

## HOMOGENEOUS FUNCTIONS

**Graham S McDonald**

A Tutorial Module for learning to solve  
differential equations that involve  
homogeneous functions

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## 1. Theory

$M(x, y) = 3x^2 + xy$  is a **homogeneous function** since the sum of the powers of  $x$  and  $y$  in each term is the same (i.e.  $x^2$  is  $x$  to power 2 and  $xy = x^1y^1$  giving total power of  $1 + 1 = 2$ ).

The **degree** of this homogeneous function is 2.

Here, we consider differential equations with the following standard form:

$$\frac{dy}{dx} = \frac{M(x, y)}{N(x, y)}$$

where  $M$  and  $N$  are **homogeneous functions** of the **same degree**.

To find the solution, change the dependent variable from  $y$  to  $v$ , where

$$y = vx .$$

The LHS of the equation becomes:

$$\frac{dy}{dx} = x \frac{dv}{dx} + v$$

using the product rule for differentiation.

Solve the resulting equation by separating the variables  $v$  and  $x$ .

Finally, re-express the solution in terms of  $x$  and  $y$ .

Note. This method also works for equations of the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) .$$

## 2. Exercises

Click on [EXERCISE](#) links for full worked solutions (there are 11 exercises in total)

### EXERCISE 1.

Find the general solution of  $\frac{dy}{dx} = \frac{xy + y^2}{x^2}$

### EXERCISE 2.

Solve  $2xy \frac{dy}{dx} = x^2 + y^2$  given that  $y = 0$  at  $x = 1$

### EXERCISE 3.

Solve  $\frac{dy}{dx} = \frac{x + y}{x}$  and find the particular solution when  $y(1) = 1$

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## EXERCISE 4.

Solve  $x \frac{dy}{dx} = x - y$  and find the particular solution when  $y(2) = \frac{1}{2}$

## EXERCISE 5.

Solve  $\frac{dy}{dx} = \frac{x - 2y}{x}$  and find the particular solution when  $y(1) = -1$

## EXERCISE 6.

Given that  $\frac{dy}{dx} = \frac{x + y}{x - y}$ , prove that  $\tan^{-1} \left( \frac{y}{x} \right) = \frac{1}{2} \ln(x^2 + y^2) + A$ , where  $A$  is an arbitrary constant

## EXERCISE 7.

Find the general solution of  $2x^2 \frac{dy}{dx} = x^2 + y^2$

## EXERCISE 8.

Find the general solution of  $(2x - y) \frac{dy}{dx} = 2y - x$

**Note.** The key to solving the next three equations is to recognise that each equation can be written in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \equiv f(v)$$

## EXERCISE 9.

Find the general solution of  $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$

## EXERCISE 10.

Find the general solution of  $x \frac{dy}{dx} = y + xe^{\frac{y}{x}}$

## EXERCISE 11.

Find the general solution of  $x \frac{dy}{dx} = y + \sqrt{x^2 + y^2}$

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### 3. Answers

1. General solution is  $y = -\frac{x}{\ln x + C}$ ,
2. General solution is  $x = C(x^2 - y^2)$ , and particular solution is  $x = x^2 - y^2$ ,
3. General solution is  $y = x \ln(kx)$ , and particular solution is  $y = x + x \ln x$ ,
4. General solution is  $1 = Kx(x - 2y)$ , and particular solution is  $2xy - x^2 = -2$ ,
5. General solution is  $x^2(x - 3y) = K$ , and particular solution is  $x^2(x - 3y) = 4$ ,
6. HINT: Try changing the variables from  $(x, y)$  to  $(x, v)$ , where  $y = vx$ ,

7. General solution is  $2x = (x - y)(\ln x + C)$ ,

8. General solution is  $y - x = K(x + y)^3$ ,

9. General solution is  $\sin\left(\frac{y}{x}\right) = kx$ ,

10. General solution is  $y = -x \ln(-\ln kx)$ ,

11. General solution is  $\sinh^{-1}\left(\frac{y}{x}\right) = \ln x + C$ .

## 4. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
$x^n$	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln x $	$\frac{g'(x)}{g(x)}$	$\ln g(x) $
$e^x$	$e^x$	$a^x$	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln \cos x $	$\tanh x$	$\ln \cosh x$
$\operatorname{cosec} x$	$\ln\left \tan \frac{x}{2}\right $	$\operatorname{cosech} x$	$\ln\left \tanh \frac{x}{2}\right $
$\sec x$	$\ln \sec x + \tan x $	$\operatorname{sech} x$	$2 \tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln \sin x $	$\operatorname{coth} x$	$\ln \sinh x $
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x) dx$	$f(x)$	$\int f(x) dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$ $(a > 0)$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a} \ln \left  \frac{a+x}{a-x} \right $ ( $0 <  x  < a$ ) $\frac{1}{2a} \ln \left  \frac{x-a}{x+a} \right $ ( $ x  > a > 0$ )
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a}$ $(-a < x < a)$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln \left  \frac{x+\sqrt{a^2+x^2}}{a} \right $ ( $a > 0$ ) $\ln \left  \frac{x+\sqrt{x^2-a^2}}{a} \right $ ( $x > a > 0$ )
$\sqrt{a^2-x^2}$	$\frac{a^2}{2} \left[ \sin^{-1} \left( \frac{x}{a} \right) + \frac{x\sqrt{a^2-x^2}}{a^2} \right]$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2} \left[ \sinh^{-1} \left( \frac{x}{a} \right) + \frac{x\sqrt{a^2+x^2}}{a^2} \right]$ $\frac{a^2}{2} \left[ -\cosh^{-1} \left( \frac{x}{a} \right) + \frac{x\sqrt{x^2-a^2}}{a^2} \right]$

## 5. Tips on using solutions

- When looking at the THEORY, ANSWERS, INTEGRALS or TIPS pages, use the [Back](#) button (at the bottom of the page) to return to the exercises.
- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct.
- Try to make less use of the full solutions as you work your way through the Tutorial.

## Full worked solutions

### Exercise 1.

RHS = quotient of homogeneous functions of same degree (= 2)

Set  $y = vx$  : i.e.  $\frac{d}{dx}(vx) = \frac{xvx + v^2x^2}{x^2}$

i.e.  $x\frac{dv}{dx} + v = v + v^2$

Separate variables  $x\frac{dv}{dx} = v^2$  (subtract  $v$  from both sides)

and integrate :  $\int \frac{dv}{v^2} = \int \frac{dx}{x}$

i.e.  $-\frac{1}{v} = \ln x + C$

Re-express in terms of  $x, y$  :  $-\frac{x}{y} = \ln x + C$

i.e.  $y = \frac{-x}{\ln x + C}$ .

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**Exercise 2.**

Standard form: 
$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

i.e. quotient of homogeneous functions  
that have the same degree

Set  $y = xv$ : 
$$\frac{d}{dx}(xv) = \frac{x^2 + x^2v^2}{2x \cdot xv}$$

i.e. 
$$x \frac{dv}{dx} + \frac{dx}{dx}v = \frac{x^2(1 + v^2)}{2x^2v}$$

i.e. 
$$x \frac{dv}{dx} + v = \frac{1 + v^2}{2v}$$

Separate variables

$(x, v)$  and integrate: 
$$x \frac{dv}{dx} = \frac{1 + v^2}{2v} - \frac{v(2v)}{(2v)}$$

$$\text{i.e. } x \frac{dv}{dx} = \frac{1 - v^2}{2v}$$

$$\text{i.e. } \int \frac{2v}{1 - v^2} dv = \int \frac{dx}{x}$$

$$\left\{ \text{Note: } \frac{d}{dv}(1 - v^2) = -2v \right\} \text{ i.e. } - \int \frac{-2v}{1 - v^2} dv = \int \frac{dx}{x}$$

$$\text{i.e. } -\ln(1 - v^2) = \ln x + \ln C$$

$$\text{i.e. } \ln[(1 - v^2)^{-1}] = \ln(Cx)$$

$$\text{i.e. } \frac{1}{1 - v^2} = Cx$$





**Exercise 3.**

Set  $y = xv$ :

$$\begin{aligned}x \frac{dv}{dx} + v &= \frac{x + xv}{x} \\ &= \frac{x}{x} (1 + v) = 1 + v \\ \text{i.e. } x \frac{dv}{dx} &= 1\end{aligned}$$

Separate variables and integrate:

$$\begin{aligned}\int dv &= \int \frac{dx}{x} \\ \text{i.e. } v &= \ln x + \ln k \quad (\ln k = \text{constant}) \\ \text{i.e. } v &= \ln(kx)\end{aligned}$$

Re-express in terms of  $x$  and  $y$ :

$$\begin{aligned} \frac{y}{x} &= \ln(kx) \\ \text{i.e. } y &= x \ln(kx) . \end{aligned}$$

Particular solution with  $y = 1$  when  $x = 1$ :

$$\begin{aligned} 1 &= \ln(k) \\ \text{i.e. } k &= e^1 = e \\ \text{i.e. } y &= x \ln(ex) \\ &= x[\ln e + \ln x] \\ &= x[1 + \ln x] \\ \text{i.e. } y &= x + x \ln x . \end{aligned}$$

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**Exercise 4.**

$$\frac{dy}{dx} = \frac{x-y}{x} : \underline{\text{Set } y = vx} : \quad \text{i.e. } x \frac{dv}{dx} + v = 1 - v$$

$$\text{i.e. } x \frac{dv}{dx} = 1 - 2v \quad \text{i.e. } \int \frac{dv}{1-2v} = \int \frac{dx}{x}$$

$$\text{i.e. } -\frac{1}{2} \ln(1 - 2v) = \ln x + \ln k$$

$$\text{i.e. } \ln \left[ (1 - 2v)^{-\frac{1}{2}} \right] - \ln x = \ln k$$

$$\text{i.e. } \ln \left[ \frac{1}{(1-2v)^{\frac{1}{2}} x} \right] = \ln k$$

$$\text{i.e. } 1 = kx(1 - 2v)^{\frac{1}{2}}$$

$$\underline{\text{Re-express in } x, y:} \quad 1 = kx \left(1 - \frac{2y}{x}\right)^{\frac{1}{2}}$$

$$\text{i.e. } 1 = kx \left(\frac{x-2y}{x}\right)^{\frac{1}{2}}$$

$$\text{(square both sides)} \quad 1 = K x^2 \left(\frac{x-2y}{x}\right) \quad , \quad (k^2 = K)$$

$$\text{i.e. } 1 = K x(x - 2y)$$

$$\underline{\text{Particular solution:}} \quad 1 = K \cdot 2 \cdot \left(2 - 2\left(\frac{1}{2}\right)\right) = K \cdot 2 \cdot 1, \quad \text{i.e. } K = \frac{1}{2}$$

$$y(2) = \frac{1}{2} \quad \text{i.e. } \begin{array}{l} x = 2 \\ y = \frac{1}{2} \end{array} \quad \text{gives} \quad 2 = x^2 - 2xy.$$

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**Exercise 5.**

Set  $y = xv$ :

$$\begin{aligned}x \frac{dv}{dx} + v &= \frac{x - 2xv}{x} \\ &= 1 - 2v \\ \text{i.e. } x \frac{dv}{dx} &= 1 - 3v\end{aligned}$$

Separate variables and integrate:

$$\int \frac{dv}{1 - 3v} = \int \frac{dx}{x}$$

$$\text{i.e. } \frac{1}{(-3)} \ln(1 - 3v) = \ln x + \ln k \quad (\ln k = \text{constant})$$

$$\text{i.e. } \ln(1 - 3v) = -3 \ln x - 3 \ln k$$

$$\text{i.e. } \ln(1 - 3v) + \ln x^3 = -3 \ln k$$

$$\text{i.e. } \ln[x^3(1 - 3v)] = -3 \ln k$$

$$\text{i.e. } x^3(1 - 3v) = K \quad (K = \text{constant})$$

Re-express in terms of  $x$  and  $y$ :

$$\begin{aligned}x^3 \left(1 - \frac{3y}{x}\right) &= K \\ \text{i.e. } x^3 \left(\frac{x - 3y}{x}\right) &= K \\ \text{i.e. } x^2 (x - 3y) &= K .\end{aligned}$$

Particular solution with  $y(1) = -1$ :

$$\begin{aligned}1(1 + 3) &= K \quad \text{i.e. } K = 4 \\ \therefore x^2 (x - 3y) &= 4 .\end{aligned}$$

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**Exercise 6.**

Already in standard form, with quotient of two first degree homogeneous functions.

$$\underline{\text{Set } y = xv:} \quad x \frac{dv}{dx} + v = \frac{x + vx}{x - vx}$$

$$\text{i.e.} \quad x \frac{dv}{dx} = \frac{x(1 + v)}{x(1 - v)} - v$$

$$= \frac{1 + v - v(1 - v)}{1 - v}$$

$$\text{i.e.} \quad x \frac{dv}{dx} = \frac{1 + v^2}{1 - v}$$



Separate variables and integrate:

$$\int \frac{1-v}{1+v^2} dv = \int \frac{dx}{x}$$

i.e.  $\int \frac{dv}{1+v^2} - \frac{1}{2} \int \frac{2v}{1+v^2} = \int \frac{dx}{x}$

i.e.  $\tan^{-1} v - \frac{1}{2} \ln(1+v^2) = \ln x + A$

Re-express in terms of  $x$  and  $y$ :

$$\tan^{-1} \left( \frac{y}{x} \right) - \frac{1}{2} \ln \left( 1 + \frac{y^2}{x^2} \right) = \ln x + A$$

i.e.  $\tan^{-1} \left( \frac{y}{x} \right) = \frac{1}{2} \ln \left( \frac{x^2 + y^2}{x^2} \right) + \frac{1}{2} \ln x^2 + A$

$$= \frac{1}{2} \ln \left[ \left( \frac{x^2 + y^2}{x^2} \right) \cdot x^2 \right] + A$$

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**Exercise 7.**

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$$

Set  $y = xv$ :

$$\begin{aligned}x \frac{dv}{dx} + v &= \frac{x^2 + x^2v^2}{2x^2} \\ &= \frac{1 + v^2}{2} \\ \text{i.e. } x \frac{dv}{dx} &= \frac{1 + v^2}{2} - \frac{2v}{2} \\ &= \frac{1 + v^2 - 2v}{2}\end{aligned}$$

Separate variables and integrate:

$$\int \frac{dv}{1 - 2v + v^2} = \frac{1}{2} \int \frac{dx}{x}$$

i.e. 
$$\int \frac{dv}{(1 - v)^2} = \frac{1}{2} \int \frac{dx}{x}$$

[Note:  $1 - v$  is a linear function of  $v$ , therefore use standard integral and divide by coefficient of  $v$ . In other words,

$$w = 1 - v$$

$$\frac{dw}{dv} = -1 \quad \text{and} \quad \int \frac{dv}{(1-v)^2} = \frac{1}{(-1)} \int \frac{dw}{w^2}.]$$

i.e. 
$$-\int \frac{dw}{w^2} = \frac{1}{2} \int \frac{dx}{x}$$

i.e. 
$$-\left(-\frac{1}{w}\right) = \frac{1}{2} \ln x + C$$

i.e. 
$$\frac{1}{1 - v} = \frac{1}{2} \ln x + C$$

Re-express in terms of  $x$  and  $y$ :

$$\frac{1}{1 - \frac{y}{x}} = \frac{1}{2} \ln x + C$$

i.e. 
$$\frac{x}{x - y} = \frac{1}{2} \ln x + C$$

i.e. 
$$2x = (x - y)(\ln x + C'), \quad (C' = 2C).$$

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**Exercise 8.**

$$\frac{dy}{dx} = \frac{2y-x}{2x-y}. \quad \text{Set } y = vx, \quad x \frac{dv}{dx} + v = \frac{2v-1}{2-v}$$

$$\therefore x \frac{dv}{dx} = \frac{2v-1-v(2-v)}{2-v} = \frac{v^2-1}{2-v}; \quad \int \frac{2-v}{v^2-1} dv = \int \frac{dx}{x}$$

Partial fractions:  $\frac{2-v}{v^2-1} = \frac{A}{v-1} + \frac{B}{v+1} = \frac{A(v+1)+B(v-1)}{v^2-1}$

$$\text{i.e. } A + B = -1$$

$$A - B = 2$$

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$$2A = 1$$


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$$\text{i.e. } A = \frac{1}{2}, \quad B = -\frac{3}{2}$$

$$\text{i.e. } \frac{1}{2} \int \frac{1}{v-1} - \frac{3}{v+1} dv = \int \frac{dx}{x}$$

$$\text{i.e. } \frac{1}{2} \ln(v-1) - \frac{3}{2} \ln(v+1) = \ln x + \ln k$$

$$\text{i.e. } \ln \left[ \frac{(v-1)^{\frac{1}{2}}}{(v+1)^{\frac{3}{2}} x} \right] = \ln k$$

$$\text{i.e. } \frac{v-1}{(v+1)^3 x^2} = k^2$$

Re-express in  $x, y$ :

$$\frac{\left(\frac{y}{x} - 1\right)}{\left(\frac{y}{x} + 1\right)^3 x^2} = k^2$$

$$\text{i.e. } \frac{\left(\frac{y-x}{x}\right)}{\left(\frac{y+x}{x}\right)^3 x^2} = k^2$$

$$\text{i.e. } y - x = K(y + x)^3.$$

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**Exercise 9.**

RHS is only a function of  $v = \frac{y}{x}$ , so substitute and separate variables.

Set  $y = xv$ :

$$x \frac{dv}{dx} + v = v + \tan v$$

i.e.  $x \frac{dv}{dx} = \tan v$

Separate variables and integrate:

$$\int \frac{dv}{\tan v} = \int \frac{dx}{x}$$

{ Note:  $\int \frac{\cos v}{\sin v} dx \equiv \int \frac{f'(v)}{f(v)} dv = \ln[f(v)] + C$  }

$$\text{i.e. } \ln[\sin v] = \ln x + \ln k \quad (\ln k = \text{constant})$$

$$\text{i.e. } \ln \left[ \frac{\sin v}{x} \right] = \ln k$$

$$\text{i.e. } \frac{\sin v}{x} = k$$

$$\text{i.e. } \sin v = kx$$

Re-express in terms of  $x$  and  $y$ :  $\sin \left( \frac{y}{x} \right) = kx.$

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**Exercise 10.**

$$\frac{dy}{dx} = \left(\frac{y}{x}\right) + e^{\left(\frac{y}{x}\right)}$$

i.e. RHS is function of  $v = \frac{y}{x}$ , only.

Set  $y = vx$ :

$$x \frac{dv}{dx} + v = v + e^v$$

$$\text{i.e. } x \frac{dv}{dx} = e^v$$

$$\text{i.e. } \int e^{-v} dv = \int \frac{dx}{x}$$

$$\begin{aligned} \text{i.e. } -e^{-v} &= \ln x + \ln k \\ &= \ln(kx) \end{aligned}$$

$$\text{i.e. } e^{-v} = -\ln(kx)$$

Re-express in terms of  $x, y$ :

$$\begin{aligned} e^{-\frac{y}{x}} &= -\ln(kx) \\ \text{i.e. } -\frac{y}{x} &= \ln[-\ln(kx)] \\ \text{i.e. } y &= -x \ln[-\ln(kx)]. \end{aligned}$$

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**Exercise 11.**

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x} + \frac{1}{x}\sqrt{x^2 + y^2} \\ &= \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}\end{aligned}$$

[Note RHS is a function of only  $v = \frac{y}{x}$ , so substitute and separate the variables]

i.e. Set  $y = xv$ :

$$\begin{aligned}x \frac{dv}{dx} + v &= v + \sqrt{1 + v^2} \\ \text{i.e. } x \frac{dv}{dx} &= \sqrt{1 + v^2}\end{aligned}$$

Separate variables and integrate:

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\{ \text{Standard integral: } \int \frac{dv}{\sqrt{1+v^2}} = \sinh^{-1}(v) + C \}$$

$$\text{i.e. } \sinh^{-1}(v) = \ln x + A$$

Re-express in terms of  $x$  and  $y$

$$\sinh^{-1}\left(\frac{y}{x}\right) = \ln x + A .$$

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