The Legendre Transformation And Grid Generation In Two Dimensions

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Abstract

The Legendre transformation is used as the basis of a method to generate irregular triangular grids in two dimensions. The method also generates piecewise linear approximations to functions. Grids are created for several functions including the solution of a two dimensional semi-geostrophic frontogenesis problem in meteorology.

An algorithm is outlined for solving the time dependent semi-geostrophic frontogenesis problem.

1 Introduction

In this report we present a method for generating an irregular two-dimensional grid and creating a piecewise linear approximation to a function on the grid by an application of the Legendre transformation. The method is also used to give an approximate representation of semi-geostrophic frontogenesis in which the numerical issues involved in the time evolution are discussed.

Section 2 deals with the methods of creating the grid and approximating functions using the Legendre transformation. In section 3 some background to the meteorological problem of semi-geostrophic frontogenesis is presented and an initial approximate solution is found. A method to integrate the solution in time is outlined in section 4. In section 5 problems associated with the boundaries of domains are discussed and a method to overcome them is proposed.

2 The Legendre Transformation in Two Dimensions

Given a function P of two variables x and z we seek a piecewise linear approximation $\hat{P}$ to P.

One method of creating such an approximation would be to form a regular triangular grid in the xz plane and then consider the plane above each triangular element which intersects the surface of P vertically above the three vertices of the triangle. The set of these triangular plates is a 'chordal' type approximation to the function P.

It may be possible to improve on this approximation by using an irregular triangular grid in the xz plane and/or seeking alternative values of the approximation above the vertices of each triangle instead of using the corresponding value of the function P. An irregular grid can be created using the Legendre transformation (cf. [1]).

In one dimension, an application of the Legendre transformation leads to the generation of an irregular distribution of nodes along an axis and a piecewise linear approximation to a function of one variable [2]. It was found that, on the interval between any two adjacent nodes x_{i-1} and x_i on the axis, the second derivative of the function to be approximated, u, was equidistributed such that

$$\int_{x_{i-1}}^{x_i} u''(x)\, dx = \text{ constant.} \tag{2.1}$$

A consequence of this was that the nodes of the approximation tended to cluster in regions where the second derivative of the function was largest. By analogy with the method in one dimension, it may be that an irregular two dimensional grid, generated using the Legendre transformation, exhibits a similar property and the nodes of the grid cluster in regions where the second derivatives of the function to be approximated are largest.

Returning to the two dimensional problem, consider the coordinates (m, θ, R), dual to the original coordinates (x, z, P). The Legendre transformation gives a relationship between these sets:

$$m = \frac{\partial P}{\partial x}, \quad \theta = \frac{\partial P}{\partial z} \tag{2.2}$$

$$x = \frac{\partial R}{\partial m}, \quad z = \frac{\partial R}{\partial \theta} \tag{2.3}$$

$$P + R = mx + \theta z. \tag{2.4}$$

So, given $P(x, z)$, m and θ can be found as functions of x and z from (2.2). Provided these functions are invertible it is possible to express x and z as functions of m and θ and, using (2.4), to find an expression for the dual function R in terms of m and θ.

One property of the Legendre transformation is that a part of a plane in one space transforms to a point in the dual space. Thus, if the surface in one space is to consist of triangular plates, the dual surface must consist of non-overlapping plates where the projections onto the $m\theta$ plane of no more than three plates meet at any point. In particular, a surface which consists of hexagonal plates is sufficient to ensure that the dual surface is made up of triangular plates. Adjacent hexagons transform to points in the dual space which may be thought of as two of the vertices of a triangle, and three hexagons whose projections onto the $m\theta$ plane have a common node transform to the three vertices of a triangle (see figure 1).

Generating a regular hexagonal grid in the $m\theta$ plane, forming a linear approximation to the function R, then performing a Legendre transformation back to the original space, gives a set of points in the xz plane which may be used as nodes for an irregular triangular grid, and a set of values for the approximation to P at each node. Then one piecewise linear approximation to P is the set of planes, one above each triangle, which pass through the approximate values of P corresponding to each node of the triangle.

A regular grid of hexagons is set up in the $m\theta$ plane. Over each hexagon the best fit plane approximation to the surface R may be found by minimising

$$\|R - mx - \theta z + P\|_2 \tag{2.5}$$

with respect to P, x and z. The best fit plane above each hexagon transforms to a point in the original space.

Consider hexagon i. To find the corresponding point (x_i, z_i, P_i) minimise

$$\|R - mx_i - \theta z_i + P_i\|_2 \tag{2.6}$$

over P_i, x_i, z_i. This leads to the equations

$$\left.\begin{array}{r} \iint_{D_i}(R - mx_i - \theta z_i + P_i)m\, dm\, d\theta = 0 \\ \iint_{D_i}(R - mx_i - \theta z_i + P_i)\theta\, dm\, d\theta = 0 \\ \iint_{D_i}(R - mx_i - \theta z_i + P_i)\, dm\, d\theta = 0 \end{array}\right\} \tag{2.7}$$

where the integrals are over the i th hexagon in the $m\theta$ plane. This gives rise to the matrix equation

$$\begin{pmatrix} \iint dm\, d\theta & -\iint m\, dm\, d\theta & -\iint \theta\, dm\, d\theta \\ -\iint m\, dm\, d\theta & \iint m^2\, dm\, d\theta & \iint m\theta\, dm\, d\theta \\ -\iint \theta\, dm\, d\theta & \iint m\theta\, dm\, d\theta & \iint \theta^2\, dm\, d\theta \end{pmatrix} \begin{pmatrix} P_i \\ x_i \\ z_i \end{pmatrix} = \begin{pmatrix} -\iint R\, dm\, d\theta \\ \iint mR\, dm\, d\theta \\ \iint \theta R\, dm\, d\theta \end{pmatrix} \tag{2.8}$$

to be solved for x_i, z_i and P_i, where the integrals are again over the i th hexagon. This is done for each hexagon.

The point in (x, z, P) space, transformed from hexagonal plate i in (m, θ, R) space, is joined by straight lines to all other points transformed from hexagonal plates neighbouring plate i. In this way a piecewise linear approximation to the function P is constructed from the triangular plates formed when three points are joined by three lines (see figure 1).

This process has been performed for several trial functions. Irregular triangular grids are generated in the xz plane and piecewise linear approximations to the functions are found. The grids generated for the trial functions

$$P(x,z) = x^2 + z^5 \text{ and } P(x,z) = e^{-8x} + e^{-8z}$$

are shown in figures 2 and 3 respectively. The nodes of the triangular grids are seen to cluster in regions of the domain where the function to be approximated, P, has largest curvature.

3 Meteorological Example

In this section we seek to model the motion of air parcels in the region of a weather front using the non dimensionalised hydrostatic Boussinesq semi-geostrophic equations on an f-plane.

From [3] in three dimensions there exists a Legendre transformation between two sets of coordinates - physical and momentum coordinates. Define x and y to be the horizontal cartesian coordinates, z to be the vertical cartesian coordinate and $P(x, y, z, t)$ to be a modified geopotential function in physical space which also depends on the time t. Then m, n, θ are the corresponding momentum coordinates - m and n are the components of geostrophic absolute momentum, θ is the potential temperature and there exists a function $R(m, n, \theta, t)$ such that the three dimensional equivalents of equations (2.2)–(2.4) are satisfied:

$$m = \frac{\partial P}{\partial x}, \quad n = \frac{\partial P}{\partial y}, \quad \theta = \frac{\partial P}{\partial z} \tag{3.1}$$

$$x = \frac{\partial R}{\partial m}, \quad y = \frac{\partial R}{\partial n}, \quad z = \frac{\partial R}{\partial \theta} \tag{3.2}$$

$$P + R = mx + ny + \theta z. \tag{3.3}$$

Using momentum coordinates, the non dimensionalised hydrostatic Boussinesq semi-geostrophic equations can be written:

$$\frac{Dm}{Dt} = y - n \tag{3.4}$$

$$\frac{Dn}{Dt} = m - x \tag{3.5}$$

$$\frac{D\theta}{Dt} = 0 \tag{3.6}$$

$$\nabla.\mathbf{v} = 0 \tag{3.7}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{3.8}$$

and $\mathbf{v} = (u, v, w)$ is the wind velocity in the physical coordinate system. In the momentum coordinate system (3.4)–(3.6) explicitly describe the motion.

Consider a large scale deformation field in the physical space where the horizontal components of velocity, u and v, are given by

$$u = -\alpha x \quad , \quad v = \alpha y \tag{3.9}$$

and α is the deformation velocity. Under these conditions the motion can be thought of as two dimensional in the plane $y = 0$.

The semi-geostrophic equations become

$$\dot{m} = -\alpha m \tag{3.10}$$

$$\dot{\theta} = 0 \tag{3.11}$$

where the dot notation denotes

$$\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z} \tag{3.12}$$

and w is the vertical component of velocity. For conservation of mass

$$\dot{A} = -\alpha A \tag{3.13}$$

must be satisfied for each parcel of air, where A is the area of the parcel.

Consider a domain Ω in the physical space

$$\Omega = \{(x, z) : 0 \leq z \leq H, -L \leq x \leq L\} \tag{3.14}$$

where $H = 10^4$, $L = 10^4$.

From [4] we use an initial distribution of potential temperature given by

$$\theta(x, z) = \theta_1 \tan^{-1}\left(\frac{5m(x, z)}{L}\right) + \left(z - \frac{H}{2}\right)\theta_2 \tag{3.15}$$

where $\theta_1 = 300, \theta_2 = 1$. The absolute momentum $m(x, z)$ satisfies

$$\frac{x - m(x, z)}{z - H/2} = -\frac{5\theta_1}{L}\frac{1}{1 + (5m(x, z)/L)^2}. \tag{3.16}$$

An expression for R can be derived using (2.3):

$$R(m, \theta) = \frac{m^2}{2} - \frac{\theta_1}{2\theta_2}\left(2\theta \tan^{-1}\left(\frac{5m}{L}\right) - \theta_1\left(\tan^{-1}\left(\frac{5m}{L}\right)\right)^2\right) + \frac{1}{2\theta_2}\left(H\theta\theta_2 + \theta^2\right) \tag{3.17}$$

which gives the initial distribution of the function in the dual momentum space. The domain of interest in this space is

$$\Gamma = \{(m(x, z), \theta(x, z)) : (x, z) \in \Omega\}. \tag{3.18}$$

We use this initial function R and discretise the domain Γ to form a linear approximation to R, then form an approximation $\hat{P}$ to P in the physical space using the Legendre transformation as outlined in section 2. The triangular grid created by projecting the triangular plates constituting $\hat{P}$ onto the xz plane represents the initial distribution of air parcels in the domain Ω.

Having found an initial approximate solution $\hat{P}$ we seek to advance the solution in time in the dual space while satisfying the condition of conservation of mass. Then, by similarly projecting the triangular plates of the time advanced approximate solution $\hat{P}$ onto the xz plane, the motion of the air parcels in Ω can be studied.

4 Approximate Solution of the Time Dependent Meteorology Problem

This section deals with finding an approximate solution of the meteorology problem in physical space after a specified time t.

Let A_i be the area of the i th triangular element formed by the projection of $\hat{P}$ onto the xz plane. From (3.13) each A_i must vary with time according to

$$A_i(t) = A_i(0)e^{-\alpha t} \tag{4.1}$$

where $A_i(0)$ is the initial area of the i th triangle. Similarly, integration of (3.10) and (3.11) gives

$$m_i(t) = m_i(0)e^{-\alpha t} \tag{4.2}$$

$$\theta_i(t) = \theta_i(0) \tag{4.3}$$

where $m_i(t)$ is the gradient in the x direction and $\theta_i(t)$ is the gradient in the z direction of the surface $\hat{P}$ above the i th triangle after time t and are the absolute momentum and potential temperature respectively of the air parcel corresponding to the i th triangle.

Therefore, in the momentum space, there are explicit equations governing the motion of the nodes of the hexagons and the nodal positions may be calculated after an arbitrary time period. The difficulty comes in constructing an approximation, $\hat{R}$, to the function R after a time period to ensure that the equivalent conservation of mass equation (4.1) is satisfied in the physical space.

The area of each triangle is a function of the positions of its three vertices. These depend on the gradient of the approximation to R in three corresponding hexagons in the momentum space. Thus the areas of the triangles are functions of the values of $\hat{R}$ at the nodes in the momentum space which determine the gradient of $\hat{R}$ over each hexagon.

Let $q_i(R)$ be the area of the i th triangle as a function of the values of $\hat{R}$ at seven of the nodes of the three hexagons which contribute to form the triangle. The set of seven nodes is chosen to consist of the one which is common to all three hexagons and six more on the boundaries of the three hexagons so that no two of the nodes are common to a single edge of any hexagon (see figure 4).

Let $\mathbf{R}$ be the vector of nodal values of the function $\hat{R}$. Then

$$q_i(R) = \mathbf{R}^T C_i \mathbf{R} \tag{4.4}$$

for an $(r \times r)$ matrix C_i where r is the number of nodes in the $m\theta$ plane.

The problem to find $\mathbf{R}$ is the minimisation of

$$\sum_i (A_i - q_i(R))^2 \tag{4.5}$$

with respect to $\mathbf{R}$, subject to the constraint that there is a plane passing through all six points corresponding to the nodes of a hexagon. This constraint may be written as

$$D\mathbf{R} = \mathbf{0} \tag{4.6}$$

where D is a $(3s \times r)$ matrix, s is the the number of hexagons in the $m\theta$ plane.

Let $\boldsymbol{\lambda}$ be the $(3s \times 1)$ vector of Lagrange multipliers. Then the problem becomes:

$$\min_{\mathbf{R},\boldsymbol{\lambda}} \left(\sum_i (A_i - \mathbf{R}^T C_i \mathbf{R})^2 + \boldsymbol{\lambda}^T D \mathbf{R} \right). \tag{4.7}$$

This leads to the equations

$$\left.\begin{aligned} -2\sum_i (A_i - \mathbf{R}^T C_i \mathbf{R}) \mathbf{R}^T (Ci + Ci^T) + \boldsymbol{\lambda}^T D = 0 \\ D\mathbf{R} = 0 \end{aligned}\right\} \tag{4.8}$$

to be solved for $\mathbf{R}$ and $\boldsymbol{\lambda}$.

The equations could be solved by Newton's Method as follows. Let

$$\begin{aligned} \mathbf{F}_1(\mathbf{R},\boldsymbol{\lambda}) &= -2\sum_i (A_i - \mathbf{R}^T C_i \mathbf{R})\mathbf{R}^T(Ci + Ci^T) + \boldsymbol{\lambda}^T D && (4.9) \\ \mathbf{F}_2(\mathbf{R},\boldsymbol{\lambda}) &= D\mathbf{R}. && (4.10) \end{aligned}$$

Then

$$\frac{\partial \mathbf{F}_1}{\partial \mathbf{R}} = -2\sum_i \Big((A_i - \mathbf{R}^T C_i \mathbf{R})(Ci + Ci^T) - (Ci + Ci^T)\mathbf{R}\mathbf{R}^T(Ci + Ci^T)\Big) \quad (4.11)$$

$$\begin{aligned} \frac{\partial \mathbf{F}_1}{\partial \boldsymbol{\lambda}} &= D^T && (4.12) \\ \frac{\partial \mathbf{F}_2}{\partial \mathbf{R}} &= D && (4.13) \\ \frac{\partial \mathbf{F}_2}{\partial \boldsymbol{\lambda}} &= 0. && (4.14) \end{aligned}$$

The Jacobian matrix is given by

$$J = \begin{pmatrix} \frac{\partial \mathbf{F}_1}{\partial \mathbf{R}} & \frac{\partial \mathbf{F}_1}{\partial \boldsymbol{\lambda}} \\ \frac{\partial \mathbf{F}_2}{\partial \mathbf{R}} & \frac{\partial \mathbf{F}_2}{\partial \boldsymbol{\lambda}} \end{pmatrix}. \quad (4.15)$$

Let $\mathbf{R}^k, \boldsymbol{\lambda}^k$ be previous guesses to the solution of (4.7). Improved values may be found by solving

$$J^k \left(\begin{pmatrix} \mathbf{R}^{k+1} \\ \boldsymbol{\lambda}^{k+1} \end{pmatrix} - \begin{pmatrix} \mathbf{R}^{k} \\ \boldsymbol{\lambda}^{k} \end{pmatrix} \right) = -\begin{pmatrix} \mathbf{F}_1^k \\ \mathbf{F}_2^k \end{pmatrix} \quad (4.16)$$

for $\mathbf{R}^{k+1}, \boldsymbol{\lambda}^{k+1}$, where the k superscript denotes quantities evaluated at $\mathbf{R}^k, \boldsymbol{\lambda}^k$. This can be simplified by using various properties of matrix (4.15), in particular the entries of matrix (4.14) are all zero and matrix (4.11) will have some zero rows and columns where the values of $\hat{R}$ corresponding to several of the nodes in the momentum space do not contribute to the area function (4.4) of any triangle in the physical space.

Let the number of zero rows and columns in (4.11) be l. Perform row and column operations on matrix (4.15) until it is in the form

$$\begin{pmatrix} A & 0 & B_1^T \\ 0 & 0 & B_2^T \\ B_1 & B_2 & 0 \end{pmatrix} \quad (4.17)$$

where A is an $(r-l \times r-l)$ matrix with no zero rows or columns, B_1 is a $(3s \times r-l)$ matrix and B_2 is a $(3s \times l)$ matrix. Performing these row and column operations on the system (4.16) gives

$$\begin{pmatrix} A & 0 & B_1^T \\ 0 & 0 & B_2^T \\ B_1 & B_2 & 0 \end{pmatrix}^k \left(\begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_0 \\ \boldsymbol{\lambda} \end{pmatrix}^{k+1} - \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_0 \\ \boldsymbol{\lambda} \end{pmatrix}^{k} \right) = -\begin{pmatrix} \hat{\mathbf{F}}_1 \\ \mathbf{F}_0 \\ \mathbf{F}_2 \end{pmatrix}^k \quad (4.18)$$

where $\mathbf{R}_1$ is $(r-l\times 1)$, $\mathbf{R}_0$ is $(l\times 1)$, $\hat{\mathbf{F}}_1$ is $(r-l\times 1)$ and $\mathbf{F}_0$ is $(l\times 1)$, $(\mathbf{R}_1,\mathbf{R}_0)^T$ is a rearrangement of $\mathbf{R}$ and $(\hat{\mathbf{F}}_1,\mathbf{F}_0)^T$ is a rearrangement of $\mathbf{F}_1$. This leads to

$$A(\mathbf{R}_1^{k+1}-\mathbf{R}_1^k)+B_1^T(\boldsymbol{\lambda}^{k+1}-\boldsymbol{\lambda}^k) = -\hat{\mathbf{F}}_1^k \tag{4.19}$$

$$B_2^T(\boldsymbol{\lambda}^{k+1}-\boldsymbol{\lambda}^k) = -\mathbf{F}_0^k \tag{4.20}$$

$$B_1(\mathbf{R}_1^{k+1}-\mathbf{R}_1^k)+B_2(\mathbf{R}_0^{k+1}-\mathbf{R}_0^k) = -\mathbf{F}_2^k \tag{4.21}$$

to be solved for $\mathbf{R}_1^{k+1},\mathbf{R}_0^{k+1}$ and $\boldsymbol{\lambda}^{k+1}$. Manipulation of these equations gives explicit equations for $\mathbf{R}_1^{k+1},\mathbf{R}_0^{k+1}$ and $\boldsymbol{\lambda}^{k+1}$:

$$\mathbf{R}_0^{k+1}=\mathbf{R}_0^k+\left(B_2^T(B_1A^{-1}B_1^T)^{-1}B_2\right)^{-1}\left(-\mathbf{F}_0^k-B_2^T(B_1A^{-1}B_1^T)^{-1}(\mathbf{F}_2^k-B_1A^{-1}\hat{\mathbf{F}}_1^k)\right) \tag{4.22}$$

$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k+(B_1A^{-1}B_1^T)^{-1}\left(\mathbf{F}_2^k+B_2(\mathbf{R}_0^{k+1}-\mathbf{R}_0^k)-B_1A^{-1}\hat{\mathbf{F}}_1^k\right) \tag{4.23}$$

$$\mathbf{R}_1^{k+1} = \mathbf{R}_1^k-A^{-1}\left(\hat{\mathbf{F}}_1^k+B_1^T(\boldsymbol{\lambda}^{k+1}-\boldsymbol{\lambda}^k)\right). \tag{4.24}$$

The iteration process is likely to be computationally expensive since matrix A, which must be inverted in the iteration, varies with the solution $\mathbf{R}$ and may need to be updated. The solutions at the previous time step could be used as the initial values for the iteration at the present time step, the iteration process being continued until the vectors $\mathbf{R}$ and $\boldsymbol{\lambda}$ change by less than some tolerance. This may converge to a solution for $\hat{R}$ after some iterations. The positions of the triangle nodes in the xz plane can be recovered from the gradient of this approximation over each hexagon, and the values of $\hat{P}$ at each node can be found from (2.8). In this way the approximate solution, $\hat{P}$, to the problem may be integrated forwards in time ([3],[4]), and by a projection of the solution on to the xz plane, the motions of air parcels may be studied.

5 Domain Boundaries

In this section we outline a problem with the domain boundaries under the Legendre transformation and suggest a method to rectify it.

In the meteorology problem, consider the domain Γ in the momentum space and the domain Ω in the physical space from which Γ is derived in (3.18). In section 3, the smallest rectangular region completely enclosing Γ is discretised into regular hexagons. Those hexagons which lie entirely within the boundary of Γ are used as the domain, Γ_0, for the transformation which creates the irregular triangular grid in the xz plane in physical space. The region covered by this triangular grid does not correspond exactly to the original domain Ω but the boundary of the region appears well conditioned. Thus the approximate solution is found on a domain which is different from the domain of the original problem.

In an attempt to make the region covered by the irregular triangular grid closer to Ω, those hexagons on the boundary of Γ_0 are extended outwards from Γ_0 so that the domain covered by the hexagons, some of them now irregularly shaped, is exactly Γ. However, under the transformation, this results in an irregular triangular grid in the xz plane with a badly conditioned boundary.

One possible, but untested, method which may create a triangular grid covering a region closer to Ω in size and shape is as follows. Discretise the physical domain Ω into regular triangles so that all internal nodes of the grid are vertices of six triangles. Use a piecewise linear approximation, $\hat{P}$, of the function P in physical space to create a grid of points in the momentum space by an application of the Legendre transformation. One such $\hat{P}$ may be the 'chordal' type approximation outlined at the beginning of section 2 or a best fit plane approximation over each triangle. An irregular hexagonal grid may be formed by joining with a straight line any two points which are transformed from triangles with a common edge. These hexagons cover an irregularly shaped domain in the $m\theta$ plane which, as an alternative to using the Γ of (3.18), may be discretised into regular hexagons. The piecewise linear best fit to R can be found in this domain and used for a Legendre transformation back to the physical domain to give an irregular triangular grid. It may be that this grid covers a region of the xz plane which is more like the rectangular Ω than is the region covered by the grid of section 3.

The test functions of section 2 were initially defined on a domain $0 \leq x \leq 1$, $0 \leq z \leq 1$. Figures 2 and 3 show that the generated triangular grids do not extend over the whole domain. Applying a method similar to the one outlined above may generate grids which cover more of the original domain.

6 Conclusions

The Legendre transformation provides a means whereby an irregular triangular grid and a piecewise linear approximation to a function defined on that grid can be created in one space from a hexagonal discretisation of the dual space and a best fit approximation to the dual function. Whether this approximation is a better representation of the function than the 'chordal' type approximation mentioned in section 2 is open to question. In the analogy with the one dimensional problem of creating an irregular grid using the Legendre transformation, the regular hexagons in the two dimension dual space correspond to equally spaced points along the axis in the one dimension dual space.

An initial piecewise linear approximate solution of the semi-geostrophic equations can be derived using this method, the projection of each linear section of the solution onto the xz plane giving an initial distribution of air parcels.

Numerically the semi-geostrophic equations of motion can be solved easily in the dual momentum space. The problem of advancing the physical solution with time is that of solving a constrained minimisation, the minimisation being to approximately satisfy conservation of mass in the physical space subject to a constraint imposed by the necessary piecewise linearity of the function in the momentum space.

Problems exist with the boundaries - the approximate solution is derived on a domain which differs from that of the original problem. A possible remedy for this is proposed.

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8 References

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Figure 1: Legendre transformation from 3 hexagonal plates to a triangular plate in the dual space.

Figure 2: Grid for $P(x, z) = x^2 + z^5$.

Figure 3: Grid for $P(x,z) = e^{-8x} + e^{-8z}$.

Figure 4: Set of seven nodes for $q_i(R)$.