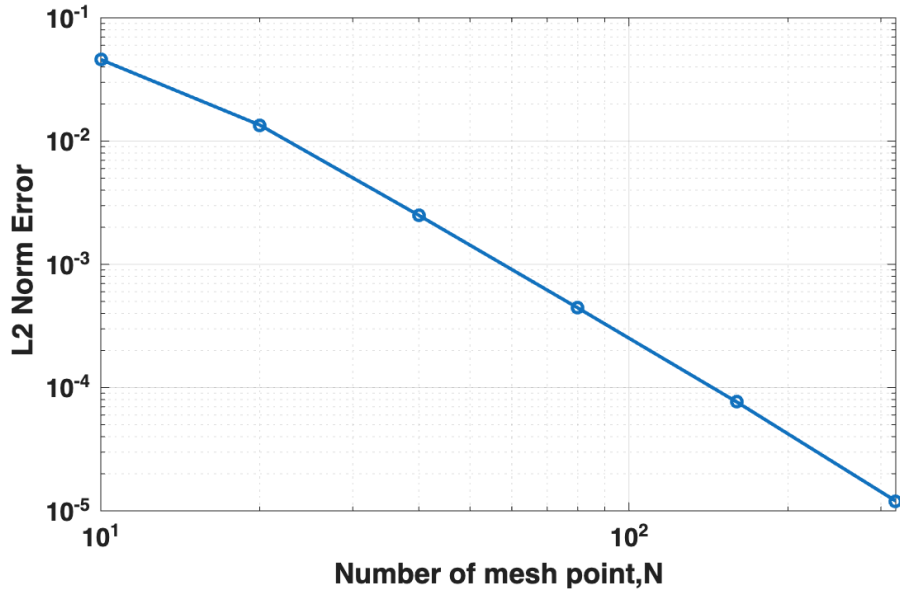


Figure 3.10: Plot of the ℓ_2 -norm error against the total number of mesh points N for $q_0 = 0.3$.

3.6 Summary

In this chapter, we have developed and tested numerical methods moving mesh finite difference methods based on velocity for solving the simple case of our model 3.1 without external forces, which is a one-dimensional non-linear diffusion equation known as the porous medium equation (PME). We implemented both mass-conserving and non-mass-conserving approaches under different boundary conditions, such as zero influx and non-zero influx scenarios.

We used three methods: a fully mass-conserving method when we have zero influx; a relative mass method for problems with external influx, which preserves mass fractions and adjusts mesh movement accordingly. We tested this method on problems with constant influx. However, we observed there was limitations of this method when the influx q_0 became large relative to the diffusion α_g , causing mesh distortion and instability. To address these limitations, we introduced a mass balance method that applies a direct local conservation law.

The third method (local mass balance) performs well for all tested values of q_0 and α_g . The only minor limitation is that the first cell can become relatively large when the influx q_0 is strong. However, we control this by adaptively adding a new mesh point whenever the first interval becomes more than twice as long as the second one. This keeps the method accurate and stable while maintaining second order convergence.

For each method, we validated the accuracy through convergence studies, confirming that the methods achieve second-order accuracy in space when

appropriate time step scaling is used. These methods form the basis for our complex scenarios when we include the external forces in the next chapters.

Chapter 4

The dynamics of oil slicks under the action of travelling surface waves

The early mathematical models of the oil spreading process on water after spills, while being asymptotically rigorous, had been based on a simplifying assumption of the so-called calm sea conditions [Fay, 1969, Fay, 1971, Hoult, 1972a, Buckmaster, 1973, Foda and Cox, 1980, Phillips, 1997]. However, they played an important role, as their analysis has allowed us to identify the main features, parameters and associated regimes of the phenomenon. The models were unable to provide practical insights on the influence of the diverse natural factors regularly present in real sea conditions, such as water waves. At the same time, it has been well-known from previous studies that water wave motion, including the surface waves, have a very profound influence on the fate of oil spills. Those studies have been focused for the last 30-40

years on the process of emulsification of the oil layer, which could turn the layer into a distribution of oil droplets, and on the subsequent transport of the droplets in the water by currents and turbulent motion [Delvigne and Sweeney, 1988, Li et al., 2008, Fingas, 2011a, Johansen et al., 2015, Nissanka and Yapa, 2016, Li et al., 2017a, Li et al., 2017b, Nissanka and Yapa, 2017, Nissanka and Yapa, 2018].

From the environmental point of view, in contrast to the oil distributed on the surface in the form of oil slicks, the emulsified fraction of oil spillage could penetrate deep into the water column leading to large volume pollutions [Boufadel et al., 2020]. Such volume distributions of oil droplets are very difficult to remove making decontamination procedures very expensive, time consuming and largely less effective, which explains the attention paid to the behaviour of the emulsified fraction.

The mechanism of the emulsification process has been long linked to the activity of the so-called breaking waves. The first empirical studies were very insightful producing many universal and practical relationships such as the distribution of droplets by their length scale at different parameters characterising the breaking waves [Delvigne and Sweeney, 1988]. This work has been serving as the foundation for the further developments of the simulation methodologies in full-scale forecasting models currently in use [Li et al., 2008, Geng et al., 2016a, Zhao et al., 2016, Li et al., 2017b, Boufadel et al., 2020, Zhao et al., 2020]. Due to the complexity of the real-sea environmental conditions, the development of full-scale forecasting models have been always empirical by nature. So that it appeared there was a gap in our understanding between the rather rigorous theoretical background developed under the

calm-sea conditions and much more realistic but largely empirical approach taken in global simulation strategies.

The idea promoted in the thesis is to partially fill that gap by demonstrating that many interesting results and insights can be rigorously obtained by dropping the assumption of the calm-sea conditions and considering surface water waves, though away from the breaking point.

The approach can be possibly justified as the emulsification process is known to be rather slow in comparison to the time scale of the gravity-viscous regime of spreading analysed in [Fay, 1969, Fay, 1971, Hoult, 1972a, Buckmaster, 1973, Foda and Cox, 1980, Phillips, 1997]. Moreover, the droplet formation models currently used require initial distribution, which can only be generated on the basis of a model modified by the presence of the wave motion. So here we would like to conduct such a study to provide comprehensive and rigorous analysis of the oil slick dynamics under the action of water waves.

In this chapter we focus on the effects of the travelling-wave perturbations, as opposed to standing-wave perturbations, which are not experiencing any breaking events, that is steep spatial gradients on the scale of the oil layer thickness, more details about the standing wave can be found in our published work in [Lukyanov et al., 2024].

The structure of this chapter is the following. We simulate the motion of oil slicks under the action of travelling wave perturbations using the numerical method described in Chapter 3 and two representative real-case scenarios, when the controlling parameter $\alpha_g \approx 10^{-2}$ and the effective Péclet number $Pe \approx 1$, see the definition (4.1) further in this chapter, that is α_g is large enough so that the effects from the diffusion and the water-wave motion are

comparable, and when $\alpha_g \approx 10^{-5}$ and $Pe \gg 1$, when the water wave motion dominates over the diffusion processes.

We use numerical results to understand how the wave properties, that is the amplitude and the frequency, affect the spreading of oil slicks. We will demonstrate different characteristic regimes of spreading showing their main features and interesting interpretations.

One interesting, though may be not very rigorous, interpretation comes from the analysis of motion of debris in the ocean, that is in fact suspended particles. We know that there is a particular mechanism, so called the Stokes drift effect, which causes the net movement of particles due to wave oscillations [Stokes, 1847, Longuet-Higgins, 1953, Falkovich, 2011]. The drift motion of oil slicks by short wavelength perturbations has been studied and compared with laboratory observations in [Christensen and Terrile, 2009], which gives us strong indication that the mechanism plays crucial role in the oil slick dynamics in general conditions.

Therefore, this would be informative to compare our model predictions with the Stokes drift effect even though we are dealing here with a liquid distribution rather than a distribution of particles. Indeed, at some conditions one may expect that the behaviour of an oil spot on the water surface should resemble that of a solid particle. The details of the Stokes drift effect in its simplest one-dimensional formulation and analysis can be found in the Appendix.

4.1 Characteristic sets of parameters

Before we analyse the spreading of oil slicks under the travelling wave perturbations, we first introduce the range of the dimensional and non-dimensional parameters that describe oil slick dynamics under the wave-induced transport in our case. The set of non-dimensional parameters has been introduced in Chapter 2 dedicated to the mathematical model. Here, these choices are made to represent realistic conditions at different stages of an oil-spill evolution.

In our study, the evolution of oil slicks on wavy water surfaces is modelled using the thin-film lubrication approximation, see Chapter 2, which assumes that the oil layer thickness H is much smaller than the characteristic horizontal scale L , such that the aspect ratio $\varepsilon = \frac{H}{L} \ll 1$. Based on this condition, we choose the dimensional parameters. We consider oil layer thicknesses H in the range $1 \text{ mm} \leq H \leq 10 \text{ mm}$, covering both initial and later stages of spreading. The horizontal length scale $L = 50 \text{ m}$ represents either the characteristic horizontal dimension of the oil slick or the wavelength of the underlying surface wave, and the characteristic horizontal velocity is set to $U = 1 \text{ m/s}$ as representative of initial spreading velocities in the gravity-viscous regime, consistent with order-of-magnitude estimates in [Hoult, 1972a]. This gives a characteristic spreading time of

$$t_0 = \frac{L}{U} = \frac{50 \text{ m}}{1 \text{ m/s}} = 50 \text{ s}.$$

This time scale indicates the time it would take for the oil to spread over the

horizontal length scale L at the characteristic velocity U . The characteristic velocity U can be also associated with the characteristic water-wave phase velocity, hence the time can be interpreted as the characteristic time of the wave to propagate the characteristic distance L .

So, for the range of H , the aspect ratio ε values are in between $2 \times 10^{-5} \leq \varepsilon \leq 2 \times 10^{-4}$, which satisfy the thin-film assumption $\varepsilon \ll 1$.

The oil density $\rho = 8.7 \times 10^2 \text{ kg/m}^3$ and the dynamic viscosity $\mu = 8.7 \times 10^{-3} \text{ Pa} \cdot \text{s}$ are chosen because they represent typical light crude oils [Shaw, 2003, Fingas, 2011a, Lukyanov et al., 2024], ensuring realistic modelling of buoyancy and spreading dynamics on water surfaces.

From these dimensional values, we estimate the following non-dimensional parameters, previously introduced in Chapter 2, that allow us to distinguish between diffusion - dominated and advection - dominated regimes and to assess how the wave motion may affect the transport and the deformation of oil slicks.

The Reynolds number, which was defined in our model in Chapter 2 as

$$\text{Re} = \frac{\text{inertia force}}{\text{viscous force}} = \frac{\varepsilon \rho U H}{\mu},$$

is then in the range $0.002 \leq \text{Re} \leq 0.2$. The Froude number, given by

$$\text{Fr}^2 = \frac{\text{inertia force}}{\text{gravity force}} = \frac{U^2}{gH},$$

is in the range $10 \leq Fr^2 \leq 100$ (at $g \approx 9.81 \text{ m/s}^2$), and

$$\alpha_g = \frac{\text{gravity force}}{\text{viscous force}} = \frac{\rho \varepsilon g H^2}{\mu U},$$

is then in the range $2 \times 10^{-2} \leq \alpha_g \leq 2 \times 10^{-5}$. The aspect ratio ε values used lie in the range $2 \times 10^{-5} \leq \varepsilon \leq 2 \times 10^{-4}$. These quantities are summarised in Table 4.1 for the initial (thick layer) and later (thin layer) stages of spreading.

In addition, we introduce non-dimensional wave parameters to quantify the influence of the wave-induced transport on oil slick spreading. In particular, the wave amplitude B_0 , when the dimensional quantity was normalised by H , the wave number k , when the dimensional quantity was normalised by L^{-1} and the angular frequency ω , when the dimensional parameter was normalised by t_0^{-1} .

We consider a characteristic set of the water-wave parameters within the studied range. The non-dimensional amplitude B_0 is set within $4 \leq B_0 \leq 8$ at $\alpha_g = 0.02$, corresponding to a dimensional free-surface elevation in between 4 cm and 8 cm if $H = 10 \text{ mm}$, and within $160 \leq B_0 \leq 320$ at $\alpha_g = 2 \times 10^{-5}$ corresponding to a dimensional free-surface elevation in between 16 cm and 32 cm if $H = 1 \text{ mm}$. So that despite the non-dimensional numbers differ substantially, the dimensional elevations are kept on a comparable level.

The non-dimensional wavenumber is set in between $4 \leq k \leq 32$, which, for $L = 50 \text{ m}$, corresponds to a dimensional wavenumber within $0.08 \text{ m}^{-1} \leq k_{\text{dim}} \leq 0.64 \text{ m}^{-1}$ and hence a wavelength of about 10 – 80 m. Using the non-dimensional dispersion relation (2.37) with these parameters, we obtain the characteristic values $\omega \approx 70$ at $k = 10$ and $\omega \approx 45$ at $k = 4$, consistent with

moderate ocean wave conditions.

The balance between wave-driven advection and effective diffusion can be measured using the effective Péclet number,

$$\text{Pe} = \frac{B_0 \varepsilon \omega}{\alpha_g} = \frac{\text{advection}}{\text{diffusion}}. \quad (4.1)$$

In our analysis of the initial stage, when $H = 10$ mm and $\alpha_g = 0.02$ at $\varepsilon = 2 \times 10^{-4}$, using $B_0 = 4$, $k = 4$, and $\omega \approx 45$, one obtains $\text{Pe}_{\min} = 1.8$, indicating that both processes are relevant, with advection becoming more significant for larger amplitudes or shorter wavelengths. Here the wave number k enters through the deep-water dispersion relation

$$\omega^2 = \varepsilon^{-1} F r^{-2} k$$

Parameter	Initial Stage (Thick Layer)	Later Stage (Thin Layer)
H [mm]	10	1
Re	0.2	0.002
$F r^2$	10	100
α_g	2×10^{-2}	2×10^{-5}

Table 4.1: Characteristic dimensional and non-dimensional parameters for early and late stages of oil slick spreading.

For the later stage, when $H = 1$ mm, with $B_0 = 160$ corresponding again to a dimensional free-surface elevation of approximately 16 cm, and the same $k = 4$ and $\omega \approx 45$, the value increases to $\text{Pe}_{\min} \approx 7000$, indicating that the wave-driven advection dominates.

In the next section, we investigate numerically this advection-dominated

regime in detail.

We also note, that in such a strongly advection-dominated regime, the transport of the oil slick, we conjecture, may be controlled, at least to some extent, by the Stokes drift mechanism. We examine this hypothesis in the next section.

4.2 Travelling wave effect in the advection regime

Now, consider the dynamics of oil slicks under the influence of perturbations (2.30). In this section, we focus on the advection-dominated regime, which corresponds to the later stages of the oil slick spreading where the layer is thinner $H = 1 \text{ mm}$ and the diffusion effects are much less important in comparison to the convection induced by the water waves, that is $\alpha_g = 2 \times 10^{-5}$ and $\text{Pe}_{\min} \approx 7000$.

We consider the same type of the travelling wave perturbations introduced earlier in Chapter 2, representing deep-water wave motion propagating in the positive x -direction:

$$B(x, t) = B_0 \sin(kx - \omega t), \quad V(x, t) = B_0 \epsilon \omega \sin(kx - \omega t), \quad (4.2)$$

where, to remind, B_0 is the amplitude of the surface wave, k is the wave number, ω is the frequency, and ϵ is the small parameter.

4.3 Numerical results and analysis

We numerically investigate how the slick evolves under an advection-driven regime by solving equation (2.30) using a moving-mesh finite difference method (implemented in Chapter 3) which is second-order accurate in space and first-order accurate in time [Lee et al., 2015], with mass conservation enforced in each cell through zero-flux boundary conditions.

In our simulation we used the boundary conditions specified in (2.31) and the non-dimensional parameters corresponding to the later stage of spreading, where the oil layer is thin and the diffusion parameter α_g is relatively small, see table 4.1.

The computational domain was set to $x \in [-1, 1]$ and the total mass of the oil slick was normalized such that initially,

$$M = \int_{-1}^1 h(x, 0) dx = 2.$$

This corresponds to an average initial non-dimensional oil thickness of 1 over a domain of length 2. Since mass is conserved in the absence of external sources, $M = 2$ remains constant for all later times. We selected the initial profile to match a realistic oil slick scenario. Specifically, we used a self-similar solution (2.46) from Chapter 2, which acts as an attractor for the spreading process in the absence of water waves. This idea should align with theoretical expectations if, say, till some moment of time, the profile had some time to evolve before the waves started to have action.

We also tested alternatives, such as cosine profiles, to check robustness and

universality of the numerical approach.

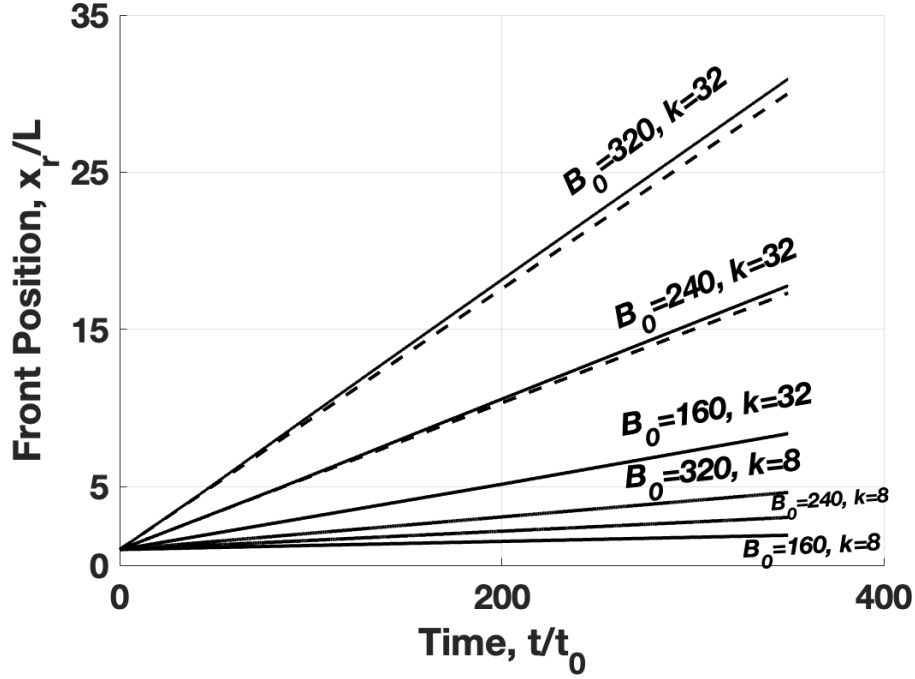


Figure 4.1: Evolution of the moving front $x_r(t)$, starting from $x = 1$, over a wavy interface for different perturbation amplitudes B_0 and wave numbers k , with fixed parameters $\alpha_g = 2 \times 10^{-5}$, $\text{Fr}^2 = 100$, and $\varepsilon = 2 \times 10^{-5}$. The initial profile is the self-similar solution (2.46). The dashed lines show the evolution curves corresponding to the Stokes drift with velocities according to (A.7).

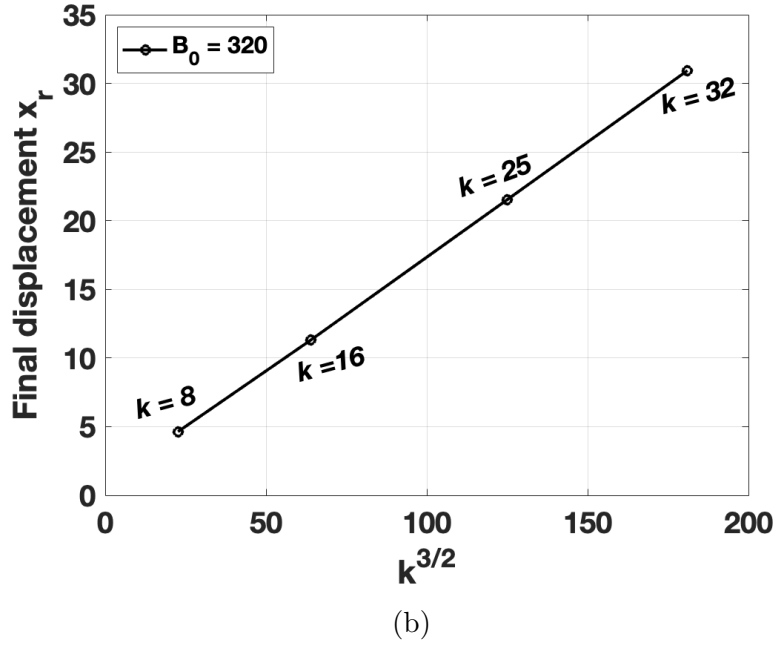
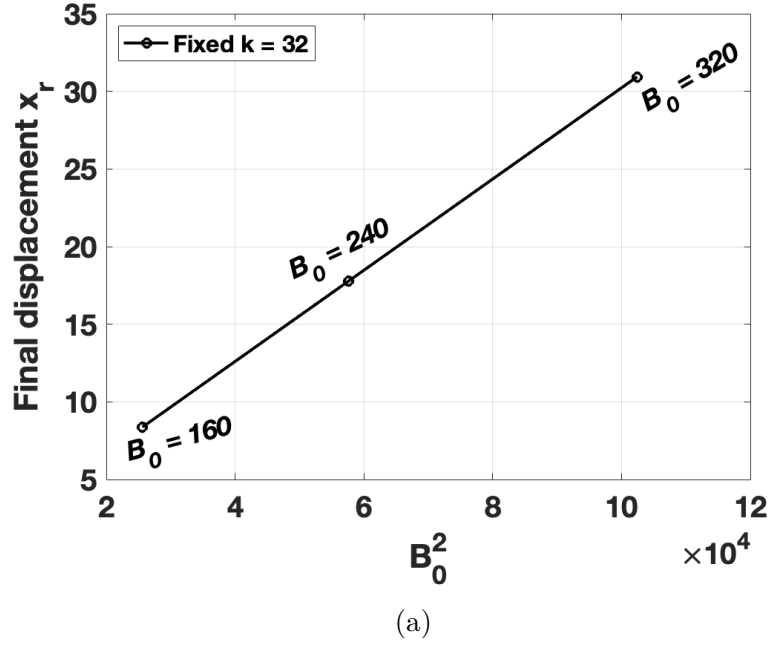


Figure 4.2: Final displacement of the oil slick front at $t/t_0 = 350$ as a function of B_0^2 for fixed wave number $k = 32$ (a), and as a function of $k^{3/2}$ for fixed amplitude $B_0 = 320$ (b). The parameters used are: $\alpha_g = 2 \times 10^{-5}$, $\text{Fr}^2 = 100$ and $\varepsilon = 2 \times 10^{-5}$.

To facilitate the parametrical study, the wave parameters were varied to see how the wave properties affect the slick motion. The amplitude B_0 is chosen from the range $160 \leq B_0 \leq 320$, corresponding to realistic wave heights of about 16 – 32 cm in physical units, typical for moderate sea conditions. The wave number k is varied from 4 to 32, which corresponds to wavelengths between 80 m and 10 m. We used 100 – 200 mesh points when $k \leq 20$ and increased the resolution for higher wave numbers $k > 20$ to capture fine details.

The results of the simulations are presented in Figures 4.1 - 4.5. One can observe in the figures how the front position $x_r(t)$ and the profile shape change with time at different representative values of the problem parameters.

Firstly, one can clearly observe that the front position on the right of the oil slick, $x_r(t)$, advances linearly over time in the advection-dominated regime, see Figure 4.1. This observation is interesting on its own, as we have already seen that in the self-similar regime, see (2.57), the front should move non-linearly as $x_r \propto t^{1/5}$, that is with vanishing velocity.

The linear dependence of the position on time might also be expected from the Stokes drift theory. If we directly fit the front displacement dependencies, Fig. 4.1, with a linear function of time, one can see that the displacement dynamics follows the Stokes drift theory prediction,

$$x(t) = x_0 + u_s t$$

,

$$u_s = \frac{1}{2} k B_0^2 \omega \varepsilon^2.$$

It is not just the linear dependence on time but also the parametric dependencies as one can figure out from Figs. 4.2, where the displacement at the final moment of time is shown as a function of B_0^2 and $k^{3/2}$, respectively. One can observe, that parametrically, the effect increases indeed quadratically with the wave amplitude B_0 , Fig. 4.2, and as $k^{3/2}$ with the wave number k , Fig. 4.2. The result is consistent with the theoretical scaling $u_s \propto B_0^2 k^{3/2}$, if one uses the dispersion relationship for the deep water waves, see also (A.8).

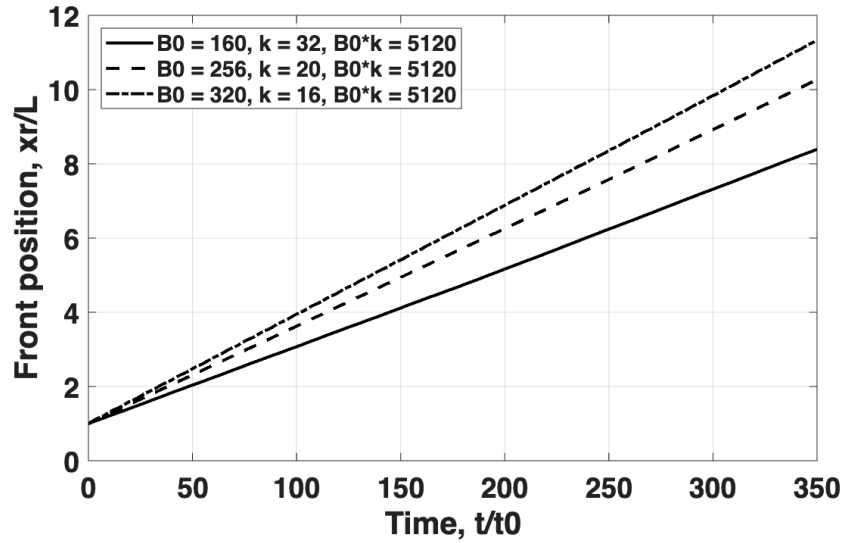
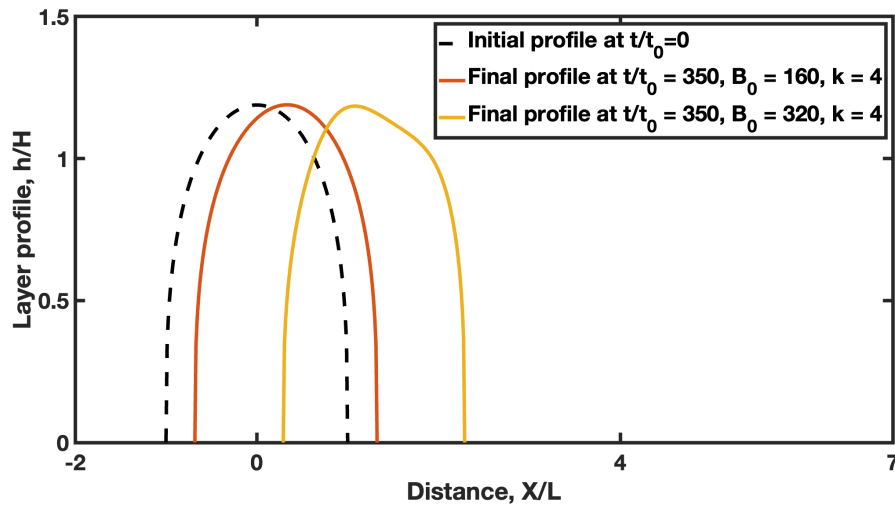
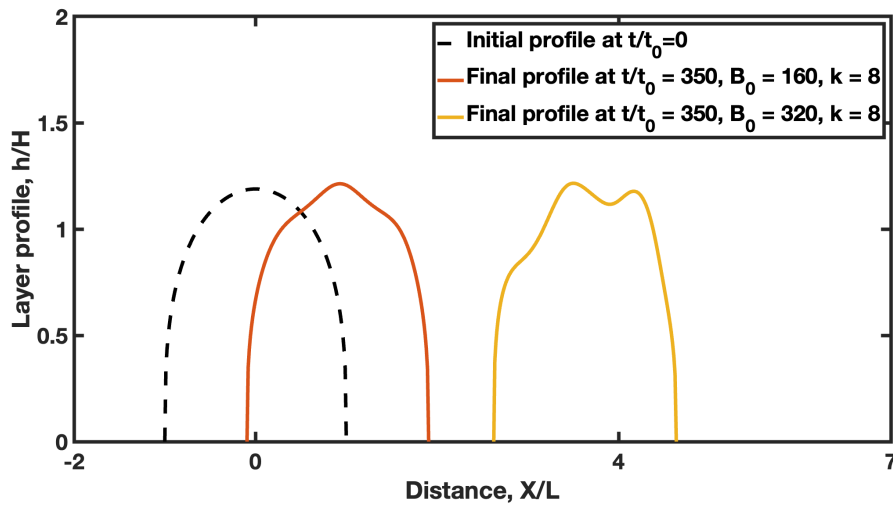


Figure 4.3: Evolution of the moving front $x_r(t)$ for different combinations of B_0 and k with fixed product $B_0 k = 5120$. The parameters are $\alpha_g = 2 \times 10^{-5}$, $Fr^2 = 100$, and $\varepsilon = 2 \times 10^{-5}$.

To examine this point more directly, we also considered cases where B_0 and k were varied while keeping the product $B_0 k$ approximately fixed. The resulting front-position curves are shown in Figure 4.3. The curves do not collapse onto a single line, indicating that the front displacement is not controlled by $B_0 k$ alone in this regime. This is consistent with the Stokes-drift scaling discussed above.

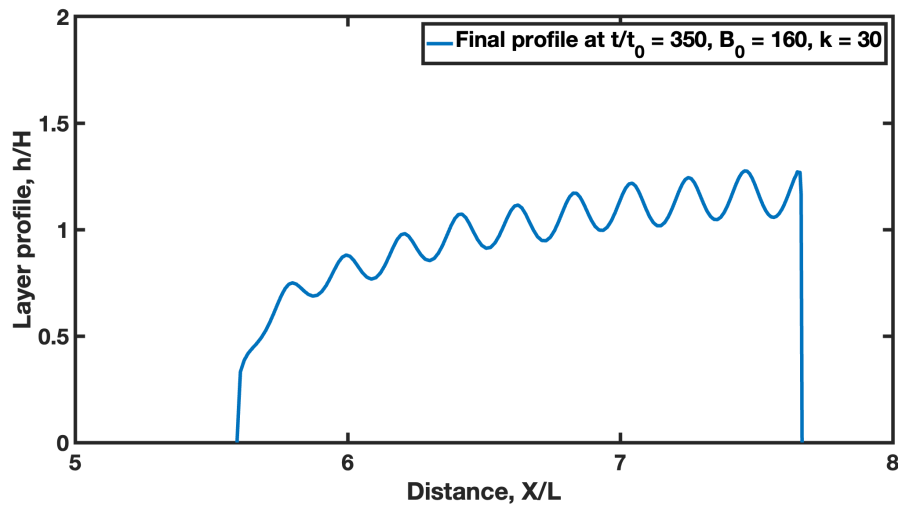


(a)

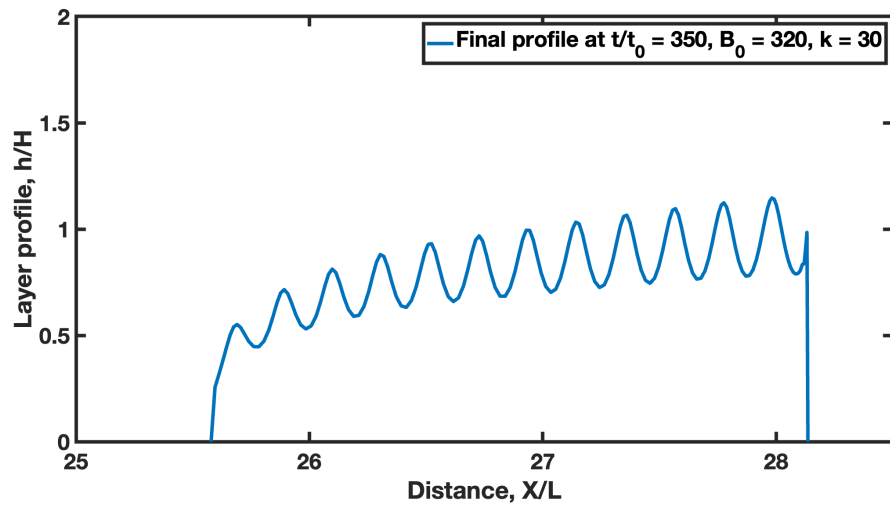


(b)

Figure 4.4: Evolution of the oil layer profiles $h(x, t)$ (2.30) at different low wave numbers k and amplitudes B_0 . The final time is $t/t_0 = 350$, with $\alpha_g = 2 \cdot 10^{-5}$ and $Fr^2 = 100$. The number of mesh nodes is $N = 320$. The initial profile is the self-similar solution (2.46).



(a)



(b)

Figure 4.5: Evolution of the oil layer profiles $h(x,t)$ (2.30) at high wave numbers k and amplitudes B_0 . The final time is $t/t_0 = 350$, with $\alpha_g = 2 \cdot 10^{-5}$ and $Fr^2 = 100$. The number of mesh nodes is $N = 320$. The initial profile is the self-similar solution (2.46).

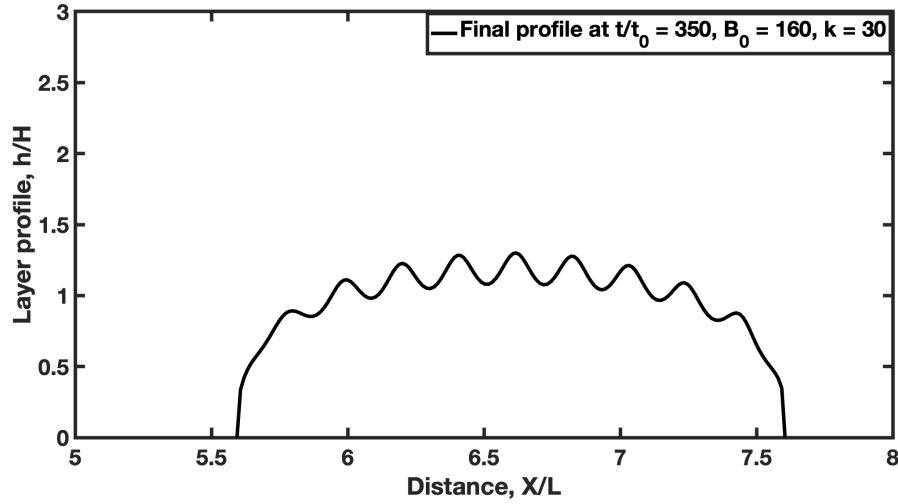


Figure 4.6: Evolution of the oil layer profiles $h(x,t)$ (2.30) at high wave numbers k and amplitudes B_0 in the absence of the elevation function $B(x,t)$. The final time is $t/t_0 = 350$, with $\alpha_g = 2 \cdot 10^{-5}$ and $Fr^2 = 100$. The number of mesh nodes is $N = 160$. The initial profile is the self-similar solution (2.46).

Consider now how the profile shape evolves during the oil slick motion. Apparently, considering the parallels found with the Stokes drift theory, the whole effect is expected to be proportional to the amplitude of the wave and its wavenumber. To verify the conjecture, we consider incremental variations of B_0 and k .

Figure 4.4 illustrates the evolution of the layer profile under travelling wave perturbations at two fixed wave numbers $k = 4$ and $k = 8$ with two amplitudes for each case $B_0 = 160$ and $B_0 = 320$. As one can observe, the profiles move almost unchanged in the shape at the lower end of the parameter space. In particular, as we see in Figure 4.4a, at a lower wave number $k = 4$ and moderate amplitude $B_0 = 160$, the final profile remains almost the same as the initial self-similar shape, with only a moderate forward drift of the slick.

When the amplitude increases to $B_0 = 320$, the profile shows a much larger displacement from its initial position having almost the initial profile. This shows that as is expected from the Stokes drift estimates, the larger is the amplitude the faster the profile is moving.

In Figure 4.4b, which corresponds to a higher wave number $k = 8$, the effect of increasing amplitude becomes more pronounced and the oil slick drifts forward more than in the $k = 4$ case, with similar shape preservation at lower B_0 but with small distortion at higher B_0 . Overall, these results show us how the drift increases with B_0 and becomes stronger at higher k , while the profile stays close to its initial shape. The width stays about 2 units in all cases, which shows that the travelling wave mainly drifts the whole profile forward without spreading. Figure 4.5 shows the evolution of the oil slick at high wave number (short waves), where we observe that the slick moves as a whole while exhibiting asymmetry at the edges, and the width of the oil slick is preserved. Moreover, we ran the same simulation as in Figure 4.5, but this time we omitted $B(x, t)$ and kept the velocity field $V(x, t)$ unchanged. We want to understand the effect of the elevation function $B(x, t)$ on the spreading specifically the effect of the term $\partial_x B$ in the right-hand side of (2.30). The result is shown in Figure 4.6. We observe that the oil slick evolves symmetrically keeping its original shape, and the profile no longer shows the asymmetry that we have seen previously in Figure 4.5. Also, the width of the slick stays nearly unchanged. This means that the distortions we see in the presence of $B(x, t)$ as shown in Figures 4.4 and 4.5 are not caused by the velocity field $V(x, t)$, but rather result from variations in the elevation function $B(x, t)$.

4.3.1 Interpretation in the limit of zero α_g

This result can be qualitatively understood by noticing that in the limit $\alpha_g \rightarrow 0$, when the diffusion is negligible, our governing equation (2.30) reduces to a first-order quasilinear PDE:

$$\frac{\partial h}{\partial t} + V \frac{\partial h}{\partial x} = -h \frac{\partial V}{\partial x}, \quad (4.3)$$

where $V(x, t)$ is the velocity field induced by the travelling wave.

This quasilinear equation describes the evolution of the oil slick in the purely advection regime, and its characteristic curves [Evans, 2010] are governed by the ordinary differential equation:

$$\frac{dx}{dt} = V(x, t), \quad (4.4)$$

which matches the velocity of the wave motion and therefore the structure of the Stokes drift equation (A.1) derived in Appendix A.

So, intuitively, this may provide another qualitative interpretation of why the motion of the oil slick follows the wave-induced drift, as it's confirmed in our numerical simulations.

However, one needs to note here the first-order partial differential equation (4.3) requires a different set of boundary conditions than that in our original model with diffusion, see (2.31), due to the lower order of the highest derivative; they do not apply here. In particular, as the boundary motion is controlled by the characteristics, we may study the initial problem only in the absence of any sources of oil steaming from the boundary of the domain.

In our case of a single harmonic wave, the solutions on the characteristics are explicitly defined by a system of equations

$$\frac{dx}{ds} = B_0 \epsilon \omega \sin(kx - \omega s), \quad (4.5)$$

where we introduce the parameter s along the characteristic curves. This parameter can be identified with the physical time t (i.e., $s = t$) and is used only for convenience in the derivation, and for the thickness $h = h(s)$ as a function of s , one obtains

$$\frac{dh}{ds} = -h B_0 \epsilon \omega k \cos(kx - \omega s). \quad (4.6)$$

This equation along the characteristics is obtained by applying the chain rule along the curves $dx/ds = V$ to the quasilinear PDE (4.3):

$$\frac{dh}{ds} = \frac{\partial h}{\partial t} \frac{dt}{ds} + \frac{\partial h}{\partial x} \frac{dx}{ds} = \frac{\partial h}{\partial t} + V \frac{\partial h}{\partial x} = -h \frac{\partial V}{\partial x}.$$

The initial conditions are $x(0) = x_0$, $x_0 \in [-1, 1]$ and $h(s)|_{s=0} = H(x_0)$, where $H(x) = h(x, 0)$ is the initial profile.

At large times $t \gg \omega^{-1}$, one can approximate the solution of the characteristic equation (4.5) by

$$x(s) \approx x_0 + u_s t, \quad (4.7)$$

where u_s is the Stokes drift velocity (A.8) from the asymptotic analysis presented in Appendix A. If we simply neglect the RHS of (4.6) at $B_0 \epsilon k \ll 1$, h should stay constant along the characteristics. Therefore, the initial thickness

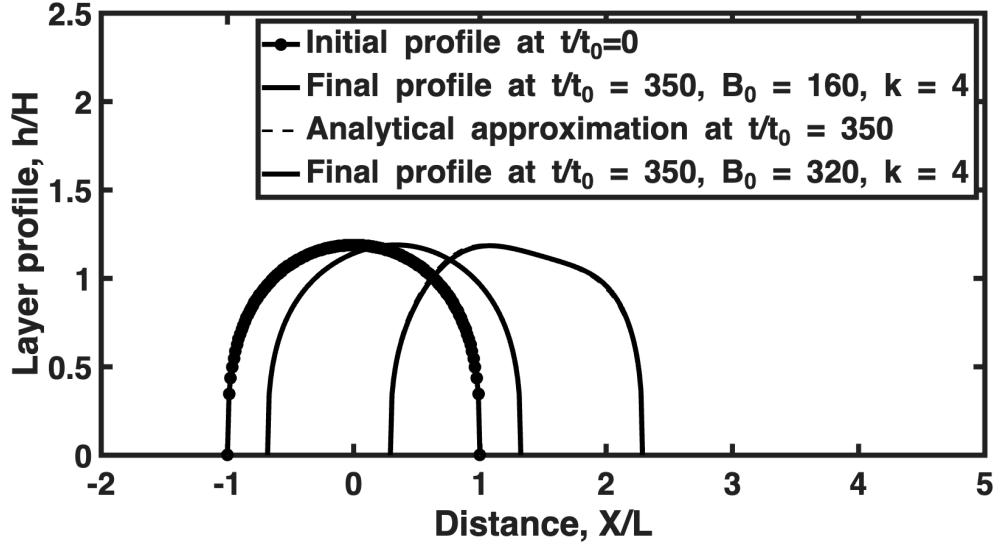
profile $H(x) = h(x, 0)$ is effectively advected by the drift, and the solution to the transport equation becomes after the substitution $x_0 = x - u_s t$

$$h(x, t) \approx H(x - u_s t). \quad (4.8)$$

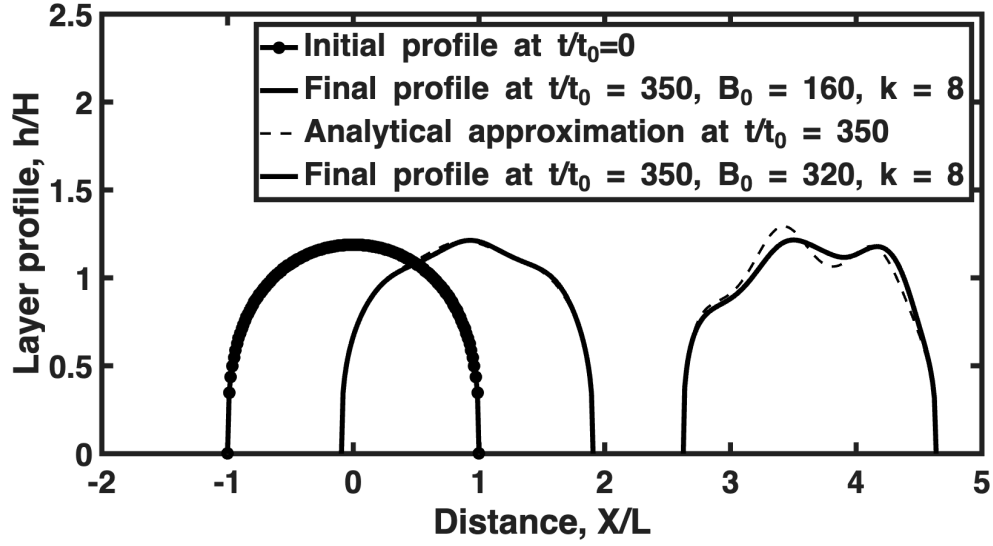
The previous result can be slightly improved if we integrate (4.6) subject to (4.7) to obtain

$$h(x, t) = H(x - u_s t) \exp \left\{ \frac{B_0 k \varepsilon \omega}{\omega - k u_s} (\sin[kx - \omega t] - \sin[k(x - u_s t)]) \right\}, \quad (4.9)$$

which at small $B_0 k \varepsilon \ll 1$ corresponds to (4.8).

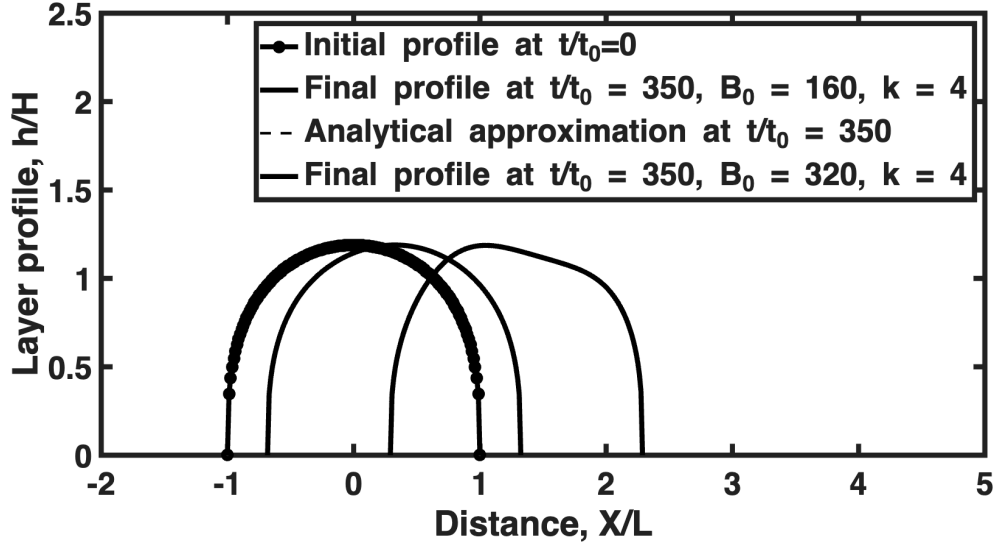


(a)

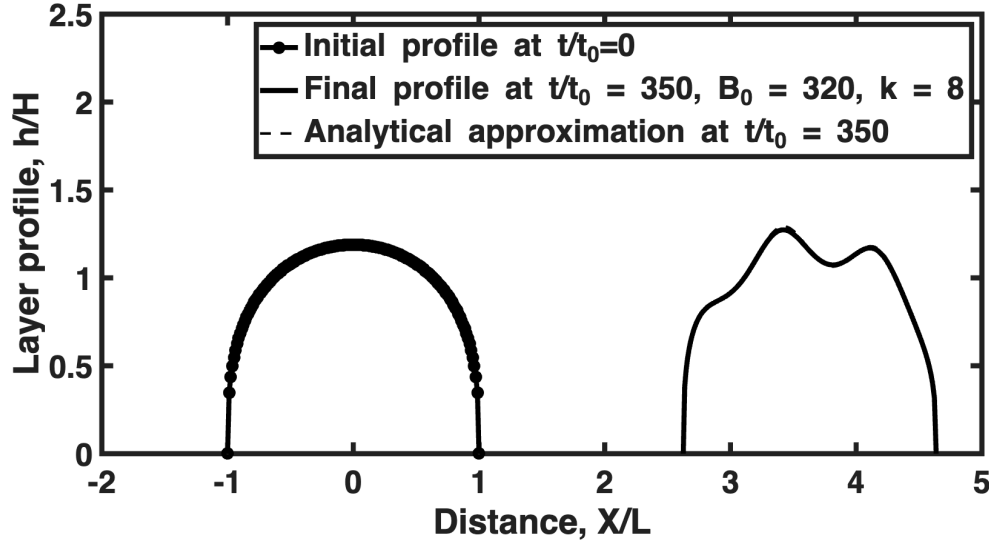


(b)

Figure 4.7: Comparison of the numerical solution (2.30) (solid line) and the analytical approximation (4.9) (dashed line) at different amplitudes B_0 and wave numbers k . The final time is $t/t_0 = 350$, with $\alpha_g = 2 \cdot 10^{-5}$ and $Fr^2 = 100$. The number of mesh nodes is $N = 160$ and 320 . The initial profile is the self-similar solution (2.46).

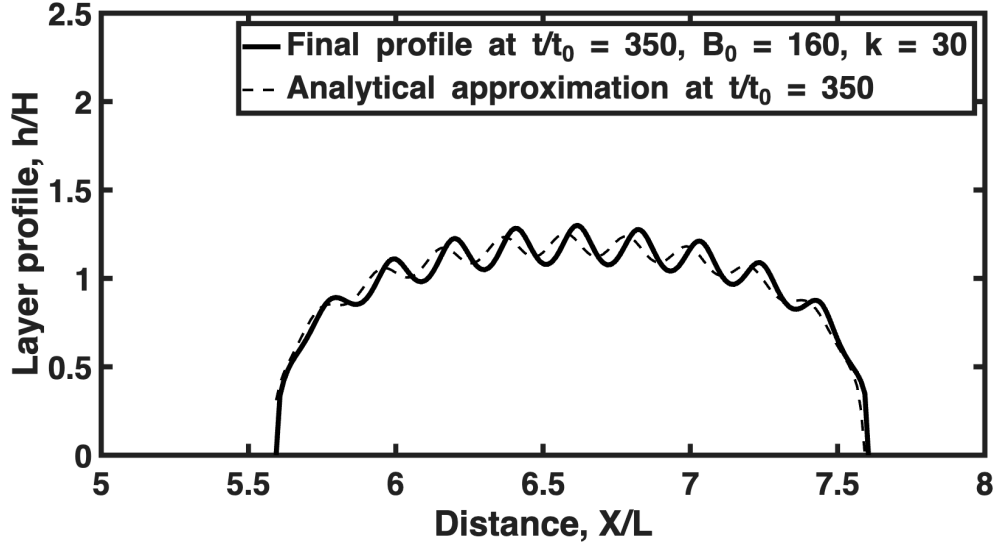


(a)

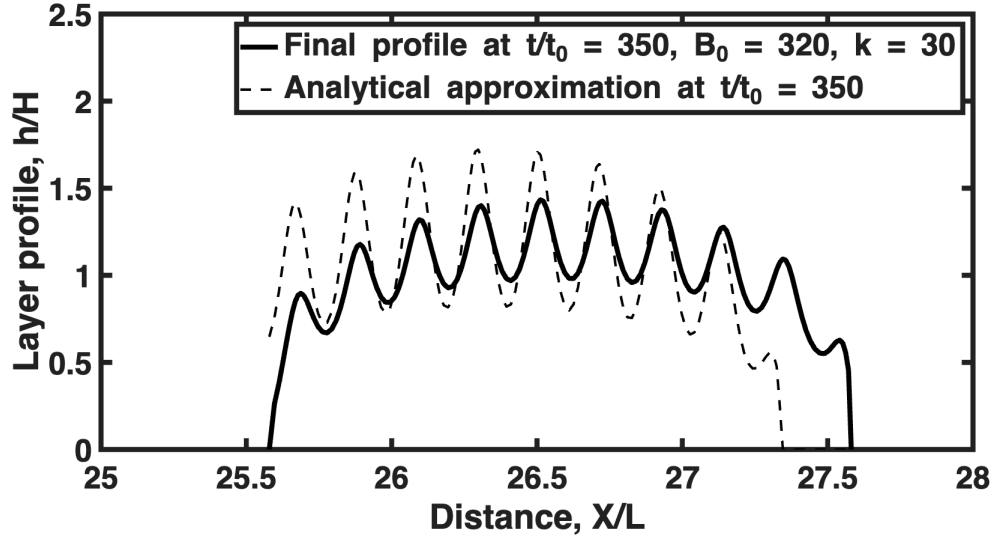


(b)

Figure 4.8: Comparison of the numerical solution (2.30) (solid line) in the absence of the elevation $B(x, t)$ and the analytical approximation (4.9) (dashed line) at different amplitudes B_0 and wave numbers k . The final time is $t/t_0 = 350$, with $\alpha_g = 2 \cdot 10^{-5}$ and $Fr^2 = 100$. The number of mesh nodes is $N = 160$ and 320. The initial profile is the self-similar solution (2.46).



(a)



(b)

Figure 4.9: Comparison of the numerical solution (2.30) (solid line) in the absence of the elevation $B(x, t)$ and the analytical approximation (4.9) (dashed line) at different amplitudes B_0 and $k = 30$. The final time is $t/t_0 = 350$, with $\alpha_g = 2 \cdot 10^{-5}$ and $Fr^2 = 100$. The number of mesh nodes is $N = 160$ and 320. The initial profile is the self-similar solution (2.46).

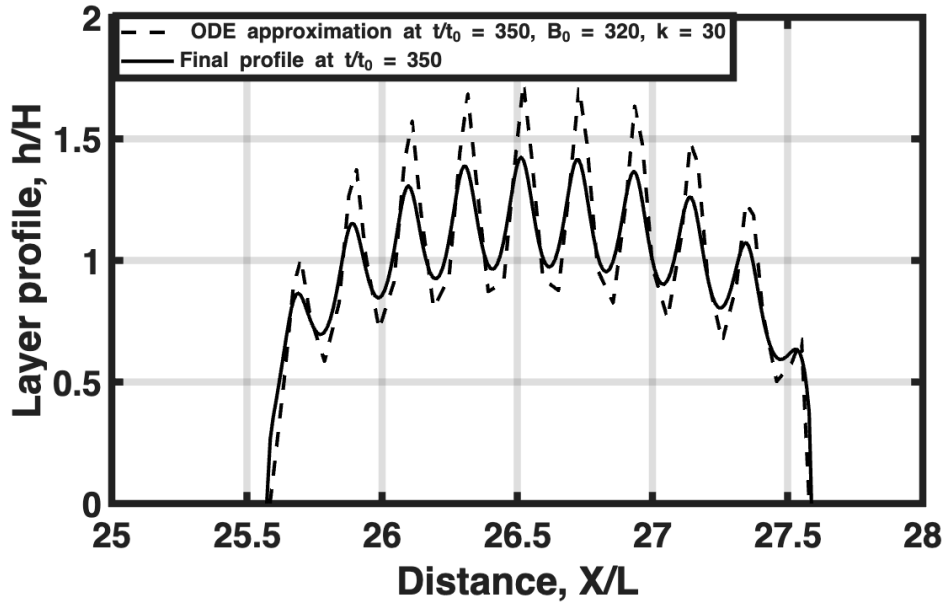
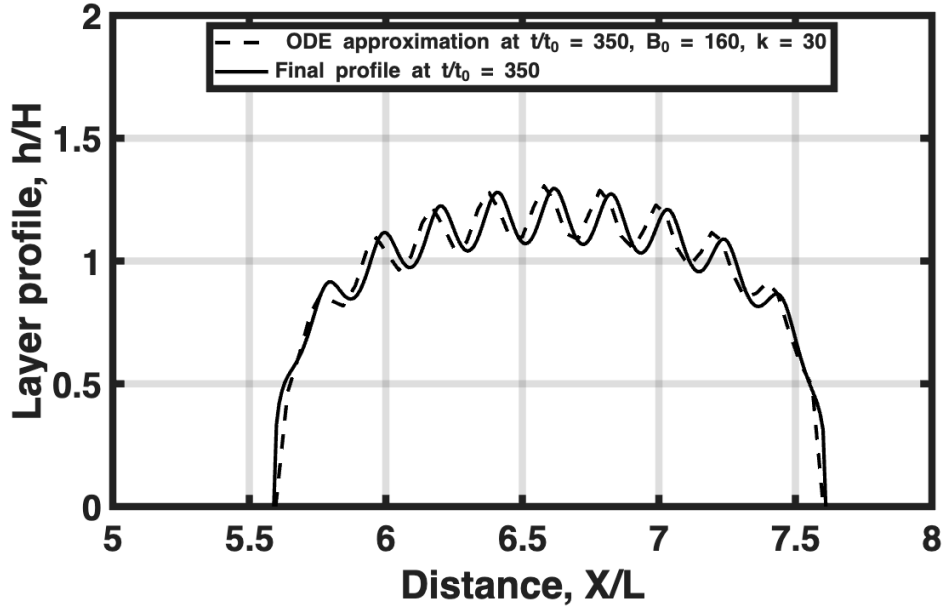


Figure 4.10: Comparison of the numerical solution (2.30) (solid line) in the absence of the elevation $B(x, t)$ with the analytical ODE approximations (dashed line) (4.5) and (4.6) at different amplitudes $B_0 = 160, 320$ and $k = 30$. The final time is $t/t_0 = 350$, with $\alpha_g = 2 \cdot 10^{-5}$ and $Fr^2 = 100$. The number of mesh nodes is $N = 160$ and 320 . The initial profile is the self-similar solution (2.46).

To further investigate the validity and limitations of the approximate solution (4.9) and the ODEs (4.5) and (4.6), we performed a direct comparison between (4.9) and the full numerical solution of our main model, which includes diffusion (2.30). We also compared the ODEs (4.5) and (4.6) with the full numerical solution of the same diffusion-inclusive model (2.30).

In these simulations, we use a small but essentially non-zero value of the non-dimensional control parameter $\alpha_g = 2 \cdot 10^{-5}$. This is necessary to understand the contribution from the diffusion over a long time period and also for the numerical method to be consistent in terms of the boundary conditions.

Our goal of this comparison is to examine whether the dynamics of the oil slick can be accurately described using only the convective, simplified equation (4.3) and to evaluate how much this simplification deviates from the numerical solution of the full PDE when weak diffusion is present. In particular, we aim to assess whether this difference manifests as a time lag or phase shift in the oil thickness profile.

In Figures 4.7–4.9, we present the comparison between the analytical approximation (4.9) and the numerical solutions of the (2.30). We consider two cases: one with the surface elevation function $B(x, t)$ and one without it.

The comparison in Figure 4.7 represents the case where the surface elevation $B(x, t)$ is included. We can see that the analytical approximation (4.9) and the numerical solution match very well at low wave numbers. Similarly, for $k = 8$ and moderate wave amplitude $B_0 = 160$, the analytical and numerical results are still very close. This shows that the simple advection model, which neglects diffusion, can capture the essential dynamics of the full numerical solution when the diffusion α_g is small.

We observe that when both the wave number and the wave amplitude increase ($k = 8$, $B_0 = 320$), a small but clear difference appears between the analytical and numerical results. We can see in Figure 4.7(b) that the analytical solution shows a slightly larger forward movement than the numerical one, which means that the simple advection model moves the oil layer a bit faster. We can explain this difference by a small delay caused by the diffusion in our full model, which makes the motion a little slower.

In contrast, Figures 4.8 and 4.9 show the results for the case where $B(x, t)$ was set to zero in (2.30). Removing the elevation term allows us to examine whether the velocity field $V(x, t)$ alone can describe the observed transport behaviour.

At moderate wave numbers ($k = 4$ and $k = 8$) and for both amplitudes ($B_0 = 160$ and $B_0 = 320$) in Figure 4.8, the analytical approximation (4.9) and the numerical results are almost the same.

However, when the wave number increases ($k = 30$), differences appear between the analytical and numerical results, as shown in Figure 4.9. So as the profile becomes highly irregular at larger values of k and also at larger B_0 , the diffusion contribution becomes visible and the small α_g approximation is less accurate then.

A similar behaviour was obtained when we compared the full numerical model with the characteristic system of ODEs (4.5) and (4.6), as shown in Figure 4.10, confirming again that the diffusion starts changing the overall transport behaviour.

In summary, we have shown in this section that in the advection-dominated regime when the diffusion contribution is relatively small in comparison to

the advection effects, 'thin' oil layers characteristic for later stages of spreading tend to drift in the direction of the wave phase velocity while largely preserving their initial width and in many cases their overall shape. This indicates that the effect of travelling wave perturbations in this regime is very similar and close to the classical Stokes drift motion of a particle when the oil slick is moving as a whole with the Stokes drift velocity [Lukyanov et al., 2024]. The speed of this drift increases with both the wave's amplitude and wave number, which matches both the theory and our numerical results. However, in the early stages of spreading, when the oil layer is thicker, diffusion plays an important role and cannot be ignored. During this time, both advection and diffusion act together and influence the oil's spreading. To explore these effects, we now turn to the diffusion-advection regime in the next section.

4.4 Travelling wave effect in the diffusion - advection regime

We now examine the diffusion-advection regime, which corresponds to the initial stages of oil slick spreading with thicker layers, $H = 10$ mm, and a larger diffusion parameter $\alpha_g = 0.02$, giving a minimum Péclet number $\min(\text{Pe}_E) = 2.8$. This value indicates a balance between viscous diffusion and wave-driven advection, and allows us to explore how their interaction influences the slick's evolution.

In this regime, we analyse the evolution of the oil slick parametrically by

varying the amplitude of the perturbations B_0 , the wave number k , and, in some cases, the initial oil slick profile to assess sensitivity to initial conditions [Lukyanov et al., 2024]. We begin our simulation with perturbation amplitudes of $B_0 = 4$ and $B_0 = 8$ corresponding to the dimensional elevations 4 cm and 8 cm respectively as in the previous consideration in Section 4.1. The wave number is set to $k = 10$ corresponding to a wavelength of ~ 30 m and $\omega \approx 71$. We also consider cases with shorter wave length and therefore with a higher wave number, $k = 16$ corresponding to a wavelength of approximately 20m.

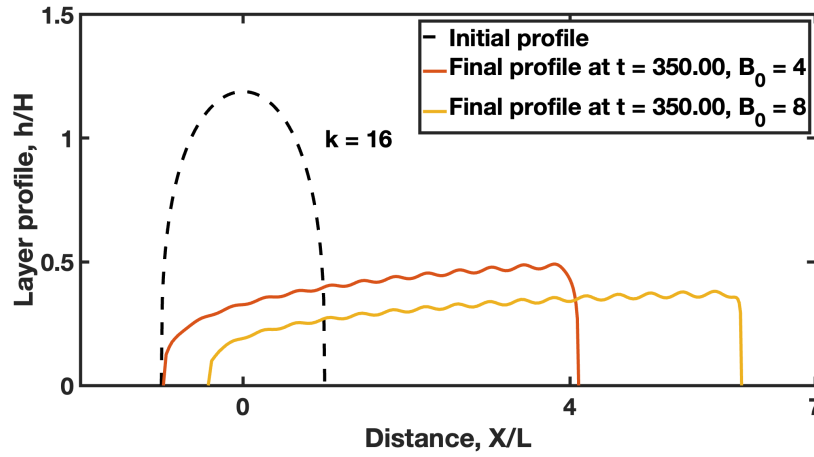


Figure 4.11: Evolution of the oil layer profiles $h(x, t)$ (2.30) at wave numbers k and different amplitudes B_0 . The final time is $t/t_0 = 350$, with $\alpha_g = 0.02$ and $Fr^2 = 10$. The number of mesh nodes is $N = 160$. The initial profile is the self-similar solution (2.46).

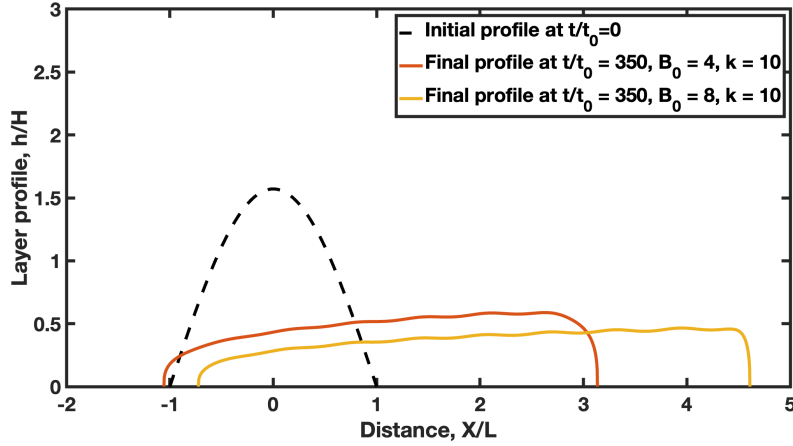


Figure 4.12: Evolution of the oil layer profiles $h(x, t)$ (2.30) at wave numbers k and different amplitudes B_0 . The final time is $t/t_0 = 350$, with $\alpha_g = 0.02$ and $Fr^2 = 10$. The number of mesh nodes is $N = 160$. The initial profile is a cosine function.

As one can observe in 4.11, the spreading process involves both drift and diffusion happening at the same time. Also, we notice that as the slick expands, its overall length increases, and both boundaries move in the direction of the wave group velocity [Lukyanov et al., 2024]. The steep leading edge is caused by the moving contact line. As the oil thickness vanishes at the front, the profile develops a sharp gradient near the contact line while being advected by the waves.

The spreading process here is asymmetric, since the right boundary of the oil slick moves faster than the left. This difference happens because wave motion and diffusion act unevenly on the two sides. While the right boundary moves and spreads faster due to the waves motion keep pushing it forward, the left boundary spreads more slowly due to diffusion. This asymmetry has been observed in previous experiments and studies [Ermakov et al., 2018], which reports experimental observations of asymmetric oil slick spreading

influenced by wind waves in real-world conditions, confirming that the numerical model successfully captures essential aspects of how oil slicks move and spread in real ocean conditions. We performed our simulations with different initial conditions $h(x, t_0) = \cos\left(\frac{\pi}{2}x\right)$ to investigate their influence on it. As shown in Figure 4.12, the final profiles exhibit similar asymmetric spreading and drift. This indicates an attractor behaviour, where diffusion moderates initial differences that converge to a similar universal final shape after approximately 350 non-dimensional time units (corresponding to about 5 hours in dimensional terms, based on the characteristic scales $L = 50$ m and $U = 1$ m/s). So, this converged form resembles, but does not exactly match, the self-similar solution in the absence of waves (2.46), indicating that wave motion preserves the attractor while introducing asymmetry from the advective drift caused by the travelling perturbation.

In our analysis, the evolution of the oil slick profile is examined by splitting it into two parts, the motion of the centre of mass of the oil slick denoted as X_{CM} and the motion of the boundaries, particularly the right boundary $x_r(t)$. The centre of mass is defined as:

$$X_{CM} = \frac{1}{M} \int_{x_l}^{x_r} hx \, dx, \quad (4.10)$$

where M is the total mass of the oil slick. By tracking X_{CM} over time, we quantify the net drift of the slick, which arises primarily from advective forces induced by the underlying wave motion.

To enable direct comparison with experimental studies on calm water surfaces [Hoult, 1972a], we apply asymptotic power-law fits to our numerical data for

large times $t \rightarrow \infty$, employing the forms:

$$x_r(t) - X_{CM} = A_p t^\alpha, \quad \text{or} \quad X_{CM} = A_p t^\alpha. \quad (4.11)$$

Here, A_p is a scaling coefficient that captures how parameters like the wave amplitude B_0 and wave number k affect the spreading and drifting rates of the oil slick, such as the speed at which the front position $x_r(t)$ advances or the centre of mass $X_{CM}(t)$ shifts over time, and α is the spreading exponent, which determines how fast the slick spreads over time.

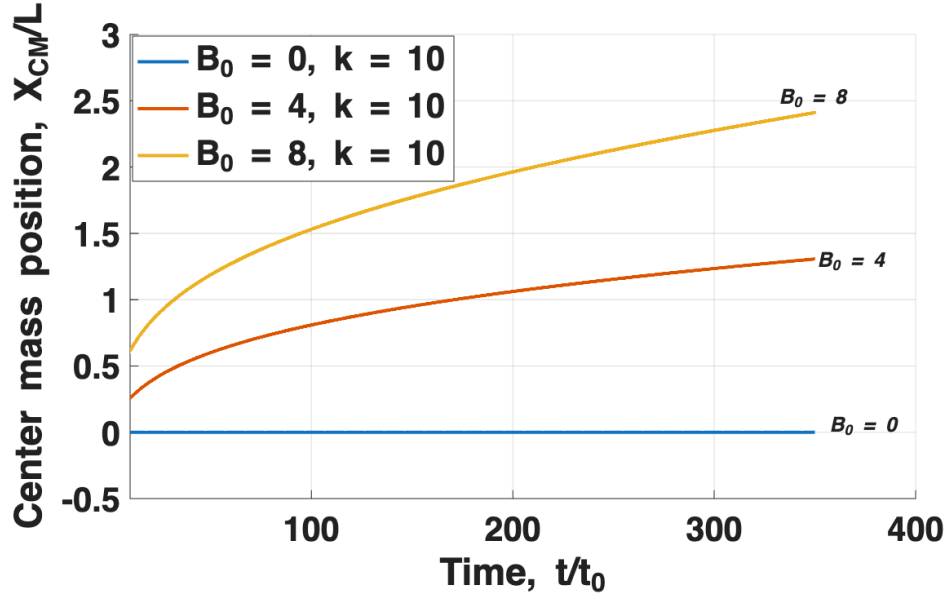


Figure 4.13: Position of the centre of mass $X_{CM}(t)$ over a wavy interface at different amplitudes B_0 and wave numbers k , with fixed parameters $\alpha_g = 0.02$, $Fr^2 = 10$, and $\varepsilon = 2 \times 10^{-4}$. The final time is $t/t_0 = 350$. The initial profile is the self-similar solution (2.46).

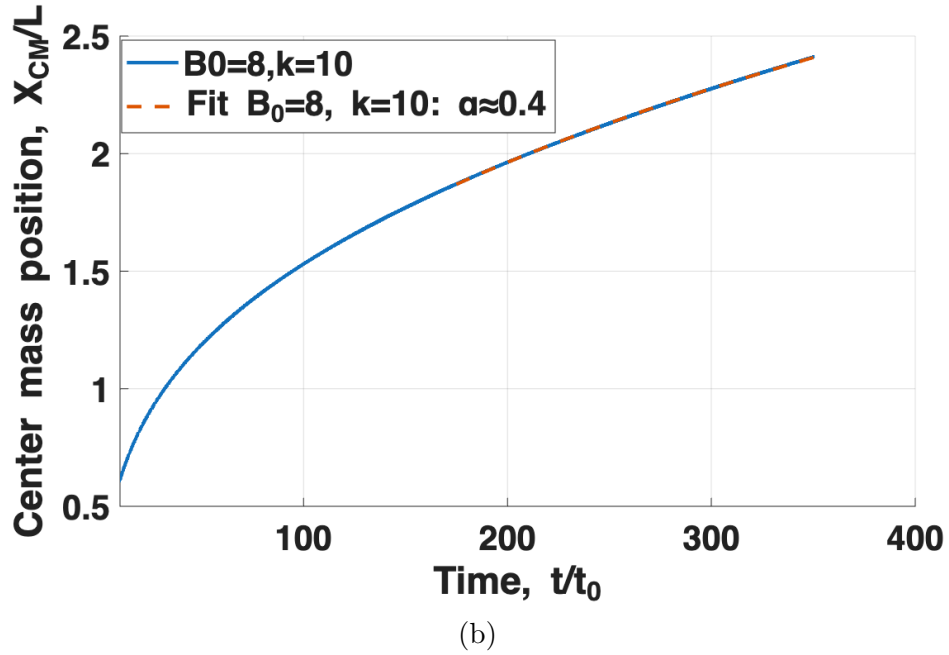
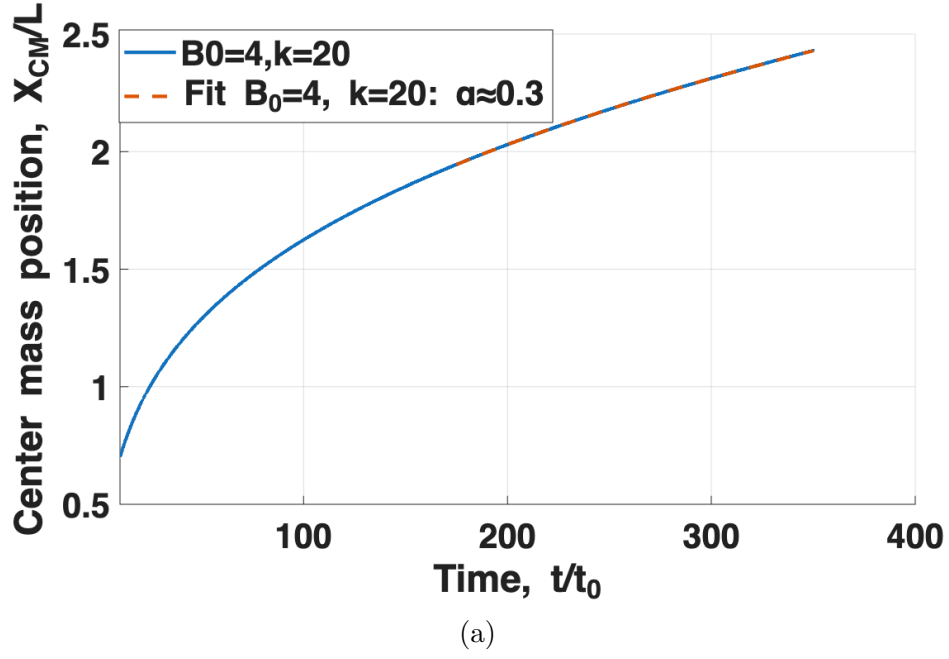


Figure 4.14: Position of the centre of mass $X_{CM}(t)$ over a wavy interface at different amplitudes B_0 and wave numbers k . The solid lines denote numerical results, and dashed lines represent power-law fits $f = f_0 t^\alpha$, with fixed parameters $\alpha_g = 0.02$, $Fr^2 = 10$, and $\varepsilon = 2 \times 10^{-4}$. The initial profile is the self-similar solution (2.46).

As illustrated in Figure 4.13, we observe that in the absence of wave forcing $B_0 = 0$, the centre of mass X_{CM} remains stationary at zero, confirming that pure diffusion leads to symmetric spreading with no net drift, consistent with classical porous medium theory [Vázquez, 2007]. In contrast, for $B_0 > 0$, the centre of mass drifts in the direction of the wave group velocity, but this drift is nonlinear over time due to diffusion-induced changes in the slick's shape and size. We also observe that this drift becomes stronger as the wave amplitude B_0 and the wave number k increase, as shown by the fitted curves in Figures 4.14a and 4.14b. This drift does not simply follow the classical Stokes drift prediction ($u_s \propto B_0^2 k^{3/2}$) as we have seen in previous section when the advection dominate the regime. Instead, our results indicate scaling with $B_0 k$, as diffusion moderates advection. This demonstrates that diffusion plays a critical role in reshaping the scaling of mass transport under travelling waves.

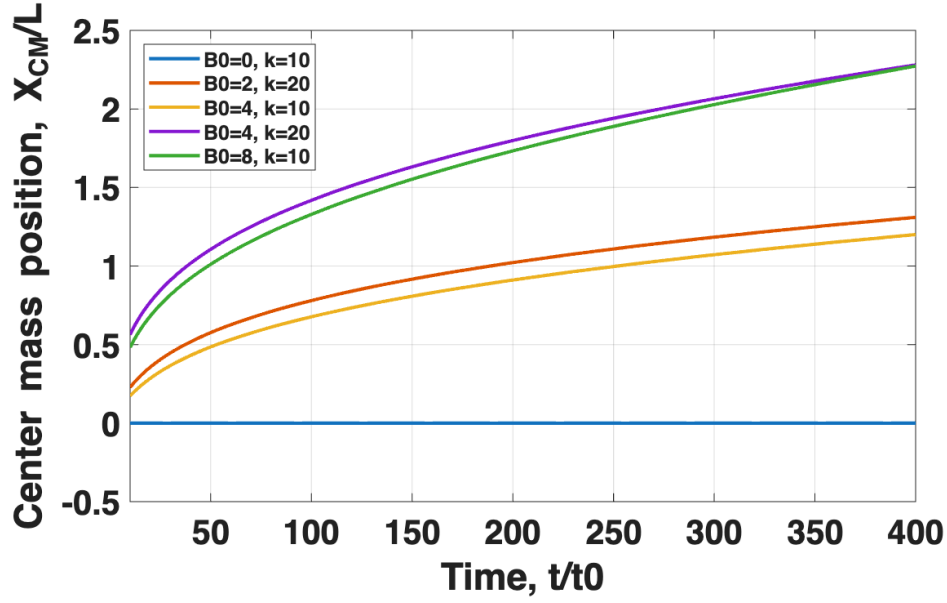


Figure 4.15: Centre of mass position $X_{CM}(t)$ for different combinations of B_0 and k , with fixed parameters $\alpha_g = 0.02$, $Fr^2 = 10$, and $\varepsilon = 2 \times 10^{-4}$. The final time is $t/t_0 = 400$. The initial profile is the self-similar solution (2.46).

To make this scaling clearer, we compare cases in which B_0 and k are varied separately while the product B_0k is kept fixed. The cases $(B_0, k) = (2, 20)$ and $(4, 10)$ both give $B_0k = 40$, while the cases $(B_0, k) = (4, 20)$ and $(8, 10)$ both give $B_0k = 80$. As shown in Figure 4.15, the curves with the same value of B_0k are very close to each other. This supports the interpretation that the drift of the centre of mass is mainly governed by the product B_0k .

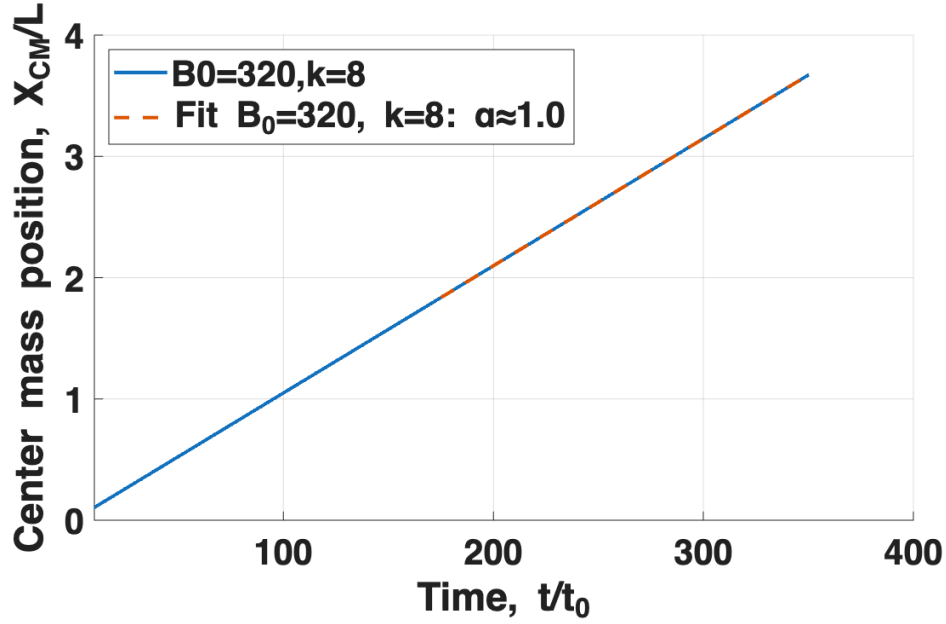


Figure 4.16: Position of the centre of mass $X_{CM}(t)$ over a wavy interface in the advection-dominated regime. The solid lines denote numerical results, and dashed lines represent power-law fits $f = f_0 t^\alpha$, with fixed parameters $\alpha_g = 2 \times 10^{-5}$ and $Fr^2 = 100$. The number of mesh nodes is $N = 160$. The initial profile is the self-similar solution (2.46).

In the advection-dominated regime Figure 4.16, the centre of mass is mainly transported by the wave-induced drift. Therefore, $X_{CM}(t)$ grows approximately linearly with time, corresponding to a fitted exponent close to $\alpha = 1$. This behaviour represents drift rather than diffusive spreading.

After analysing the centre of mass motion, we extend our examination to the relative front position $x_r(t) - X_{CM}(t)$, which tells us how far the front of the slick is from the centre of mass position. Our simulations show that the relative front position grows with time faster when waves are present compared to the no wave case ($B_0 = 0$).

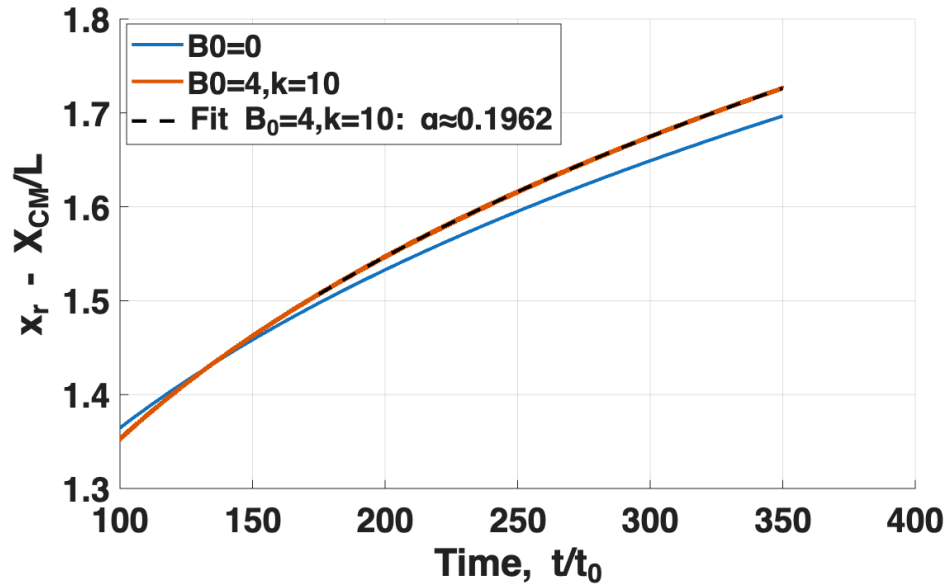


Figure 4.17: Relative front position $x_r(t) - X_{CM}(t)$ initially at $x_r(t) = 1$ over a wavy interface at different amplitudes B_0 and wave numbers k . The solid lines are numerical solutions while the dashed lines represent power-law fits $f = f_0 t^\alpha$, with fixed parameters $\alpha_g = 0.02$, $Fr^2 = 10$, and $\varepsilon = 2 \times 10^{-4}$. The initial profile is the self-similar solution (2.46).

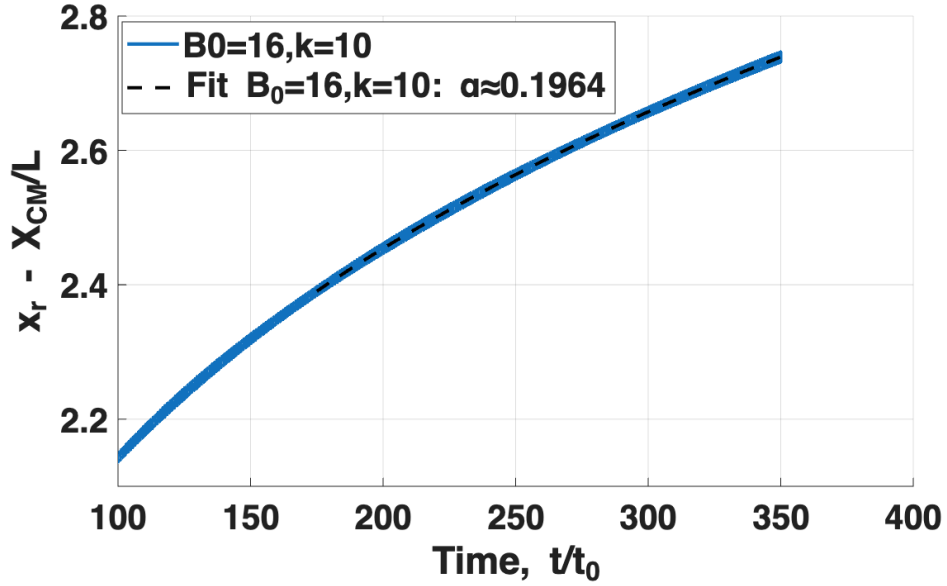
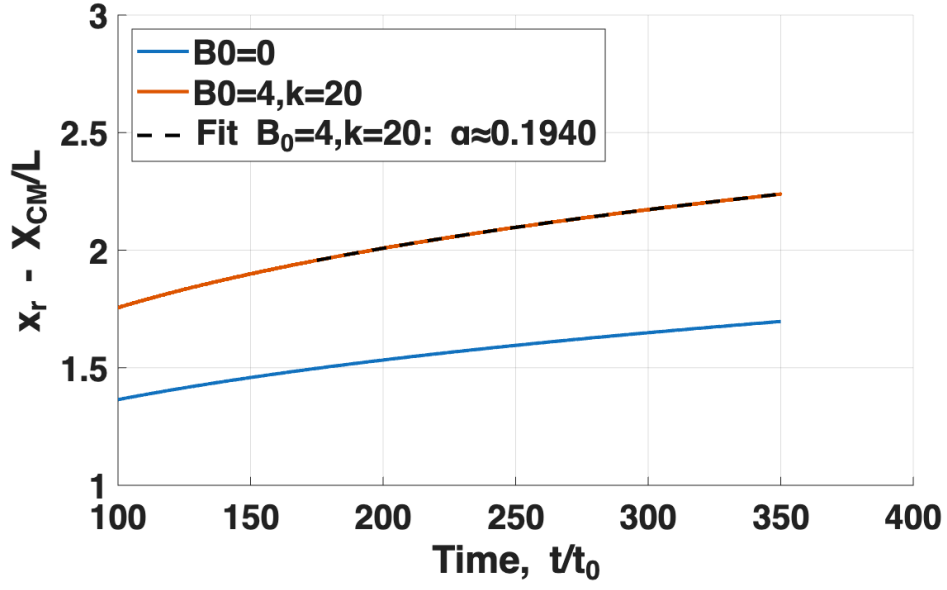


Figure 4.18: Relative front position $x_r(t) - X_{CM}(t)$ initially at $x_r(t) = 1$ over a wavy interface at different amplitudes B_0 and wave numbers k . The solid lines are numerical solutions while the dashed lines represent power-law fits $f = f_0 t^\alpha$, with fixed parameters $\alpha_g = 0.02$, $Fr^2 = 10$, and $\varepsilon = 2 \times 10^{-4}$. The initial profile is the self-similar solution (2.46).

In Figure 4.17 and Figure 4.18 , we show how the relative front position changes under different wave conditions, compared with the calm-water baseline. As we showed in Chapter 3 in absence of the wave, the slick spreading is symmetric without any front advancing faster than the centre. This behaviour is consistent with the calm-water spreading law, where laboratory studies have reported a power-law exponent of about $\alpha \approx 0.38$ for viscous spreading [Hoult, 1972b], with diffusion spreading the slick evenly in all directions. But here, when waves are present, we find that advection drives the front forward asymmetrically while diffusion smooths the profile. This leads to a smaller temporal spreading exponent of about $\alpha \approx 0.19 - 0.20$, so that the relative front position grows in time approximately as

$$x_r(t) - X_{CM}(t) \propto t^{0.2}.$$

At lower wave amplitude $B_0 = 4$, we observe that longer waves $k = 10$ show less growth than shorter waves $k = 20$. In the high-amplitude case $B_0 = 16, k = 10$, we observe that the front extends even further, showing that wave forcing enhances the overall spreading several times compared to calm water. When we double the wave number k , the front advances about 1.4 times further, consistent with the $\sqrt{2}$ increase that we observe from our simulations. This diffusive scaling can be expressed as $(B_0 k)^{1/2}$, which we see directly in the numerical results as shown in Figure 4.19. The rescaled curves show a similar late time trend, indicating an approximate universal behaviour under this scaling.

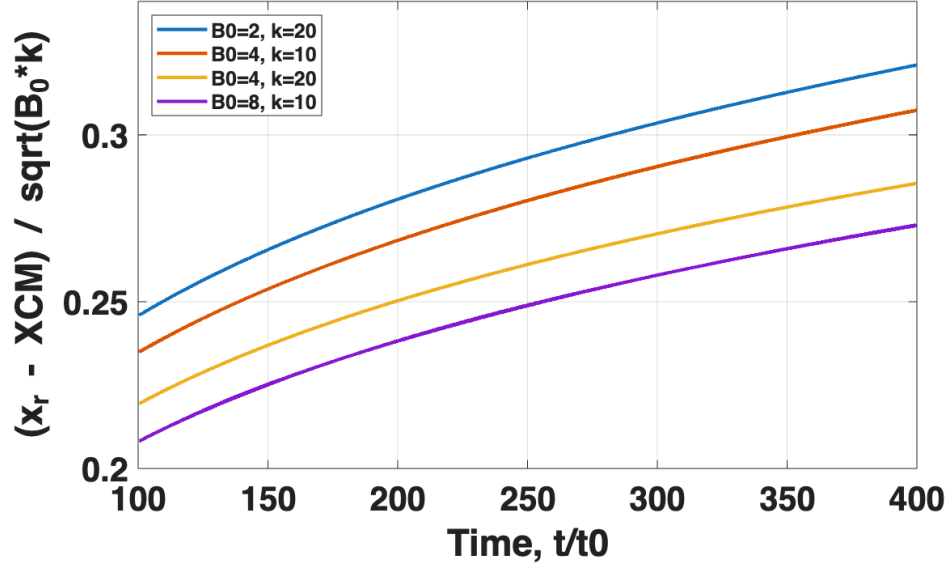


Figure 4.19: Rescaled relative front position $(x_r(t) - X_{CM}(t))/(B_0 k)^{1/2}$ for different combinations of B_0 and k , with fixed parameters $\alpha_g = 0.02$, $Fr^2 = 10$, and $\varepsilon = 2 \times 10^{-4}$. The final time is $t/t_0 = 400$. The initial profile is the self-similar solution (2.46).

In dimensional terms, if the wave amplitude is around 8 cm, a 100 m slick can extend to about 350 m in about five hours. If the amplitude is increased to 16 cm, which is still modest, the slick can extend to about 500 m over the same period [Lukyanov et al., 2024]. This means that waves push the front forward, while diffusion spreads the slick out, creating a longer and uneven shape that shows the combined effects of advection and diffusion.

4.5 Summary

In this chapter, we investigated the dynamics of oil slicks under the influence of travelling surface wave perturbations using the thin-film lubrication approximation (2.30). We analysed two characteristic regimes based on the rel-

ative importance of advection and diffusion: an advection-dominated regime corresponding to the later stages of spreading when the oil layer is thin, and a diffusion-advection regime corresponding to the early stages when the layer is thicker. We began by introducing a characteristic parameter set representing both early and late stages of spreading. This included dimensional and non-dimensional parameters.

In the advection-dominated regime, we showed that the oil slick drifts forward with minimal distortion of its shape, and the motion of the front closely follows the classical Stokes drift velocity. The numerical results confirmed that the drift speed increases nonlinearly with both the wave amplitude and wave number, consistent with theoretical predictions. In the diffusion-advection regime, the spreading became asymmetric, with the front advancing faster than the rear due to the combined influence of wave-induced advection and viscous diffusion. This led to a net drift of the centre of mass and a slower, power-law-like growth of the front position relative to the centre, with the displacement scaling approximately as $x_r(t) - X_{CM}(t) \propto t^{0.2}$. This shows how diffusion reduces the effect of wave-driven transport and leads to behaviour that is clearly different from classical diffusion or Stokes drift predictions. Additionally, different initial profiles evolved toward similar final shapes, indicating that the spreading tends toward an attractor, even under the influence of travelling surface waves.

Chapter 5

The dynamics of oil spills over a calm sea water surface in the presence of oil discharge.

In the preliminary analysis of the mathematical model derived in Chapter 2, we have already seen that in the reduced form relevant to calm sea conditions (when $B = 0$ and $V = 0$) the model equations (2.30) possess self-similar particular solutions. These solutions not only provide useful insights into the nonlinear behaviour but also represent the asymptotic forms to which the solutions of the initial-value problem tend as time elapses [Aronson, 1986, Barenblatt, 1996], consistent with self-similar solutions found in related viscous gravity current models [Grundy and Rottman, 1986, Huppert, 1982].

The solutions we have analysed so far correspond to isolated oil slicks without an external source. In this chapter, we extend the analysis to the case where

an external oil influx is present at x_0 , representing a continuous discharge through the boundary in time. We investigate how such an influx, characterised by a time-dependent boundary flux following a power-law behaviour, modifies the spreading dynamics and the self-similar structure of the flow. Specifically, we aim to demonstrate that even in the presence of an external source the system evolves asymptotically toward a self-similar regime. Our analysis is performed for semi-confined geometries, as is illustrated in panel (b) of Fig. 2.2, where oil is injected at the left boundary and spreads toward a free-moving front where the oil runs out. This setup builds on prior studies of viscous gravity currents with variable inflow, where the volume of fluid increases as a power law in time, leading to self-similar spreading behaviours [Grundy and Rottman, 1986, Huppert, 1982, Lister and Kerr, 1989], in particular by adapting their lubrication-theory approximations and self-similar exponents for gravity-viscous spreading (e.g., front position scaling as $x_r \propto t^\beta$) from unbounded or inclined-plane configurations to a thin-film oil-on-water model in semi-confined domains in this study with fixed boundaries and point like sources, while incorporating realistic oil parameters and geometric constraints to explore confinement effects. We use numerical methods to approximate the solutions, in order to plot how the initial data evolves into the self-similar behaviour over time. This sets the stage for Chapter 6, where we explore how surface waves influence the spreading of the oil layer in the presence of the external influx.

5.1 Problem formulation

We consider a one-dimensional semi-confined compact domain $x \in [x_l, x_r(t)]$, where the left boundary $x_l = 0$ represents a fixed oil source, and $x_r(t)$ represents the right boundary, corresponding to the moving front of the oil slick. This is illustrated in Fig. 5.1.

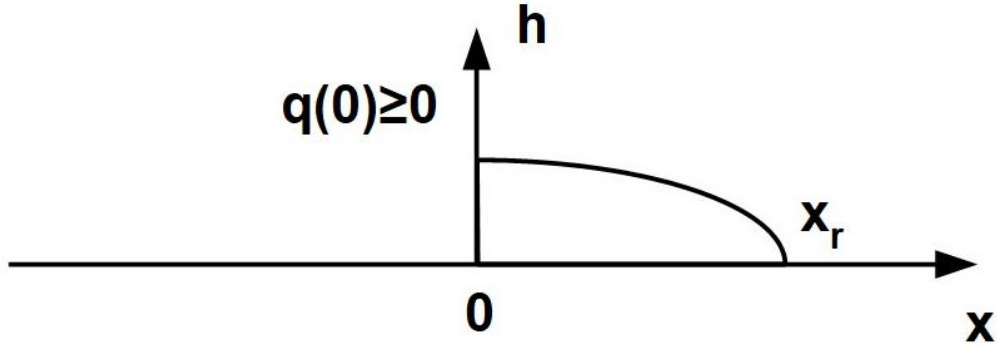


Figure 5.1: Illustration of the semi-confined geometry.

The governing equation, disregarding surface waves (2.30), is reduced to a nonlinear diffusion equation of the form (2.43) with boundary conditions (2.45).

That is

$$\frac{\partial h}{\partial t} = A_g \frac{\partial}{\partial x} \left(h^3 \frac{\partial h}{\partial x} \right)$$

with the boundary conditions

$$h(x_r(t), t) = 0, \quad (5.1)$$

and

$$-A_g h^3 \frac{\partial h}{\partial x} \Big|_{x=0} = q_0(t), \quad (5.2)$$

where $A_g = \alpha_g/3$, $q_0(t)$ prescribe the influx of oil at the fixed source boundary $x = 0$. It is assumed that the external feeding (influx of the oil) obeys the following power law over time, $q_0(t) = q_0 t^\gamma$, where q_0 is a constant measuring how strong the discharge is, and γ describes how the oil discharge changes with time. By continuity the outflow rate at the boundary is assumed to be the same as the flow rate immediately inside the boundary (otherwise, there is a discontinuous jump in its value from inside the boundary to its value on the boundary, not considered in this study).

We use this specific power-law form for the oil influx in our study, following the classical approach used in [Huppert, 1982, Lister, 1992]. This form is particularly important because it is the only one that maintains scaling invariance and keeps the governing equation (2.43) self-similar in time, which allows analytical treatment and direct comparison across different influx regimes.

In addition, it covers a wide range of physically relevant situations, for example, a constant influx rate when $\gamma = 0$, a gradually decaying discharge when $-1 < \gamma < 0$, and an increasing influx when $\gamma > 0$.

To study the self-similar behaviour of the solution, we examine the governing equation (2.43) together with the boundary conditions under scaling transformations $w = x t^{-\beta}$ and $\phi(w) = h t^\alpha$, as well as $q_0(t) = q_0 t^\gamma$, which reduce (2.43) to an ordinary differential equation together with the time-independent inflow boundary condition, provided that

$$\beta w \frac{d\phi}{dw} + \alpha \phi + A_g \frac{d}{dw} \left(\phi^m \frac{d\phi}{dw} \right) = 0. \quad (5.3)$$

where $A_g = \alpha_g/m$ is the diffusion coefficient with $m = 3$ in our case, together

with the boundary conditions

$$\phi(w)|_{w=\hat{w}} = 0 \quad \text{at} \quad (x(t) = x_r(t)). \quad (5.4)$$

and

$$\phi^3 \frac{d\phi}{dw} \Big|_{w=0} = -\frac{q_0}{A_g} \quad (x_l(0) = 0 \quad \text{at} \quad x(t) = 0). \quad (5.5)$$

This boundary condition at $w = 0$ is obtained directly by substituting the self-similar transformations into the original influx condition at the fixed boundary $x = 0$ given by equation (5.2).

In the case of (2.43) the scaling parameters satisfy the relations

$$\alpha = -\frac{2\gamma + 1}{5}, \quad \beta = \frac{4 + 3\gamma}{5}$$

derived from balancing terms in the original PDE and the boundary condition (see Chapter 2, Eq. (2.28)). These exponents depend only on the time dependence of the influx γ .

To confirm that the rate of flow at the free boundary is γ recall that in the untransformed problem $\hat{w}$ in the domain $w \in [0, \hat{w}]$ is found by means of the integral constraint

$$(\beta - \alpha) \int_0^{\hat{w}} \phi \, dw = (\gamma + 1) \int_0^{\hat{w}} \phi \, dw = q_0.$$

This relation derived in Section 2.5.3. In particular, equation (2.58) gives the balance between the rate of change of the total mass and the imposed influx, while equation (2.59) gives $\beta - \alpha = \gamma + 1$. Physically, by mass balance the

rate of change of $M(t)$ must be equal to the flux of oil entering the domain at $x = 0$. Differentiating with respect to time and using the boundary condition at the source (5.5), this gives

$$\frac{dM}{dt} = q_0(t) = q_0 t^\gamma.$$

Integrating in time leads to

$$M(t) = \frac{q_0}{\gamma + 1} t^{\gamma+1}.$$

Since the constant q_0 represents the strength of the oil source, we can redefine it to include the factor $1/(\gamma+1)$ for simplicity (i.e. $q_0 = \frac{q_0}{\gamma+1}$). This redefinition does not affect the scaling behaviour, and we can therefore write the total mass as

$$M(t) = q_0 t^{\gamma+1}.$$

Using the self-similar variables

$$M(t) = t^{\beta-\alpha} \int_0^{\hat{w}} \phi(w) dw.$$

Then, equating the two expressions for $M(t)$ and using the exponents $\beta - \alpha = \gamma + 1$ we get the integral constraint

$$(\gamma + 1) \int_0^{\hat{w}} \phi(w) dw = q_0. \quad (5.6)$$

Since $\phi(w) > 0$ throughout the interior of the domain, the integral $\int_0^{\hat{w}} \phi(w) dw$ is positive. Therefore, for the integral constraint to hold when $\gamma > -1$, it

is necessary that $q_0 > 0$. We immediately observe that under the assumption that the external flux represents an oil source within the domain, a self-similar solution exists only when $\gamma > -1$.

Unfortunately, the boundary value problem described by (5.3) together with boundary conditions (5.4)–(5.5) proves difficult to treat analytically, since $\alpha \neq \beta$ in general. For this reason, we turn to numerical solution of the problem to determine the self-similar profiles $\phi(w)$.

5.2 Scaling behaviour during the spreading dynamics in the presence of the source

The boundary of the domain $(0, x_r(t))$ in the self-similar regime follows

$$x(t) = wt^\beta$$

This relation comes directly from the definition of the similarity variable $w = xt^{-\beta}$. The rightmost point, where the oil layer ends, corresponds to $w = \hat{w}$, so the front position of the oil slick evolves as

$$x_r(t) = \hat{w}t^\beta.$$

Hence, the spreading distance of the oil slick grows as a power of time, and the exponent β determines the rate of expansion.

In the theoretical self-similar scaling, the characteristic exponents α and β are determined by the source exponent γ .

The relation between the spreading exponent β and the source exponent γ can be used as a diagnostic link. By observing how fast the oil front expands, which gives us an estimate of β we can infer the behaviour of the discharge rate, i.e., whether the source is constant, weakening, or strengthening over time.

Moreover, if we compare the parameter $\beta = \frac{4+3\gamma}{5}$ with its value in the absence of the source $\beta_0 = \frac{1}{5}$, we can see that for any $\gamma > -1$ it follows that $\beta > \beta_0$. This result is intuitively expected since the spreading rate should increase in the presence of an external source. Indeed, integrating (5.3) and using the boundary conditions (5.4) and (5.5), we obtain that at any $w \leq \hat{w}$ and $\gamma > -1$

$$A\phi^3 \frac{d\phi}{dw} = -\beta w\phi - (\gamma + 1) \int_w^{\hat{w}} \phi dz < 0.$$

This relation indicates that the profiles $\phi(w)$ must in general be monotonically decreasing. Therefore, the time evolution of the profiles $h(x, t)$ satisfies $\max(h) \propto t^{-\alpha}$, consistent with the behaviour of an isolated slick case. The sign of α determines the layer's thickness or thinness in time.

As expected, oil layer thickness $h(x, t)$ increases in time at $\gamma > -0.5$ as $\alpha < 0$, and decreasing in time at $-1 < \gamma < -0.5$ as $\alpha > 0$. This indicates that the oil layer grows thicker over time, which includes all realistic cases of constant or increasing oil influx $\gamma \geq 0$.

Evolution of initial profiles

We now examine how initial profiles evolve in time under different influx regimes of the exponent γ , beginning with the representative case of constant

influx for $\gamma = 0$, that is when the oil influx is constant over time $q_0(t) = q_0$. The exponents are, for example $\alpha = -0.2, \beta = 0.8$, which means that the oil thickness gradually thickens as $h \propto t^{0.2}$, while the front position advances as $x_r \propto t^{0.8}$.

Hence the spreading rate of the oil slick is accelerated compared to the isolated case where $\beta_0 = 1/5 = 0.2$ [Huppert, 1982].

For positive $\gamma > 0$ (e.g., $\gamma = 1, 2$) the oil influx grows as $q_0(t) = q_0 t^\gamma$. We therefore get

$$\alpha = -0.6, \quad \beta = 1.4.$$

$$\alpha = -1, \quad \beta = 2.$$

for $\gamma = 1$ and $\gamma = 2$ respectively. Spreading becomes dramatically faster in these cases, with the total mass scaling as $t^{\gamma+1}$ driving rapid front advance. We also observe a critical case when $\gamma = -1$, where the influx decays as $q_0(t) = q_0 t^{-1}$. Here, the exponents are $\alpha = \beta = \beta_0 = 0.2$ which matches the exponents found in the isolated case β_0 , and as a result, the problem has an analytical self-similar solution (as derived in Chapter 2). Therefore, the dynamics are expected to be similar to the isolated case, though not identical, since the boundary conditions (2.60) and (5.5) are different. The initial influx accelerates spreading, but as $q(t) \rightarrow 0$, the solution approaches the no-source self-similarity. Numerical results for these evolutions are presented in the next section.

5.3 Numerical results

In this section, we investigate the dynamics of oil spreading over a calm sea surface in the presence of an external discharge, using our PDE model (2.30) derived in Chapter 2. Our goal is to observe whether the system evolves towards the self-similar behaviour predicted by the reduced ODE formulation for various values of the source exponent γ .

We proceed in two main steps. First, we solve the reduced ODE (2.49) numerically for each γ , obtaining the self-similar profile $\phi(w)$. Then, we solve (2.30) using the moving mesh finite difference method introduced in Chapter 3. Specifically, we employ the mass balance method described in Section 3.4, which ensures accurate conservation of the total mass of the oil throughout the simulation. The numerical scheme is first-order accurate in time and second-order accurate in space, and the estimated errors converge with second-order accuracy. In our simulation, we consider a particular set of dimensional and non-dimensional parameters corresponding to the initial phases of the oil spill dynamics. We take the thickness of the oil layer in between $1 \text{ mm} \leq H \leq 10 \text{ mm}$ and the non-dimensional parameters $\alpha_g = 2 \cdot 10^{-2}$ and $\alpha_q = 2.5 \cdot 10^{-3}$. The initial domain is chosen to be $[0, 1]$ at the initial time $t_0 = 1$ with the total initial amount of the oil $I_0 = 1$ to obtain initial layer thickness of the order of one. The simulations are conducted with $N = 40$ discretized nodes with the time step of $\Delta t = 1 \times 10^{-5}$. We notice that the first cell can become quite stretched during the simulation, especially when there is a positive inflow at $x = 0$. To ensure accuracy near the inlet, we add a new mesh point whenever the first interval becomes more

than twice as long as the second one. When this happens, we split the first cell into two smaller ones and share the mass between them to keep the total mass the same. The number of mesh points increases automatically as new ones are added, but only near the source. This keeps the mesh size finite near the source, where the largest changes occur, while the rest of the domain remains unchanged. After splitting, the new second cell is treated as a standard interior cell whose local mass is subsequently kept constant over time by the moving-mesh algorithm as described in Section 3.4.

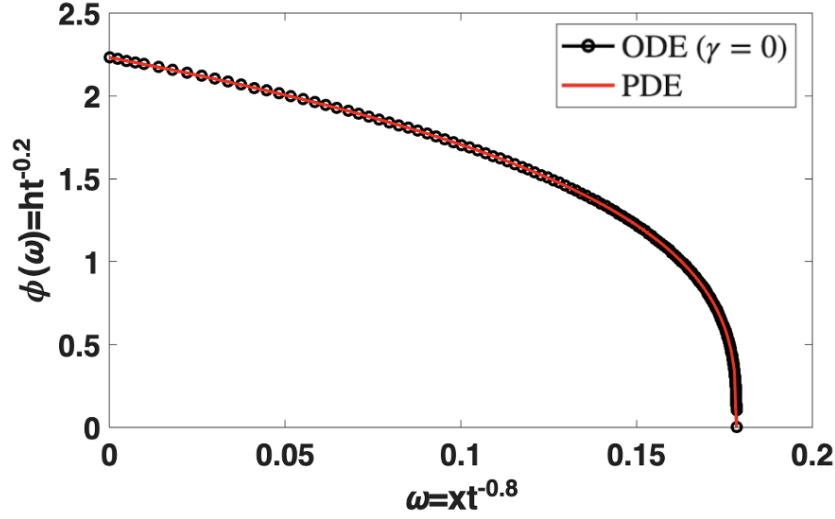
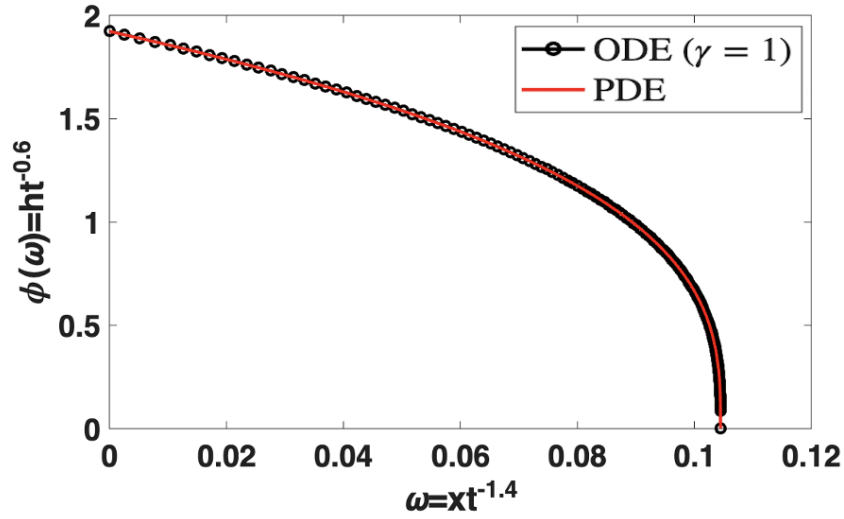
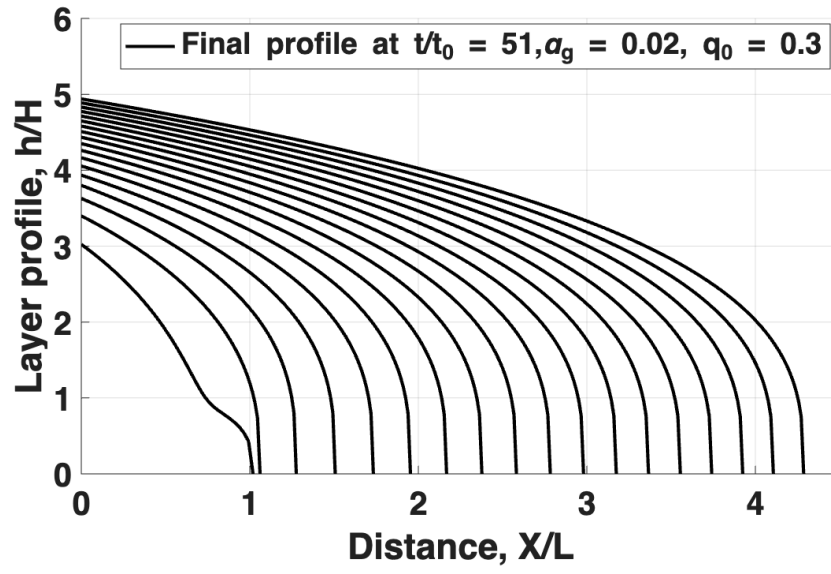
(a) Constant influx $\gamma = 0$ (b) Linear influx $\gamma = 1$

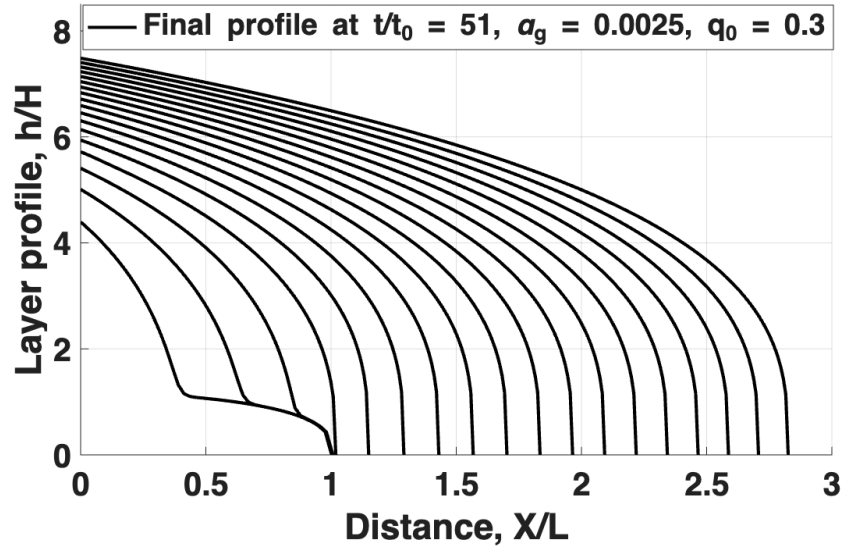
Figure 5.2: A comparison between self-similar solutions of the ODE (5.3) (circles) and PDE (2.30) solutions (solid line), both expressed in terms of self-similar variables ϕ and ω in the presence of external influx $q_0(t) = q_0 t^\gamma$ at different values of γ , and at $q_0 = 0.3$ and $t/t_0 = 51$. Parameter $\alpha_g = 2 \cdot 10^{-2}$.

To validate the numerical method, we check with the analytic self-similar

solution as a benchmark. We compare the self-similar ODE solution (5.3) with the scaled numerical profiles obtained from the full PDE (2.30) for two representative cases of the source exponent $\gamma = 0$ and $\gamma = 1$. The comparison is made in the self-similar variables $w = x t^{-\beta}$ and $\phi = h t^\alpha$. As shown in Figure 5.2, an excellent agreement between the self-similar ODE solution and the scaled PDE profiles is observed for both cases $\gamma = 0$ and $\gamma = 1$, with the scaled profiles collapsing onto a single curve, confirming that the spreading dynamics have reached the self-similar regime at $t/t_0 = 51$.

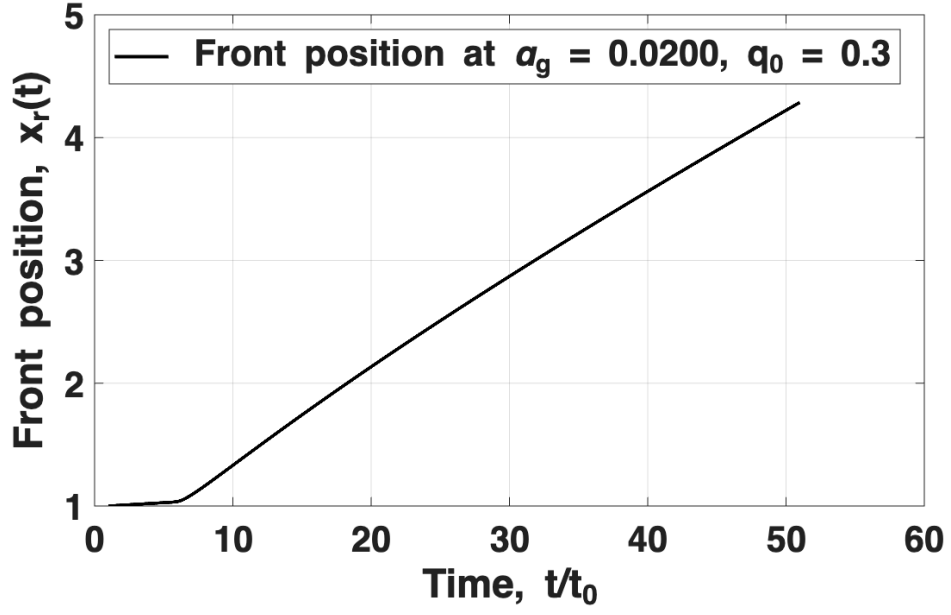


(a)

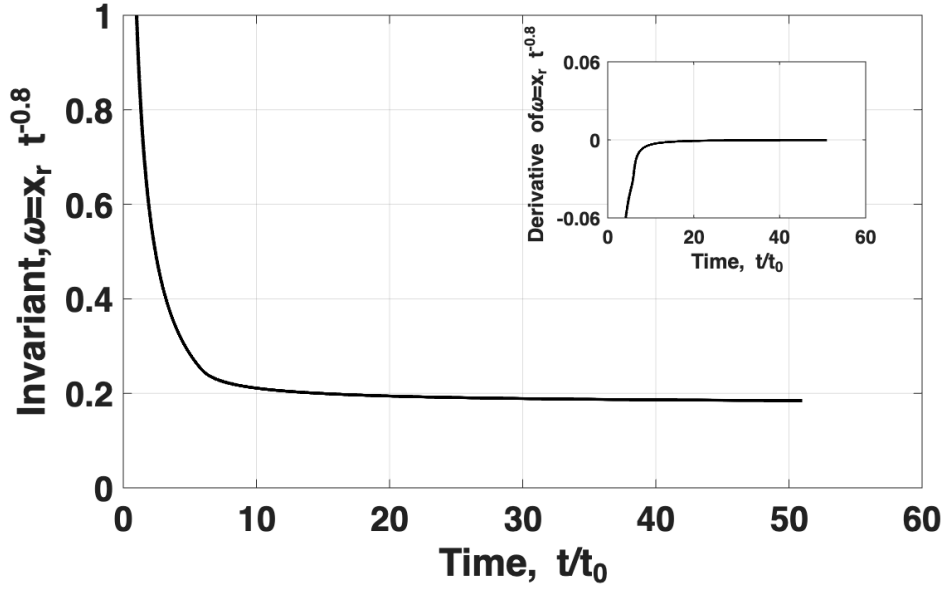


(b)

Figure 5.3: Evolution of the oil layer dynamics in the presence of an external source $q(t) = q_0 t^\gamma$ with $\gamma = 0$ and $q_0 = 0.3$ at $t/t_0 = 51$ for $\alpha_g = 2 \times 10^{-2}$, $I_0 = 1$, and $Fr^2 = 10$, and for $\alpha_g = 2.5 \times 10^{-3}$ and $Fr^2 = 20$.

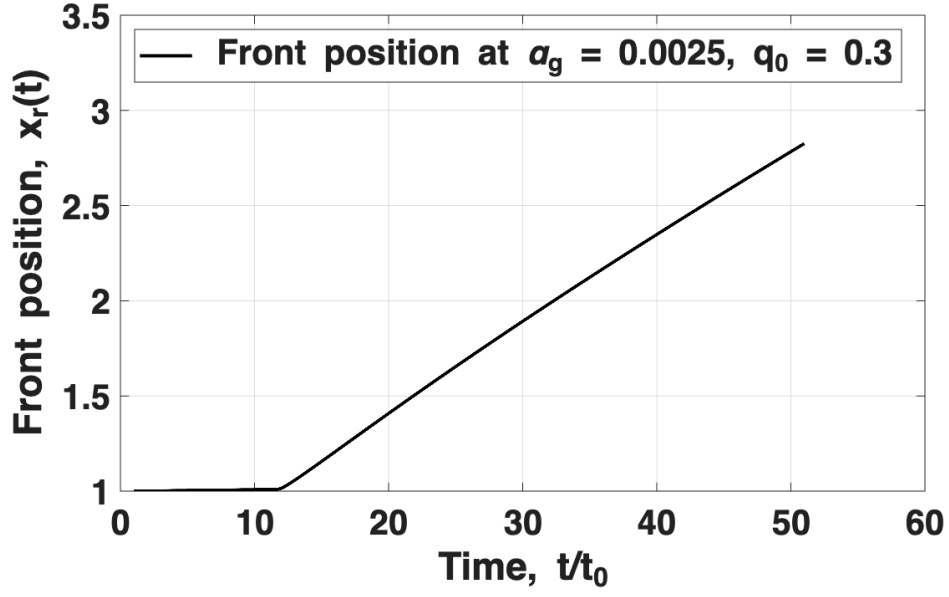


(a)

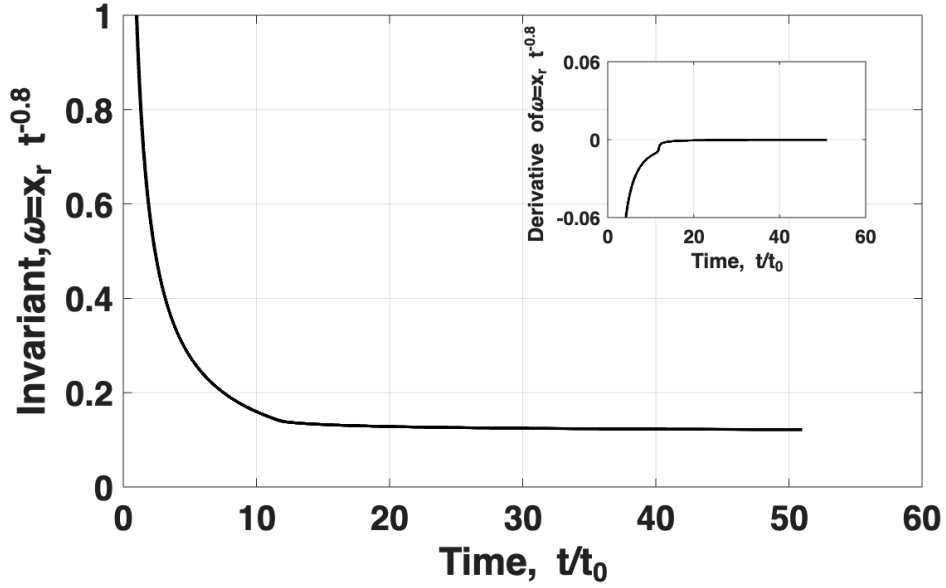


(b)

Figure 5.4: Evolution of the moving front $x_r t^{-0.8}$ for $\alpha_g = 2 \times 10^{-2}$, $I_0 = 1$, $Fr^2 = 10$, and $q_0 = 0.3$. The inset shows the time derivative of the front position presented in self-similar variables.



(a)



(b)

Figure 5.5: Evolution of the moving front $x_r t^{-0.8}$ for $\alpha_g = 2.5 \times 10^{-3}$, $I_0 = 1$, and $Fr^2 = 20$. The inset shows the time derivative of the front position presented in self-similar variables.

Figures 5.3, 5.4 and 5.5 illustrate the time evolution of the oil layer thickness $h(x, t)$ and the motion of the advancing front $x_r(t)$ for $\alpha_g = 2 \times 10^{-2}$ and

$\alpha_g = 2.5 \times 10^{-3}$ under a constant influx, corresponding only to $\gamma = 0$. We set the influx to a sufficiently large value, $q_0 = 0.3$, to observe the transient behaviour, ensuring that during the simulation time interval the total variation of the oil amount substantially exceeds its initial value [Hozan et al., 2025]. We observe that as the simulation progresses, the continuous addition of oil at the fixed boundary $x = 0$ leads to an accumulation of mass near the source, resulting in an increase in the layer thickness in this region. Simultaneously, the oil spreads outward under the influence of the diffusion, driving the front $x_r(t)$ to advance progressively. The profiles maintain a monotonic decrease from the peak height near the inlet to zero at the front, in agreement with the self-similar solutions from the reduced ODE (5.3).

As shown in Figures 5.4a and 5.5a, we observe that there is an initial slow propagation mode, similar to the no-source case ($x_r \propto t^{0.2}$) during which the external flux has no influence on the moving front. This is followed by a short transition period around $t/t_0 \approx 5 - 10$, which marks the turning point where the influx begins to dominate, accelerating the front to the self-similar scaling $x_r \propto t^{0.8}$. For $\alpha_g = 2 \times 10^{-2}$ the transition occurs earlier around $t/t_0 \approx 5$ with faster overall spreading and the front extending by $t/t_0 = 51$. In contrast, for $\alpha_g = 2.5 \times 10^{-3}$, the transition is delayed to $t/t_0 \approx 10$. In Figures 5.4b and 5.5b, one can observe the evolution of the invariant combination $w = x_r t^{-0.8}$ in terms of self-similar variables $w = x t^{-\beta}$, which approaches a constant value in the long-time limit, confirming the realisation of self-similar behaviour. This convergence is further verified by the time derivative of the invariant (shown in the insets), which steadily decays to zero after the transient phase. These observations confirm the theoretical

self-similar scaling derived from the reduced ODE formulation, showing that the influx driven dynamics dominate after the transient phase and lead to the predicted asymptotic regime.

We also check the dependence of the spreading exponent on the source strength q_0 . For this test, we keep $\gamma = 0$, for which the theoretical value is

$$\beta = \frac{4 + 3\gamma}{5} = 0.8.$$

For each value of q_0 , the front position is fitted in the late-time region by

$$x_r(t) = Ct^{\beta_{\text{numerical}}}.$$

Equivalently,

$$\log x_r = \log C + \beta_{\text{numerical}} \log t,$$

so that $\beta_{\text{numerical}}$ is obtained from the slope of the log-log plot. The fit is performed over the interval $35 \leq t/t_0 \leq 51$, after the main transient behaviour has decreased.

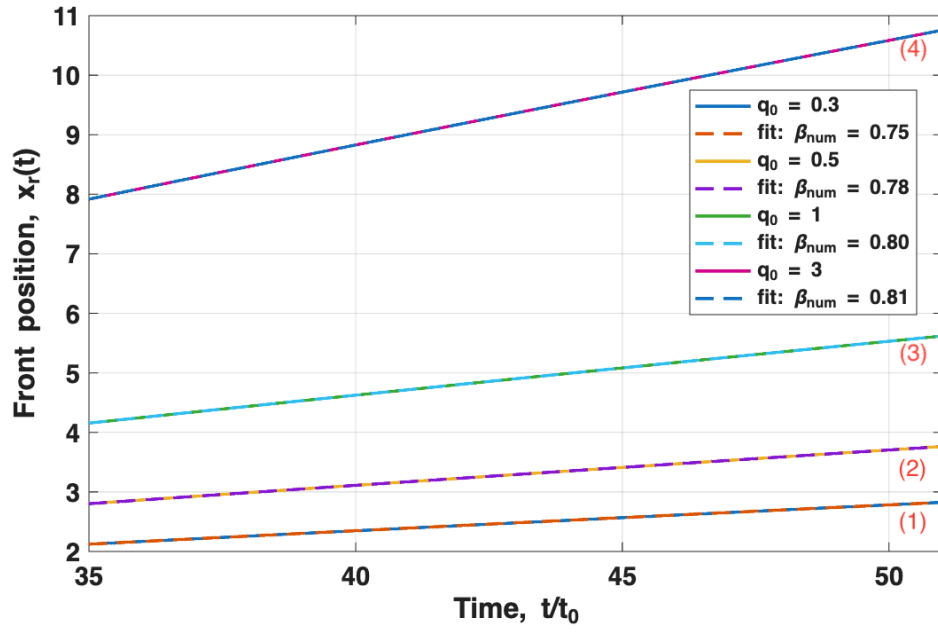


Figure 5.6: Late-time fitting of the front position for different values of q_0 at $\gamma = 0$ for $\alpha_g = 2.5 \times 10^{-3}$, $I_0 = 1$, and $Fr^2 = 20$. The dashed curves show the fitted power law $x_r(t) = Ct^{\beta_{\text{numerical}}}$.

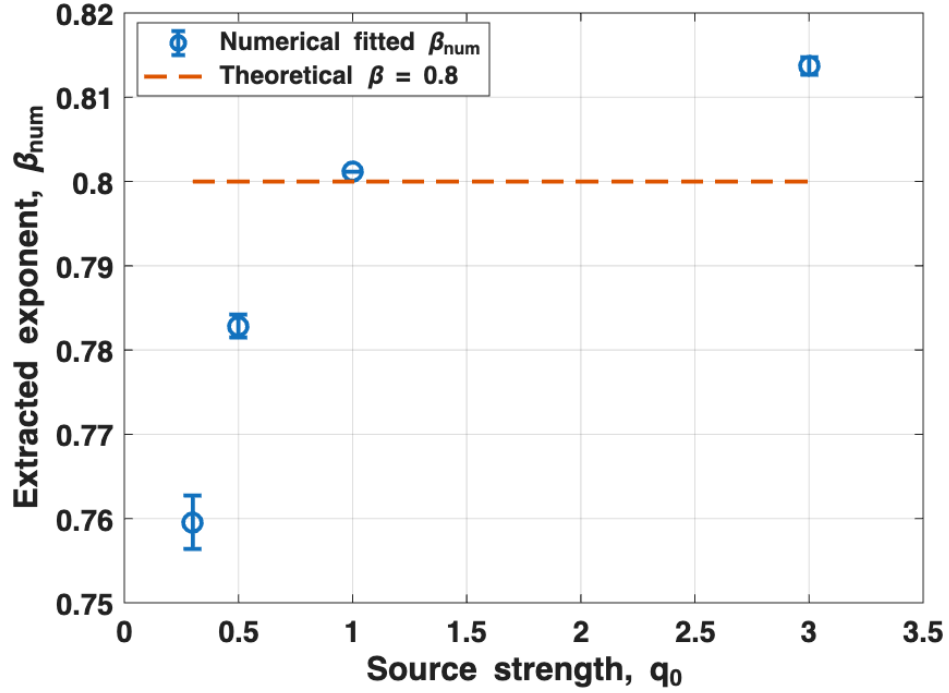


Figure 5.7: Extracted exponent $\beta_{\text{numerical}}$ as a function of q_0 for $\gamma = 0$, with $\alpha_g = 2.5 \times 10^{-3}$, $I_0 = 1$, and $Fr^2 = 20$. The symbols show the fitted numerical values, and the error bars indicate the maximal fitting error. The dashed line is the theoretical value $\beta = 0.8$.

In Figures 5.6 and 5.7, the extracted values of $\beta_{\text{numerical}}$ are compared with the theoretical prediction $\beta = 0.8$ for $\gamma = 0$. For the tested range $q_0 \in [0.3, 3]$, the fitted values remain close to the theoretical prediction, although some finite-time variation with q_0 is observed.

The numerical results therefore indicate consistency with the theoretical prediction that the asymptotic exponent is controlled by γ , while q_0 changes only the constant C in $x_r(t) = Ct^\beta$. No clear breakdown of the scaling behaviour is observed for the tested values of q_0 .

5.4 Summary

In this chapter, we investigated the dynamics of oil spreading over a calm sea surface in the presence of a power-law external oil influx $q(t) = q_0 t^\gamma$ in a semi-confined geometry with a fixed source boundary and a free-moving front. We demonstrated that, despite the presence of a time-dependent source, the system still evolves toward a self-similar regime. This was shown analytically through the derivation of the scaling exponents and numerically by solving both the full PDE (2.30) and the corresponding reduced ODE (5.3).

Our results show that the spreading front follows a power-law relation, $x_r(t) \propto t^\beta$, where the theoretical self-similar scaling predicts that the exponent β is controlled by the influx exponent γ , while the source strength q_0 affects the prefactor of the solution. The numerical simulations are consistent with this behaviour, showing a clear transition from a diffusion-dominated regime to an influx-driven phase, where the dynamics reaches a self-similar regime.

This transition happened earlier for larger value α_g , indicating that stronger gravity–viscous effects accelerate the evolution. Moreover, the agreement that we observe between the scaled profiles obtained from the full PDE and those computed from the reduced ODE confirms the validity of the self-similar formulation. As the invariant quantity $w = x_r t^{-\beta}$ stabilizes and its time derivative approaches zero, this provides strong evidence that the system evolves toward the predicted asymptotic regime.

The analytical and numerical approaches that we have developed here form the baseline for the next stage of this study in Chapter 6, where our focus will shift to understanding how surface wave motion alters the behaviour of

the spreading layer in the presence of external discharge and to explore its interplay with influx and potential confinement effects.

Chapter 6

The dynamics of oil spills in semi-confined geometries in the presence of water-wave perturbations.

In this chapter, we investigate a more realistic setting where a continuous oil influx interacts with surface waves in a semi-confined geometry.

This situation is common in coastal environments, harbours, or containment regions where the presence of fixed boundaries modifies the oil layer response to wave motion and may lead to localized or confined spreading [Alves et al., 2015, Shi et al., 2020].

This study extends the previous chapters, in which we studied the effect of wave motion on oil slicks in the absence of external influx (Chapter 4) and the dynamics of oil spreading under a continuous influx in calm water conditions

(Chapter 5). These two setups capture important aspects of oil slick evolution but do not show the full complexity of real-world spill scenarios.

6.1 Problem formulation

We now formulate the problem using the governing equation (2.30) and boundary conditions (2.45) introduced earlier in Chapter 2. The one - dimensional setup is similar to the one used in the previous Chapter 5 and shown in Fig. 2.2 from Chapter 2, such that the source of oil is at the restricting boundary, at $x = 0$, and the flow domain extends in the positive x direction. It is illustrated in Fig. 6.1 as well for convenience.

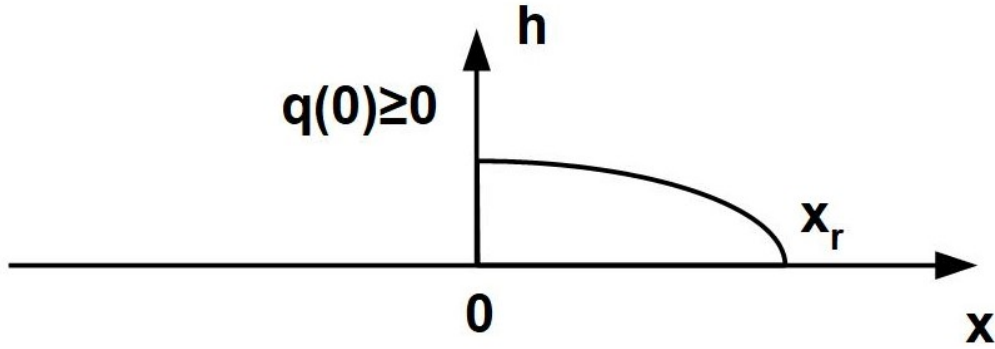


Figure 6.1: Illustration of the semi-confined geometry.

The domain is defined as $x \in [0, x_r(t)]$, where $x = 0$ is a fixed boundary with a prescribed oil influx $q(0, t)$, expressed by the boundary condition

$$q(0, t) = q_0(t), \quad \text{with} \quad q_0(t) = q_0 t^\gamma,$$

where Q_0 is a constant oil influx rate, and γ controls the time dependence of

the influx and $x = x_r(t)$ is a free boundary that evolves in time according to the front dynamics.

This setup allows us to examine how the combined effects of continuous influx, confinement, and wave-induced motion determine the evolution of the oil slick front.

Specifically, we examine how the direction and parameters of wave motion affect the rate and extent of spreading and whether confinement effects occur under certain conditions.

We work with the same single harmonic wave mode, as it was introduced earlier, but here we consider it travelling in either positive or negative direction as the oil front moves with or against it, with phase velocity

$$V_\Phi = \pm \frac{\omega}{k}$$

$$B(x, t) = B_0 \sin(kx \pm \omega t)$$

$$V(x, t) = B_0 \varepsilon \omega \sin(kx \pm \omega t).$$

Here, B_0 is the wave amplitude and k is the wave number. This setup allows us to isolate and compare two main scenarios: one where the wave moves in the same direction as the oil flow (co-propagating, $V_\phi > 0$), and another where it moves toward the fixed boundary at $x = 0$ (counter-propagating, $V_\phi < 0$).

By varying both the wave direction and wave parameters B_0 and k we aim to understand how these factors affect the spreading behaviour and whether

wave motion can slow down or even confine the movement of the front of the oil slick.

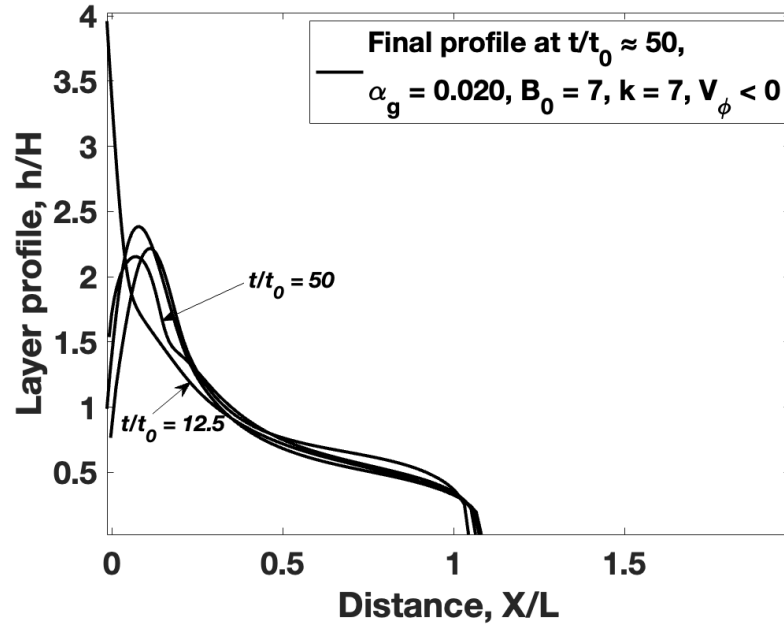
6.2 Numerical analysis

In our implementation, we use the finite difference method and the mass balance-based moving mesh method described in Chapter 3.

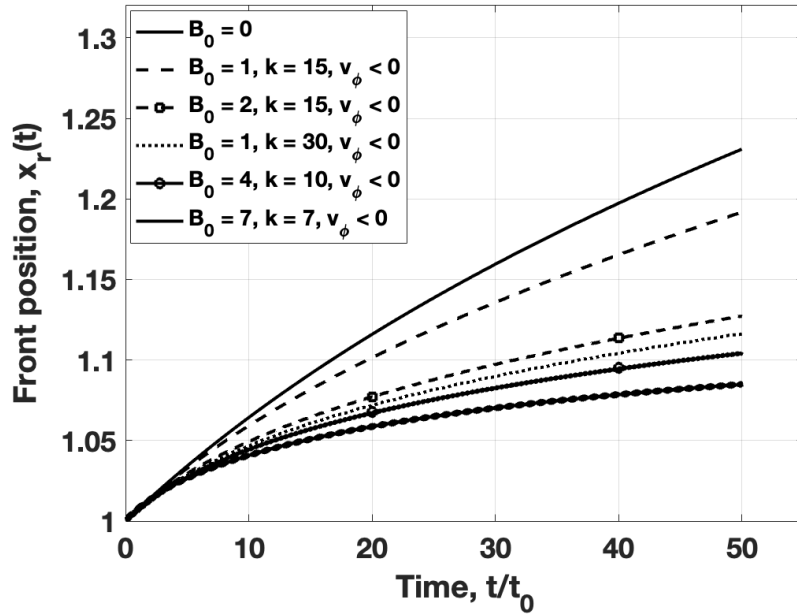
The time integration is first-order accurate, and the spatial discretisation is second-order accurate.

The method ensures that we correctly track the incoming external influx at $x = 0$.

We start the simulations with a self-similar oil profile that has a compact shape defined over the range $x \in [0, 1]$, and we use $N = 100$ mesh points at the beginning. The time step Δt was chosen to ensure numerical stability throughout all simulations, as required for explicit time integration methods.



(a)



(b)

Figure 6.2: Illustration of oil layer dynamics under wave motion for $V_\phi < 0$ in a semi-confined geometry with $q_0 = 0$, $I_0 = 1$, $\alpha_g = 0.02$, and $Fr^2 = 10$. (a) shows the evolution of oil layer profiles for $B_0 = 7$, $k = 7$, and $\omega \approx 60$. (b) shows the evolution of the moving front $x_r(t)$ over time for different values of B_0 , k , and ω .

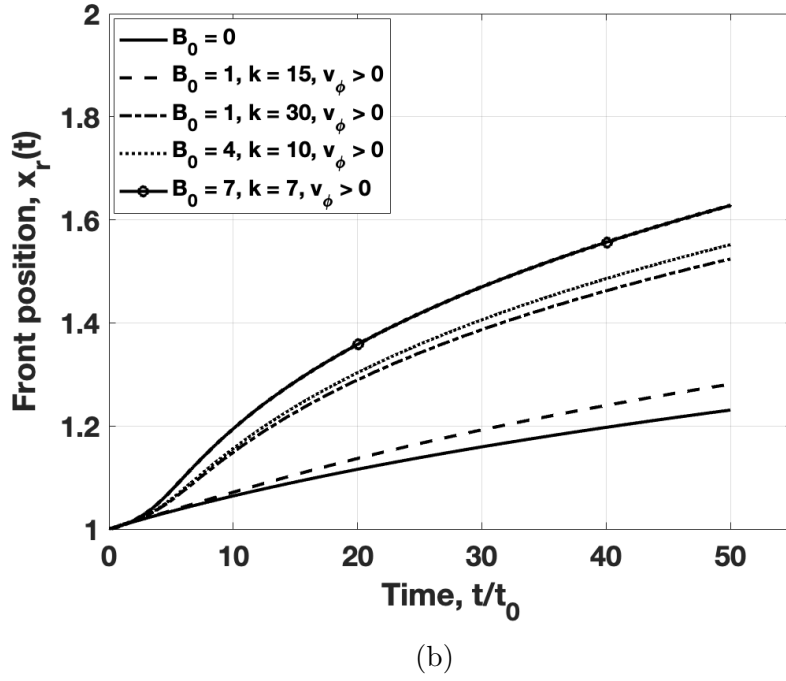
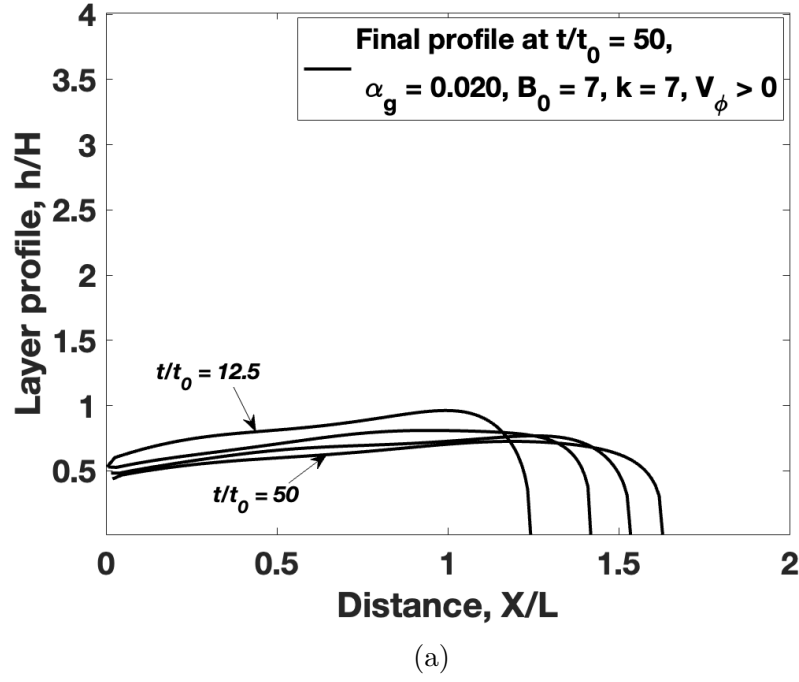


Figure 6.3: Illustration of oil layer dynamics under wave motion for $V_\phi > 0$ in a semi-confined geometry with $q_0 = 0$, $I_0 = 1$, $\alpha_g = 0.02$, and $Fr^2 = 10$. (a) shows the evolution of oil layer profiles for $B_0 = 7$, $k = 7$, and $\omega \approx 60$. (b) shows the evolution of the moving front $x_r(t)$ over time for different values of B_0 , k , and ω .

First, we examine the scenario without an external source $q_0 = 0$, that is when there is no external influx. The undisturbed evolution of the dynamics is determined by the initial amount of oil and the diffusion parameter α_g .

We set $\alpha_g = 0.02$ to capture the effect of diffusion, which is most important during the early stages of the spreading when the typical oil layer thickness in dimensional units is $H \approx 10$ mm.

In Figures 6.2 and 6.3, we illustrate the evolution of the oil layer $h(x, t)$ and the moving front $x_r(t)$ in $V_\phi < 0$ and $V_\phi > 0$. As we can see in Figure 6.2 (a)–(b), even under conditions where the diffusion counteractive process is strong and the respective Péclet number is around $Pe_E \approx 1$, the advance of the domain boundary can be completely inhibited by the oncoming wave ($V_\phi < 0$) at relatively low amplitudes B_0 .

In dimensional units, this corresponds to water waves of about 10 cm amplitude. Compared with the no-wave case ($B_0 = 0$) shown in Figure 6.2(b), the oncoming-wave cases have a much smaller slope in the front-position curves. This indicates that the front velocity is strongly reduced and approaches a nearly stationary state over the simulated time interval. Thus, even though the oil tends to spread outward due to diffusion, the movement of the water surface caused by the wave pushes the oil back toward the source.

The interaction between the opposing advection and diffusion produces a nearly stationary front, confirming that even moderate amplitude waves can effectively suppress the advance of the oil layer in semi-confined domains.

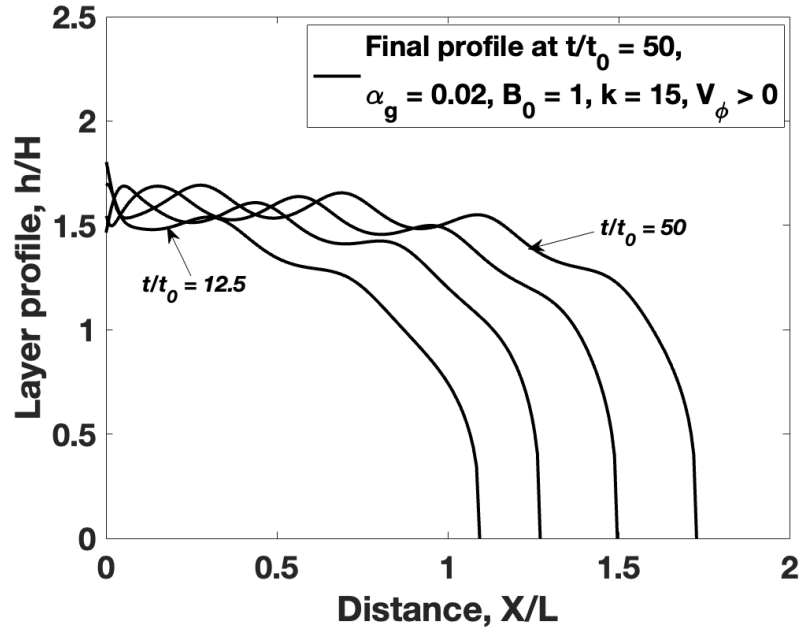
In contrast, when the waves move in the same direction as the slick ($V_\phi > 0$), the spreading becomes faster rather than slower. As shown in Figure 6.3(b),

the front position curves go above the no-wave case. Larger values B_0k cause more spreading; for example, $x_r(t)$ reaches around 1.5 or more by $t/t_0 = 50$ when $B_0 = 7$ and $k = 7$. This is because the wave-induced drift moves in the same direction as the oil slick, which strengthens the spreading and allows the slick to thin and cover a larger area.

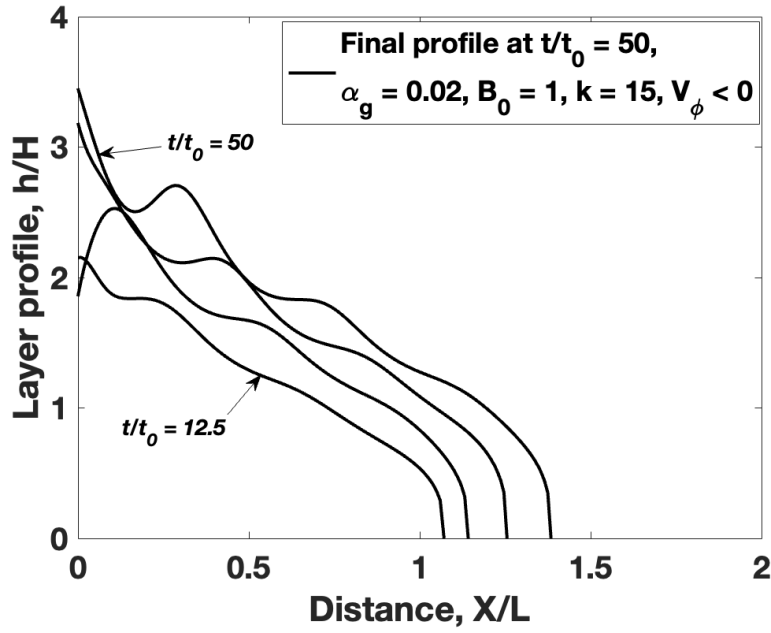
We observe a similar behaviour here as we found in the isolated slick case discussed in Chapter 4 [Lukyanov et al., 2024]. In that case, the results suggest that the front response in the diffusion-advection regime, $\text{Pe}_E \approx 1$, is mainly controlled by the combined parameter $(B_0k)^{1/2}$, rather than by B_0 or k separately.

We also observe this behaviour in the semi-confined geometry, Figure 6.2, where the change in the final front position is small when comparing parameter pairs such as $(B_0, k) = (1, 30)$ and $(2, 15)$, which both yield approximately the same product $B_0k \approx 30$, and hence the same value of $(B_0k)^{1/2}$. If we keep B_0k constant, we get similar front behaviour, even though the amplitude and wavelength are different. This suggests that $(B_0k)^{1/2}$ provides a useful numerical scaling parameter for comparing the effect of different wave amplitudes and wave numbers.

Therefore, waves with shorter wavelengths (larger k) and fixed B_0 tend to produce a stronger effect on the front motion [Hozan et al., 2025].



(a)



(b)

Figure 6.4: Evolution of the oil layer profiles in the presence of an external source with $\gamma = 0$, $q(t) = q_0$, and $q_0 = 0.03$, under wave motion in both positive and negative directions, $V_\phi > 0$ and $V_\phi < 0$, for $I_0 = 1$, $\alpha_g = 0.02$, and $Fr^2 = 10$. The wave parameters are $B_0 = 1$, $k = 15$, and $\omega \approx 87$. The initial profile is the self-similar solution (2.46).

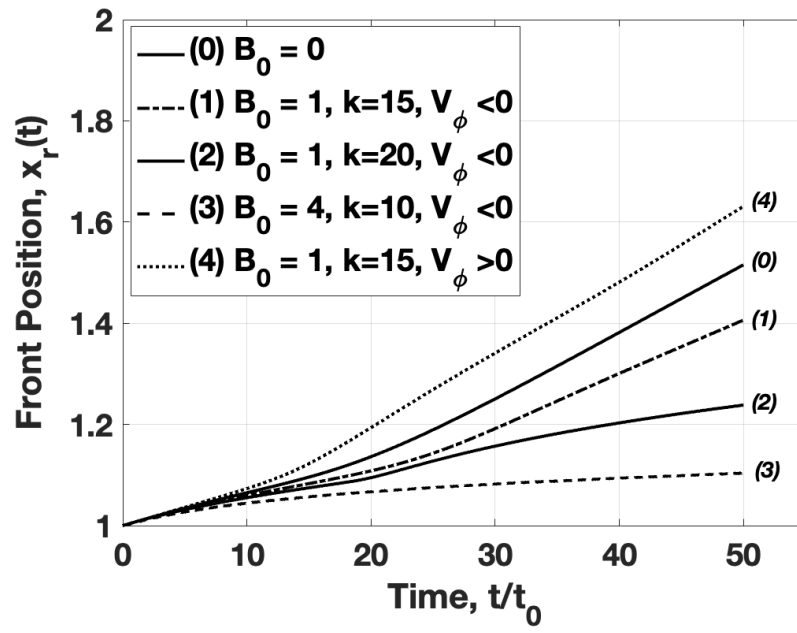
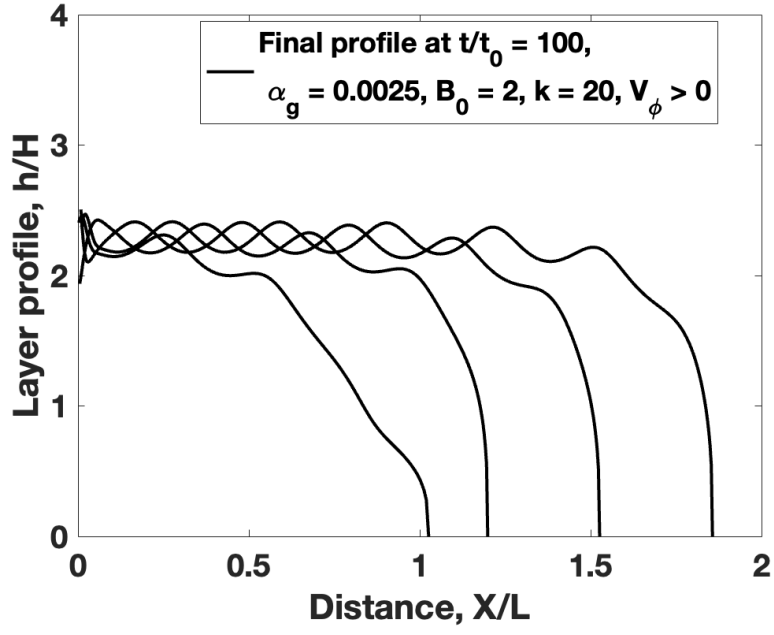
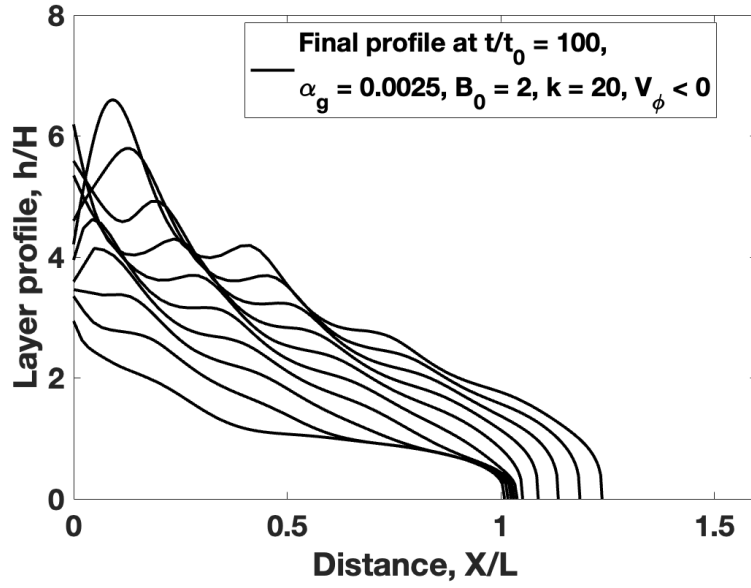


Figure 6.5: Evolution of the front position $x_r(t)$ in the presence of an external source with $\gamma = 0$ and $q_0 = 0.03$, under wave motion in the positive and negative directions, for $I_0 = 1$, $\alpha_g = 0.02$, and $Fr^2 = 10$, using different values of B_0 , ω , and k . The initial profile is the self-similar solution (2.46).



(a)



(b)

Figure 6.6: Evolution of the oil layer profiles in the presence of an external source with $\gamma = 0$, $q(t) = q_0$, and $q_0 = 0.03$, under wave motion in both positive and negative directions, $V_\phi > 0$ and $V_\phi < 0$, for $I_0 = 1$, $\alpha_g = 0.0025$, and $Fr^2 = 20$. The wave parameters are $B_0 = 2$, $k = 20$, and $\omega = 100$. The initial profile is the self-similar solution (2.46).

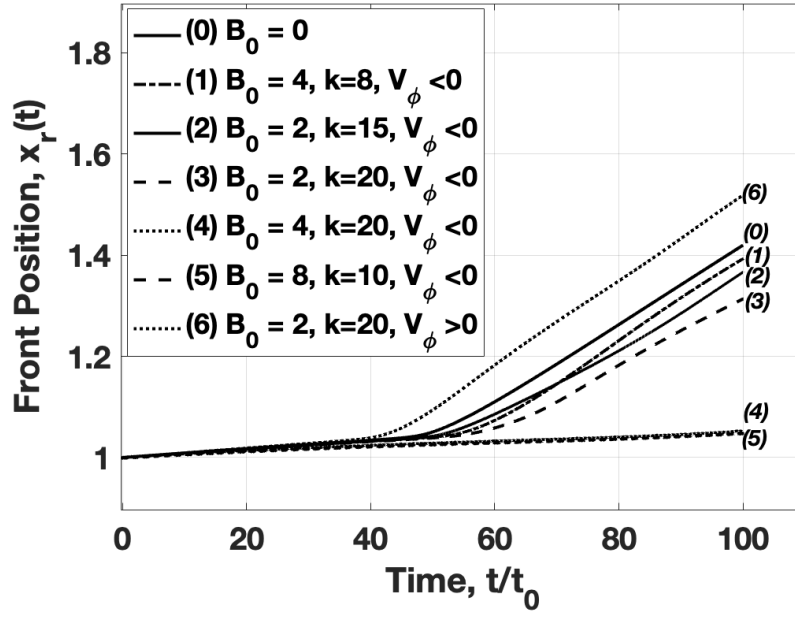


Figure 6.7: Evolution of the front position $x_r(t)$ in the presence of an external source with $\gamma = 0$ and $q_0 = 0.03$, under wave motion in both positive and negative directions, $V_\phi > 0$ and $V_\phi < 0$, for $I_0 = 1$, $\alpha_g = 0.0025$, and $Fr^2 = 20$. The initial profile is the self-similar solution (2.46).

We now consider the case where an external oil source is present while wave motion is acting, as shown in Figures 6.4–6.7. In both cases, the external source is constant, $q(t) = q_0 = 0.03$, so that the total amount of oil entering the domain during the simulation becomes comparable to, or larger than, the initial amount $I_0 = 1$.

The wave parameters were chosen such that the product $B_0 k \approx 40$ which represents a similar of wave parameters to that we used in the case without a source $q_0 = 0$. This allows us to make a direct comparison between the two scenarios with and without external influx.

As seen in Figures 6.4 (a) and (b), the profile evolution $h(x, t)$ at $B_0 = 1$ and $k = 15$ for both wave directions is shown for $\alpha_g = 0.02$, corresponding

to an oil thickness of $H \approx 10$ mm and $Pe_E \approx 1$. The small scale oscillations visible in the oil layer profiles are induced by the imposed travelling surface wave. These oscillations arise due to the periodic advection caused by the wave-induced velocity field $V(x, t)$. Since the imposed wave has wave number k , its wavelength is

$$\lambda = \frac{2\pi}{k}.$$

Therefore, the oscillations in the oil profile have a characteristic spatial scale comparable to $2\pi/k$. This is consistent with the fact that they are forced by the imposed wave motion, while their amplitude and exact shape are modified by oil layer diffusion and the moving front dynamics.

Under co-propagating wave conditions where $V_\phi > 0$, where the waves travel in the same direction as the oil front, the profiles show a continuous forward movement over time. Compared with the no wave curve $B_0 = 0$ in Figures 6.5 and 6.7, the front position grows faster, showing that co-propagating waves facilitate the spreading of the oil layer.

In contrast, under oncoming wave conditions where $V_\phi < 0$, the profile becomes compressed near the source boundary $x = 0$, indicating accumulation and suggesting a temporary trapping or confinement effect due to wave-induced backflow. We also observe consistent behaviour in Figures 6.6(a) and (b) for both wave directions, with $B_0 = 2$ and $k = 20$. Here, the smaller value $\alpha_g = 0.0025$ is chosen to represent a thinner oil layer, $H \approx 5$ mm, rather than being a result obtained from the simulation. This parameter choice weakens the gravity-viscous spreading and increases the relative importance of wave-induced advection, giving $Pe_E \approx 8 \gg 1$.

Under oncoming waves, the slick accumulates near the inlet initially at an early time but gradually spreads over time, while co-propagating waves, we notice the profile moves forward and becomes more extended, indicating efficient redistribution of incoming mass.

We now examine the evolution of the front position $x_r(t)$ for different values of wave amplitude and wave number in Figure 6.5 for $\alpha_g = 0.02$, as well as in Figure 6.7 for $\alpha_g = 0.0025$. In the absence of the waves $B_0 = 0$, the front starts moving forward following the self-similar scaling as $x_r \propto t^{0.8}$ after a short initial period (see Chapter 5). This tells us that the front position is mainly controlled by the influx rate q_0 for long times, not by diffusion α_g .

We can see this more clearly by comparing the no-wave results at the two different diffusion values. Even though α_g is quite different between the two cases, the front moves in almost the same way. Co-propagating waves $V_\phi > 0$ accelerate the advance, particularly at higher wave numbers. Conversely, oncoming waves $V_\phi < 0$ initially suppress the front motion and produce a quasi-steady phase.

This happens because the incoming waves push the oil back toward the source, causing accumulation near the inlet. But after enough oil builds up, the front eventually starts to move again. This delay is more noticeable at lower α_g .

For example, in the low-diffusion case with $\alpha_g = 0.0025$, the delay is more pronounced, and front advance resumes only after sufficient oil mass has built up. This behaviour is similar to the no-influx case we discussed earlier, but here the continuous influx guarantees that the front will eventually move forward.

6.3 Summary

In this chapter, we examined the dynamics of oil spreading in a semi-confined domain when both wave motion and a continuous external influx are present. Our numerical results show that wave motion can significantly affect the front dynamics of the oil slick, either enhancing or suppressing the spreading depending on the wave direction. In particular, oncoming waves can produce a quasi-steady state, where the front is temporarily halted, while co-propagating waves accelerate the advance.

These effects are observed across a range of parameters, and the simulations suggest that the combined parameter $(B_0 k)^{1/2}$ provides a useful numerical scaling for comparing the effect of wave amplitude and wave number on the front motion.

Our numerical method allowed us to resolve fine features of the front and profile evolution, including accumulation zones near the inlet under opposing wave action. The continuous influx was shown to eventually overcome the inhibitory effect of waves, allowing the front to resume its propagation.

Chapter 7

Conclusions and outlook.

In this thesis, we investigated the evolution of thin oil layers spreading over wavy water surfaces. The focus of the study was on the influence of wave motion and geometric constraints on the process of spreading and the drift of oil slicks.

Our approach was based on theoretical and numerical analysis of the model obtained by using the thin-film approximation. The numerical simulations have been performed using a moving mesh finite difference method [Baines et al., 2005, Lee et al., 2015]. This method helped us accurately track changes in the shape of the oil layer and effectively handle moving boundaries.

The method, second-order accurate in space, was validated using the Barenblatt self-similar solutions of the nonlinear diffusion equation known as the porous medium equation. The technique worked well for both mass-conserving and non-mass-conserving problems, adapting dynamically to complex boundary conditions and preserving essential physical quantities such as mass and flux.

In the first part of this study, we explored how travelling water waves influence the spread of an oil slick. The numerical simulations have revealed distinct regimes of oil slick behaviour depending on the relative strength of advection and diffusion which were categorised by the effective Péclet number Pe_E

$$Pe = \frac{B_0 \varepsilon \omega}{\alpha_g} = \frac{\text{advection}}{\text{diffusion}},$$

where, to remind, the parameter

$$\alpha_g = \frac{\rho g_0 \varepsilon H^2}{\mu U},$$

measures the strength of the gravity-viscous diffusion, B_0 is the wave amplitude and ω is the wave frequency.

It was established that travelling waves generate clear drift motion of the oil layer along the direction of wave propagation and enhance the rate of the oil slick widening.

In particular, when parameter α_g is small and parameter Pe_E is then large, advection dominates, and the oil layer drifts as a whole in the same direction as the wave group velocity. This drift motion of the slick in this regime resembles the motion of a single solid particle following the Stokes drift, that is the slick profile is keeping its width and initial profile thickness.

For higher α_g values and hence lower Pe_E , diffusion becomes more important, the drift motion starts to differ from the typical Stokes drift behaviour, and the thickness profile of the oil slick begins to take on a universal shape similar to the self-similar solutions (2.46) of the non-linear advection-diffusion equation in our model but without the wave action.

In all cases, the spreading rate increases with both B_0 and k , showing that short-wavelength travelling waves strongly enhance the transport and thinning of the oil layer.

The second part of the study focused on how oil spreads in one-dimensional semi-confined geometries, taking into account the influence of boundary limits and an external oil source.

The results showed that having an external source significantly changes the overall spreading behaviour. In particular, the source increases both the spreading rate and the characteristic exponent of the front motion, regardless of the source strength q_0 . This behaviour suggests that these exponents can serve as useful indicators of external source activity in practical oil spill analysis.

It was also shown that the geometric constraints played crucial role in the spreading dynamics. In particular, it has been clearly demonstrated that the interaction of the oil slick with oncoming water waves (moving towards the restricting boundary) can strongly affect the front motion even at relatively low amplitude (elevation of the surface) of the wave on the scale of ten centimetres.

Without an external source, the forward propagation of the front can be strongly suppressed by the oncoming wave. This suppression refers to the inhibition of the outward motion of $x_r(t)$, rather than a fully resolved compression of the whole slick towards the fixed boundary. We also considered stronger oncoming waves by increasing B_0 or k . In these cases, the oil layer ruptured near the fixed boundary at $x = 0$ before a clear retreat of $x_r(t)$ could be observed. Therefore, these simulations do not establish compression

of the whole slick towards the fixed boundary. By contrast, in the presence of a source, the front motion is only temporarily inhibited and the system reaches a quasi-steady regime before the front starts to move again.

These findings imply that oil slicks may become partially or fully trapped by opposing wave motion, even for relatively small wave amplitudes, whereas co-propagating waves enhance spreading and accelerate the front motion.

7.0.1 Future work

As for the future work, the approach has strong potential to be generalised into a model in two-dimensions. This extension would allow for a more realistic representation of oil slick dynamics, capturing effects such as irregular boundaries and multi-directional wave interactions that are not possible in one-dimensional settings [Moskowitz, 1964].

Moreover, based on our separate published work [Lukyanov et al., 2024], where the effect of standing waves has been numerically analysed, there is an indication of possible topological changes in two-dimensional settings. Basically, in contrast to travelling waves studied in this thesis, the standing waves were shown to be able to cause the oil layer to thin out. That is potentially in the full two-dimensional settings, one may expect break up in certain areas, especially when α_g is low and the effective Péclet number Pe_E is high. This may act as an early step in the process of emulsification.

Chapter 8

Appendix

8.1 Appendix A: Stokes drift

In this section, we re-approach the theory of the well-known Stokes drift phenomenon, which is used in the interpretation of some of our simulation results [Stokes, 1847, Longuet-Higgins, 1953, Falkovich, 2011, van den Bremer and Breivik, 2017].

The theory of the Stokes drift effect serves to provide insights into dispersion of colloid particles under the action of the surface waves. While the similarity between the colloid particle dynamics and the dynamics of an oil slick may not be always the case, there are particular conditions, when the oil slick motion follows the results found for particles [Christensen and Terrile, 2009].

8.1.1 Stokes drift effect in the presence of a single harmonic perturbation: asymptotic analysis and numerical simulation results

The Stokes Drift effect can be illustrated using the one-dimensional model of motion of a single colloid particle in the velocity field created by a travelling harmonic wave [Falkovich, 2011]. The particle motion induced by a single harmonic travelling wave, assuming that the particle follows the velocity field, obeys the evolution equation of the form

$$\frac{dx}{dt} = u_0 \sin(kx - \omega t) \quad (\text{A.1})$$

where $x(t)$ is the location of a fluid particle, together with some initial condition $x(0) = x_0$. Here, k is the wave number and ω is the wave frequency. The main motivation for interpretation and comparison between the slick evolution and the Stokes drift motion comes from the similarity between the evolution equation for the boundary motion (2.33) in the limit $\alpha_g \rightarrow 0$, that is

$$\frac{dx_r}{dt} = V(x_r, t) = B_0 \varepsilon \omega \sin(kx_r - \omega t) \quad (\text{A.2})$$

and equation (A.1) if we associate $u_0 = B_0 \varepsilon \omega$. While we never set parameter α_g exactly to zero in the full scale simulations of (2.30), nevertheless, we consider some cases with substantially diminished contribution from diffusion at $\alpha_g \approx 10^{-5}$. Therefore, the interpretation may be useful.

Equation (A.1) has an implicit analytical solution. Indeed, introducing $\varphi = kx - \omega t$, differentiating φ with respect to time t and substituting the result

in the original equation one gets

$$\frac{d\varphi}{dt} = k \frac{dx}{dt} - \omega,$$

then

$$\frac{d\varphi}{dt} = ku_0 \sin \varphi - \omega,$$

which can be integrated exactly by the separation of variables to obtain

$$\int \frac{d\varphi}{ku_0 \sin \varphi - \omega} = t + C,$$

where C is the integration constant determined by the initial condition $\varphi(0) = \varphi_0 = kx_0$.

The integral depends on the absolute value of the parameter $|u_0 k / \omega|$, so that there are two cases, and we ignore the special case $|u_0 k / \omega| = 1$:

- Case 1: $u_0^2 k^2 > \omega^2$

$$\int \frac{d\varphi}{ku_0 \sin \varphi - \omega} = \frac{1}{\sqrt{u_0^2 k^2 - \omega^2}} \ln \left| \frac{-\omega \tan(\varphi/2) + u_0 k - \sqrt{u_0^2 k^2 - \omega^2}}{-\omega \tan(\varphi/2) + u_0 k + \sqrt{u_0^2 k^2 - \omega^2}} \right| = t + C \quad (\text{A.3})$$

and

$$C = \frac{1}{\sqrt{u_0^2 k^2 - \omega^2}} \ln \left| \frac{-\omega \tan(\varphi_0/2) + u_0 k - \sqrt{u_0^2 k^2 - \omega^2}}{-\omega \tan(\varphi_0/2) + u_0 k + \sqrt{u_0^2 k^2 - \omega^2}} \right|.$$

- Case 2: $u_0^2 k^2 < \omega^2$

$$\int \frac{d\varphi}{ku_0 \sin \varphi - \omega} = \quad (\text{A.4})$$

$$\frac{2}{\sqrt{\omega^2 - u_0^2 k^2}} \arctan \left(\frac{-\omega \tan(\varphi/2) + u_0 k}{\sqrt{\omega^2 - u_0^2 k^2}} \right) = t + C,$$

where C is given by

$$C = \frac{2}{\sqrt{\omega^2 - u_0^2 k^2}} \arctan \left(\frac{-\omega \tan(\varphi_0/2) + u_0 k}{\sqrt{\omega^2 - u_0^2 k^2}} \right).$$

As one can notice, the integration always provides implicit expressions in terms of φ , which cannot be easily inverted to obtain explicit relationships useful in practice and in interpretations.

The only observable limiting case is at

$$\left| \frac{u_0 k}{\omega} \right| \gg 1,$$

when substituting into (A.3)

$$\tan(\varphi/2) \propto \exp(-u_0 k t).$$

Therefore, it follows directly that asymptotically, at $|u_0 k|t \gg 1$, $\varphi \approx 0$, and the dynamics is independent of the amplitude u_0 , that is

$$x(t) \approx \frac{\omega}{k} t. \tag{A.5}$$

The implicit solutions, while being impractical, indicate that there should be two qualitatively different particle motion regimes. This can be illustrated numerically. The characteristic behaviour of the numerical solutions to (A.1) is demonstrated in Fig. 8.1. The solutions have been readily obtained by

standard Matlab ordinary differential equation packages with the relative error tolerance 10^{-9} .

First of all, one can observe that despite the fact that the right-hand side of equation (A.1) is a bounded periodic function of its argument, the displacement demonstrates unbounded motion with time.

At $|u_0 k / \omega| \gg 1$, the solution is indeed found to be independent of u_0 , and it closely follows the asymptotic form suggested by (A.5). The minor deviation is due to the initial evolution when the asymptotic solution is not supposed to hold. One can also observe that the characteristic behaviour of the solution is controlled by the parameter $|u_0 k / \omega|$. The solution qualitatively changes behaviour at the critical value $|u_0 k / \omega| = 1$.

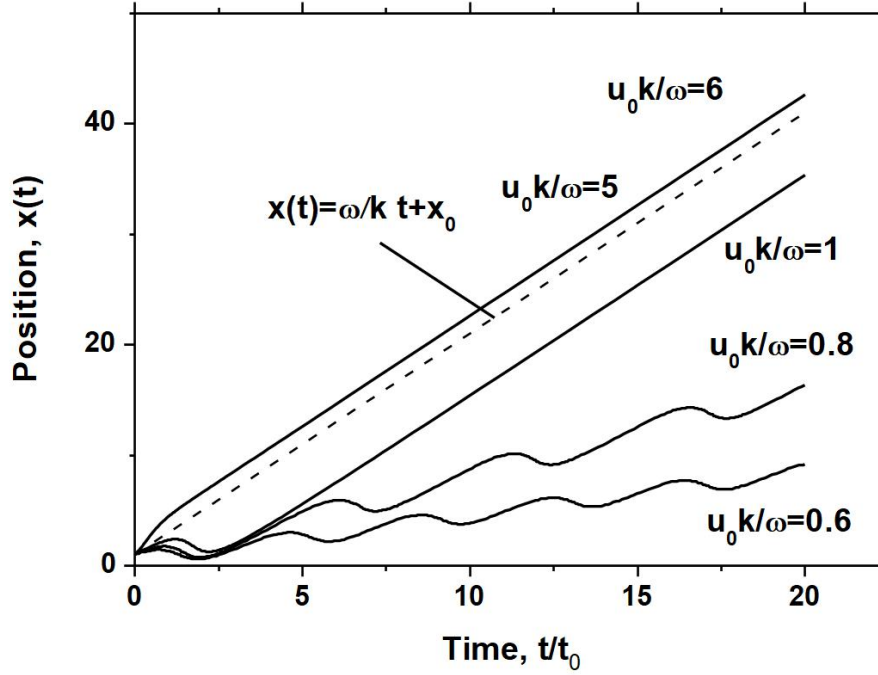


Figure 8.1: Displacement $x(t)$ as a function of time at different values of parameter $u_0 k / \omega$. The solutions of (A.1) (solid lines) were obtained numerically at $x_0 = 1$. The dashed line is the asymptotic form (A.5) at $x_0 = 1$, $\omega = 2$ and $k = 1$.

As our interest in the Stokes effect is mainly due to the similarity with (A.2), the characteristic values of $u_0 |k / \omega| = B_0 \varepsilon |k|$ can be roughly estimated as $|u_0 k / \omega| \ll 1$ in the characteristic region of our interest, see characteristic parameters in Chapter 4.

In that limit, following [Falkovich, 2011], it is possible to find simple, explicit analytical forms by iterations, which will be useful for the interpretation.

Equation (A.1) is solved by iterations setting in the first approximation $x^{(0)} = x_0$, where x_0 is the initial position. Then, we simply have

$$\frac{dx}{dt} = u_0 \sin(kx_0 - \omega t),$$

which is also subject to $x(0) = x_0$

As a result of solving the simplified equation, one gets

$$x^{(1)} = x_0 + \frac{u_0}{\omega} \{ \cos(kx_0 - \omega t) - \cos(kx_0) \}.$$

This solution shows purely oscillatory motion without any net drift.

Similarly, in the second approximation

$$\frac{dx}{dt} = u_0 \sin(kx^{(1)} - \omega t),$$

which is also subject to $x(0) = x_0$

The result in the second approximation step is

$$\begin{aligned} x^{(2)}(t) \approx & x_0 + \frac{u_0}{\omega} \{ \cos(kx_0 - \omega t) - \cos(kx_0) \} - \\ & \frac{1}{4} \frac{ku_0^2}{\omega^2} \{ \sin(2kx_0 - 2\omega t) + \sin(2kx_0) - 4 \sin(kx_0 - \omega t) \cos(kx_0) \} + \\ & \frac{1}{2} \frac{ku_0^2}{\omega} t + O \left(\left(\frac{u_0 k}{\omega} \right)^3 \right), \end{aligned} \quad (\text{A.6})$$

where the last, secular term with the linear dependence on time provides the estimate for the Stokes drift velocity

$$u_s = \frac{1}{2} \frac{u_0^2 k}{\omega}. \quad (\text{A.7})$$

Using the deep-water wave relationships (2.36) and (2.37), in particular by associating the amplitude $u_0 = B_0 \varepsilon \omega$, one gets

$$u_s = \frac{1}{2} k B_0^2 \varepsilon^2 \omega = \frac{\varepsilon^{3/2}}{2Fr} B_0^2 k^{3/2} \propto B_0^2 k^{3/2}. \quad (\text{A.8})$$

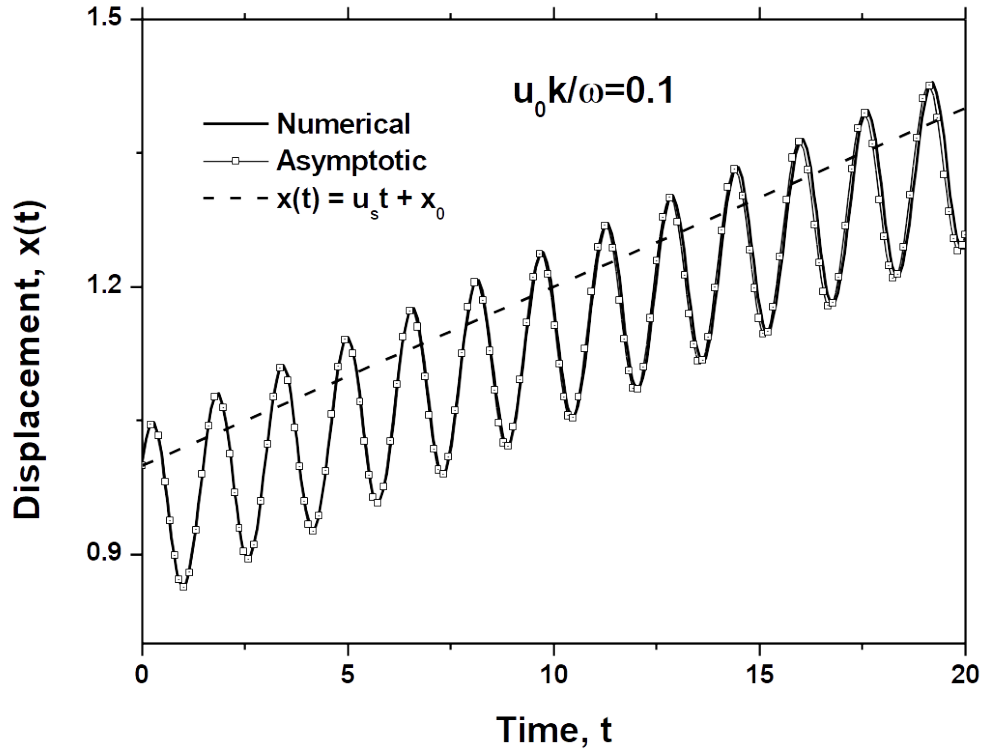


Figure 8.2: Displacement $x(t)$ as a function of time at $u_0 k / \omega = 0.1$, that is $u_0 k / \omega \ll 1$. The bold solid line is the numerical solution of (A.1), the symbols with line is the asymptotic approximation (A.6), and the dashed line is the Stokes drift $x(t) = u_s t + x_0$ according to (A.7). Parameters $u_0 = 0.4$, $k = 1$, $\omega = 4$, and $x(0) = x_0 = 1$.

Here, the second equality follows from the dispersion relation $\omega^2 = \varepsilon^{-1} F r^{-2} k$, so that $\omega = \varepsilon^{-1/2} F r^{-1} k^{1/2}$. Therefore, $\varepsilon^2 k \omega = \varepsilon^{3/2} F r^{-1} k^{3/2}$.

Equation (A.8) shows that larger k , that is smaller wavelength and larger frequency, and larger wave amplitude should facilitate the drift.

This will now be informative to briefly assess and validate the limitations of the asymptotic solution (A.6) by comparing with the full numerical solution

of equation (A.1) for different values of the parameter $\frac{u_0 k}{\omega}$.

The comparison is illustrated in Figs 8.2 and 8.3.

In the first figure, Fig. 8.2, the control parameter $\frac{u_0 k}{\omega} = 0.1$ is small, therefore, as it might be expected, the match between the analytical and numerical solutions is very good.

On the other hand, in the second figure, Fig. 8.3, parameter $\frac{u_0 k}{\omega} = 0.8$ is of the order of one, and the match is not looking even qualitatively good.

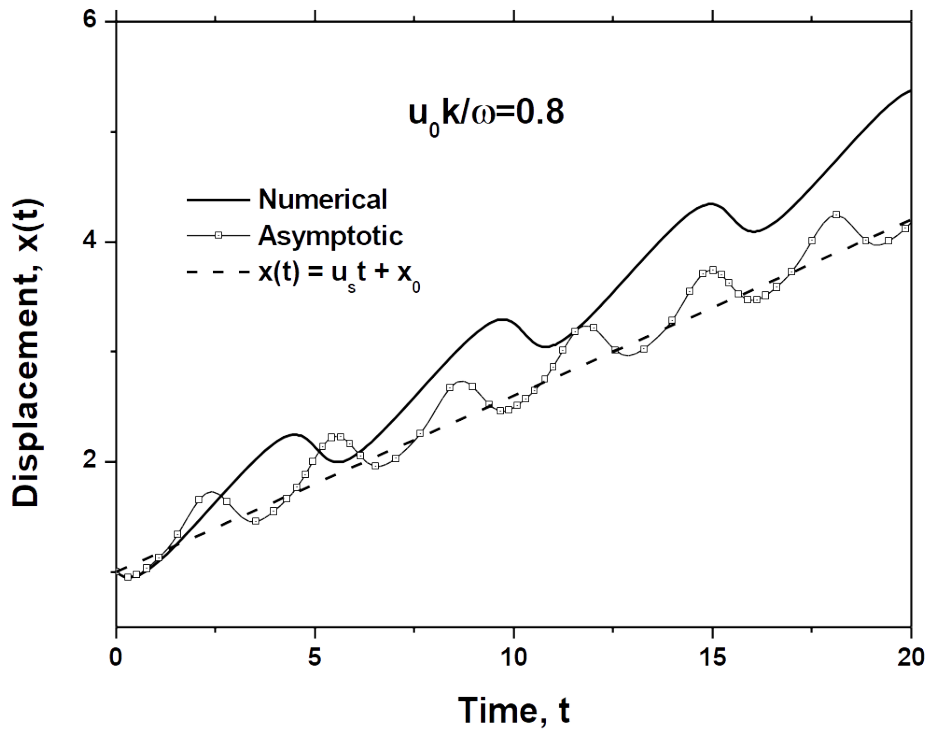


Figure 8.3: Displacement $x(t)$ as a function of time at $u_0 k / \omega = 0.8$, that is $u_0 k / \omega \approx 1$. The solid line is the numerical solution of (A.1), the symbols with line is the asymptotic approximation and the dashed line is the Stokes drift $x(t) = u_s t + x_0$ according to (A.7). Parameters $u_0 = 0.4$, $k = 4$, $\omega = 2$, and $x(0) = x_0 = 1$.

8.1.2 Stokes drift effect in the presence of several harmonics: asymptotic analysis and numerical solutions

The result for the Stokes drift velocity can be generalised to a linear combination of different harmonics. Indeed, consider the particle motion under the action of a linear superposition of two arbitrary travelling waves, which is described by the evolution equation

$$\frac{dx}{dt} = u_0 \sin(kx - \omega t) + u_1 \sin(k_1 x - \omega_1 t). \quad (\text{A.9})$$

The asymptotic solution can be found by the similar approach as that used in the single harmonic wave case by using iterations and assuming that both $|u_0 k / \omega| \ll 1$ and $|u_1 k_1 / \omega_1| \ll 1$. In the first approximation setting $x^{(0)} = x_0$, where x_0 is as before the initial position of the particle, one gets

$$x^{(1)} = x_0 + \frac{u_0}{\omega} (\cos(kx_0 - \omega t) - \cos(kx_0)) + \frac{u_1}{\omega_1} (\cos(k_1 x_0 - \omega_1 t) - \cos(k_1 x_0))$$

by solving

$$\frac{dx}{dt} = u_0 \sin(kx_0 - \omega t) + u_1 \sin(k_1 x_0 - \omega_1 t). \quad (\text{A.10})$$

In the second approximation,

$$\begin{aligned} x^{(2)}(t) \approx & x_0 + \frac{u_0}{\omega} (\cos(kx_0 - \omega t) - \cos(kx_0)) + \\ & \frac{u_1}{\omega_1} (\cos(k_1 x_0 - \omega_1 t) - \cos(k_1 x_0)) + \frac{1}{2} \left(\frac{k u_0^2}{\omega} + \frac{k_1 u_1^2}{\omega_1} \right) t + \dots \end{aligned} \quad (\text{A.11})$$

where in the second order expression we have only kept second order terms linearly proportional to time, while omitting, for simplicity, the oscillatory and bounded second order contributions.

The secular term with linear dependence on time provides the estimate for the Stokes drift velocity

$$u_s = \frac{1}{2} \left(\frac{k u_0^2}{\omega} + \frac{k_1 u_1^2}{\omega_1} \right), \quad (\text{A.12})$$

which is simply a linear combination of the results for a single harmonic.

To validate this result, we numerically solve the differential equation (A.9) in the parameter range such that $|u_0 k / \omega| \ll 1$ and $|u_1 k_1 / \omega_1| \ll 1$.

The result is shown in Fig. 8.4, where one can observe that the analytical and numerical results match very well, confirming that the asymptotic method accurately captures the Stokes drift when multiple wave components are present.

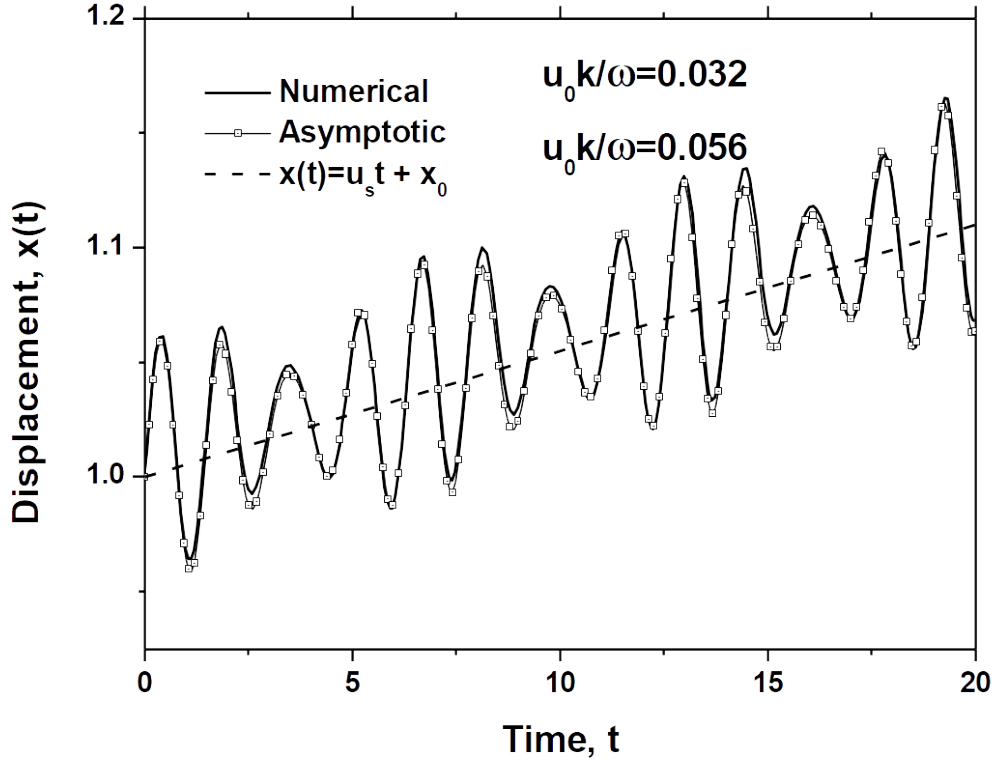


Figure 8.4: Displacement $x(t)$ as a function of time in the case of several generating harmonics. Numerical solution of (A.9) is shown by the solid blue line, asymptotic solution (A.11) is shown by the dashed red line and the Stokes drift (A.12) velocity is shown by a bold black line. Parameters used are $u_0 = 0.08$, $u_1 = 0.15$, $k = 2$, $k_1 = 1.5$, $\omega = 5$, $\omega_1 = 4$, with initial condition $x_0 = 1$.

We note that in the case of a standing wave, which is a linear combination of two waves travelling in the opposite directions, that is having positive and negative terms, equivalently if we change k to $-k$, as one may expect, the total drift $u_s = 0$.

8.2 Appendix B: Radially symmetric extension case

The results from the one dimensional case can be generalized to the radially symmetric scenario, which provides a more realistic approximation for point-source oil spills. The radial exponents are obtained here by applying the same self-similar scaling argument used in the one-dimensional case to the radially symmetric geometry. This follows the standard approach for nonlinear diffusion and viscous gravity spreading [Huppert, 1982, Lister, 1992].

In the absence of an external source, the exponents are:

$$\beta_{0r} = \frac{1}{2 + 2m}, \quad \alpha_{0r} = \frac{2}{2 + 2m},$$

such that the domain boundary (radius) evolves as $R(t) \propto t^{\beta_{0r}}$. When an external flux q is present at some small radius $r_0 \ll R(t)$ with $2\pi q r_0 = q_0 t^\gamma$, the exponents become:

$$\beta_r = \frac{1 + m(\gamma + 1)}{2 + 2m}, \quad \alpha_r = -\frac{2\gamma}{2 + 2m}.$$

As in the 1D case, for $\gamma > -1$, $\beta_r > \beta_{0r}$, indicating accelerated spreading due to the source.

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