# Module M1.1 Arithmetic and algebra



Exit module

# 1 Opening items

## 1.1 Module introduction

This module is about basic algebra and arithmetic. It assumes that you already have some familiarity with these topics and concentrates on giving you an opportunity to practise the skills you have already acquired, and to sharpen-up your use of the very precise terminology of mathematics.

As you work through this module remember that it has been designed as a tool to be used by *you* for your own benefit. You should not waste time on topics with which you are already fully familiar, but nor should you shy away from topics you have not completely mastered. Use the questions in the module to probe your own knowledge and ability, and use the text to rectify any deficiencies that you detect. By reinforcing your understanding of basic mathematics you will be preparing yourself for some of the more demanding mathematical challenges that you will inevitably meet as your studies progress.

The overall thrust of this module is towards *problem solving*—abstract principles and general definitions are certainly included, but their role is to support your development as a confident *user* of mathematics.

Section 1 contains the various *Opening items* that are a common feature of all *FLAP* modules. These are designed to give you a clear view of the module's content and the prior knowledge that it assumes, so that you can assess for yourself the extent to which you need to study the module and the degree to which you are prepared for such a study. Sections 2 and 3 contain the basic teaching material of the module, dealing with mathematical ***expressions*** and with relations between expressions, particularly ***equality*** and ***inequality***. Section 2 starts with a review of the basic ***operations*** (addition, subtraction, multiplication and division) together with their symbolic representation and order of priority in written expressions. This topic is taken further in Subsection 2.2 which concentrates on ***brackets*** and their use in simplifying and expanding expressions. The manipulation of algebraic and arithmetic ***fractions*** (a common source of errors) is discussed in Subsection 2.3, and the topic of ***powers***, ***roots*** and ***reciprocals*** is covered in Subsection 2.4. The writing and rearrangement of simple ***equations*** is dealt with in Subsections 3.1 and 3.2, while Subsection 3.3 looks at ***proportionality***. (Both ***direct*** and ***inverse*** proportionality are considered, together with the equations that reflect such relationships.) Subsection 3.4 deals with the subject of ***inequalities*** and reviews the rules for their manipulation. As is usual in *FLAP*, the module ends with a section of *Closing items* (Section 4) that includes a summary, a list of the things that you should be able to do on completing the module, and an *Exit text* designed to let you assess your achievements and alert you to any remaining difficulties. The answers to all the numbered questions included in the module can be found at the end of the module in Section 5.

***Study comment*** Having read the introduction you may feel that you are already familiar with the material covered by this module and that you do not need to study it. If so, try the *Fast track questions* given in Subsection 1.2. If not, proceed directly to *Ready to study?* in Subsection 1.3.

## 1.2 Fast track questions

***Study comment*** Can you answer the following *Fast track questions*? The answers are given in Section 5. If you answer the questions successfully you need only glance through the module before looking at the *Module summary* (Subsection 4.1) and the *Achievements* listed in Subsection 4.2. If you are sure that you can meet each of these achievements, try the *Exit test* in Subsection 4.3. If you have difficulty with only one or two of the questions you should follow the guidance given in the answers and read the relevant parts of the module. *However, if you have difficulty with more than two of the Exit questions you are strongly advised to study the whole module.*

### Question F1

$z = [2xy - (x + y)^2]^2$

(a) Evaluate $z$ when $x = 1$ and $y = 2$.

(b) Expand the expression for $z$ by removing all the brackets.

## Question F2

Write each of these expressions as a single fraction expressed in as simple a form as possible:

(a) $2mu - \dfrac{2}{u}d$ (b) $\dfrac{2l^2}{3h} + \dfrac{h}{2}$ (c) $\dfrac{3l(h+f)^2}{2a^2} \div \dfrac{3a\left(h^2 - f^2\right)}{-2l^2}$

## Question F3

A rocket of mass $m_1$ fired vertically at speed $\upsilon$ from the surface of a planet of mass $m_2$ and radius $r$ can escape from the planet provided its kinetic energy $m_1\upsilon^2/2$ is greater than $Gm_1m_2/r$. Derive an inequality, with $\upsilon$ as its subject, that must be satisfied if the rocket is to escape.

***Study comment*** Having seen the *Fast track questions* you may feel that it would be wiser to follow the normal route through the module and to proceed directly to *Ready to study?* in Subsection 1.3.

Alternatively, you may still be sufficiently comfortable with the material covered by the module to proceed directly to the *Closing items* .

If you have completed both the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.

## 1.3 Ready to study?

***Study comment*** Relatively little background knowledge is required to study this module, which deals with very basic topics in arithmetic and algebra. Nonetheless, it is assumed that you have previously performed a wide range of calculations involving the *addition*, *subtraction*, *multiplication* and *division* of *positive* and *negative quantities*, and that you have already encountered terms such as *fraction*, *power*, *square*, and *square root*. (In any case, these terms are defined in the *Glossary*.) In addition it is assumed that you are accustomed to using a calculator and that you have one available while working on this module.

Formulae from many different branches of physics have deliberately been used to illustrate the mathematical points made in this module. As a result you may find that you are asked to deal with some unfamiliar physical quantities. If this should happen, don't worry. You will find that the physical properties of the quantities concerned are of no real importance in this module and that any unfamiliar quantity can be treated as a simple mathematical 'unknown'. Moreover, if you are so inclined, you can consult the *Glossary* for further information about any unfamiliar quantity.

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# 2 Expressions

## 2.1 Terms, operations and priority

### *Basic operations*

The four basic **operations** of arithmetic are addition (+), subtraction (–), multiplication (×) and division (÷ or / ). Other operations include squaring (e.g. $x^2 = x \times x$) and taking square roots (e.g. $\sqrt{x^2} = x$). A mathematical **expression** is a combination of numbers and letters (which generally represent numbers) linked together by various operations. $5 \times 4 - 3$ and $3a - b/c$ are both examples of expressions ☞; the former is purely **arithmetic** (i.e. involving numbers only), the latter **algebraic** (i.e. involving numbers and letters).

When working with expressions it is crucially important that they should be unambiguous. Thus, when dealing with the expression $5 \times 4 - 3$, for example, you need to know whether it means ‘multiply 5 by 4 *then* subtract 3’ or ‘subtract 3 from 4 *then* multiply by 5’ — the outcome is different in the two cases. The potential ambiguity is removed by insisting on a standard order of priority for carrying out the basic operations.

> *Standard order of priority for basic operations*:
>
> multiplication and division, then addition and subtraction

**Question T1**

Using the standard order of priority evaluate the following expressions:

(a) $3 + 6/2 - 4$

(b) $4.2 \times 3.0 - 1.4$

(c) $ab + c/2$, where $a = 2$, $b = 1/2$ and $c = -4$. ❏

***A note on terminology*** For future reference, here are the definitions of some words that are often used in connection with the basic operations.

- The number that results from adding two (or more) numbers is their **sum**.
- The number that results from subtracting one number from another is their **difference**.
- The number that results from multiplying two numbers is their **product**.
- The number that results from dividing one number by another is their **quotient**.
- The numbers that are multiplied together in a product are **factors** of that product.
- The number by which a given number is divided in a quotient is called the **divisor**.

Each of these words may be applied to algebraic expressions in the same way that it is applied to arithmetic expressions.

## *Terms and ordering*

It is often useful to regard a given expression as consisting of several distinct **terms** each of which may be a number, a letter or even another expression. Strictly speaking, an expression should be treated as a *sum* of terms, so $5 \times 4 - 3$ should be viewed as a positive term $5 \times 4$ *added* to a negative term $-3$. However, in practice the word *term* is often used more loosely and an expression such as $3ab$ is spoken of as a *product* of terms.

The terms making up an expression can usually be ordered in a variety of different ways. For instance, $5 \times 4 - 3$ could be equally well written as $-3 + 5 \times 4$, though it may *not* be written $3 - 5 \times 4$, and $3ab$ can be rewritten as $3ba$ or even $a3b$ or $ab3$, though these last two would be regarded as ugly and unusual. The conventional way of ordering terms within an expression is best picked up by seeing examples such as those contained in this module. The basic rules that govern reordering are explored in the following question.

✦ Is the ordering of two terms, $a$ and $b$, immaterial if they are (a) multiplied, (b) added, (c) subtracted, (d) divided?

### *Using brackets to indicate priority*

How would you write an expression corresponding to the instruction ‘subtract 3 from 4 *then* multiply the result by 5’? You certainly can’t write it as $4 - 3 \times 5$ because (according to the standard order of priority) that would mean ‘multiply 3 by 5 and *then* subtract the result from 4’. One way to get round this problem is to use **brackets** to separate parts of an expression and to give the evaluation of one part priority over another. The standard order of priority still applies *within* the brackets but the bracketed part of the expression is treated as a single entity and its evaluation is given precedence over all other operations. Thus, the instruction ‘subtract 3 from 4 *then* multiply the result by 5’ can be written $(4 - 3) \times 5$, or more conventionally $5(4 - 3)$.

Quite often a bracketed expression includes a term that also involves brackets. In such a case one pair of brackets appears inside another pair as in $5(3 + 2(2 - 4))$. The two pairs of brackets are said to be **nested** and are dealt with by evaluating the contents of the innermost brackets first and then moving outwards. Thus

$$5(3 + 2(2 - 4)) = 5(3 + 2(-2)) = 5(3 - 4) = 5(-1) = -5$$

Sometimes brackets of different shapes ☞ are used to make it clearer which brackets are to be paired — the previous example might have been written $5[3 + 2(2 - 4)]$ — but this is not always done, even for complicated expressions.

**Question T2**

Evaluate the following expressions:

(a) $(6 + 3/(9 - 7))/2$

(b) $2[(3a - 4)b + 2a(3b + 1)]$, where $a = 1$ and $b = -1$

(c) $x\,\{[x - 2(x + 3)/(x - 1)] + [1 + (x + 3)/2]/x\}$, where $x = 2$. ❑

When expressions involve *powers*, such as squares and square roots, as well as brackets ☞ the overall order of priority is as follows.

> *Overall order of priority for operations*:
>
> 1 First deal with anything in brackets
>
> 2 then deal with powers (including roots and reciprocals)
>
> 3 then multiply and divide
>
> 4 then add and subtract.

**Question T3**

Evaluate the following expressions:

(a) $2 + 4/3^2$

(b) $2 + (4 - 3a)^2$, where $a = 1$

(c) $(2 + (2 + x)^2)^2$, where $x = 2$. ❏

## *Operations on your calculator*

Many calculators are designed to carry out processes in the correct order if you key in a calculation exactly as it is written. Explore your own calculator to find out whether it does this — use the examples given above, or invent some for yourself and do the calculations 'by hand' as well as on your calculator.

You may think that if your calculator does it all for you, there is no need to take much notice of the order in which operations should be carried out. But when you are manipulating algebraic expressions, and rearranging equations, it is important to know in exactly what order you can do, or undo, processes. (Manipulating algebraic expressions features throughout this module.) Also, it is always possible to make a mistake when using a calculator, so it is often useful to carry out a rough calculation to make sure that the answer on your calculator is sensible.

## 2.2 Expanding and simplifying

In the previous subsection, brackets were used to separate one part of an expression from the rest. Sometimes it is useful either to rewrite an expression by removing brackets (in other words, to **expand** the expression), or to use brackets to group parts of an expression together (in other words to **simplify** the expression).

### *Expanding*

$(3 + 4) \times 5$ means 'add 3 to 4, *then* multiply by 5'. In this case, 3 and 4 have been added, and their sum multiplied by 5, so the calculation could have been *expanded* as $3 \times 5 + 4 \times 5$. In cases like this, each and every term inside the brackets has to be multiplied by the factor outside. If the factor enclosed by brackets is a difference, rather than a sum, the expression can be expanded in exactly the same way. For example: $4 \times (7 - 5) = 4 \times 2 = 8$, and this could be rewritten as $4 \times 7 - 4 \times 5 = 28 - 20 = 8$.

If an expression is a product of two or more factors enclosed by brackets, it can still be expanded, but each term inside one pair of brackets must be multiplied by each term inside every other pair of brackets. For example: $(2 - 3)(4 - 5) = 2 \times 4 - 2 \times 5 - 3 \times 4 + 3 \times 5$. ☞

Note that when writing out an expansion it is advisable to treat it systematically, by working along from left to right as above, otherwise you risk forgetting or duplicating some of the terms.

Numbers were used in the examples above because they make it is easier to show what is happening, but in practice there is not often much point in expanding purely numerical expressions since it rarely makes the calculation any easier. Expanding algebraic expressions can, however, often help to simplify an expression or equation.

❖ Expand the following expressions:

(a) $-3(4x - 2)$, (b) $2(3x + 2y) - 4y$, (c) $3(x + 2(2x + 5))$.

Brackets can be removed from a division calculation in the same way as for multiplication. Each term inside the brackets must be divided by the divisor outside. For example:

$$(6 + 9)/3 = 6/3 + 9/3 = 2 + 3 = 5$$

**Question T4**

Expand the following expressions so that in each case the resulting expression does not involve brackets:

(a) $3x(4 + 3y) + 2x$, (b) $(x + 2y)^2$, (c) $(x + y)(x - y)$, (d) $(3x + 6y)/2$, (e) $(p + q)/q$,
(f) $(2xy)^2/2$. ❏

## *Collecting terms together*

Terms with a **common factor** (that is, a number or expression that is a factor of each) can be collected together using brackets. This is the reverse of the process shown above, and can often be used to simplify an expression, or at least to rewrite it in a more compact form. For example, all the terms in the expression $8x + 6y + 10$, have 2 as a common factor, so it could be written as $2(4x + 3y + 5)$. As another example: the expression $6x + 4x^2 + 8xy$ has $2x$ as a common factor, so could be rewritten as $2x(3 + 2x + 4y)$. The terms $2x$ and $4y$ also share a common factor of 2, so nested brackets could be used to collect these terms together, $2x(3 + 2(x + 2y))$.
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### Question T5

By finding the common factors, simplify the following expressions:

(a) $3x^2y + 6xy$, (b) $8x^3 + 4x^2y + 2xy^2 + 2x$. ❏

The following expansions arise frequently and are worth remembering

$$(a + b)^2 = a^2 + 2ab + b^2 \quad (1)$$

$$(a - b)^2 = a^2 - 2ab + b^2 \quad (2)$$

$$(a + b)(a - b) = a^2 - b^2 \quad (3)$$

## 2.3 Fractions

The word **fraction** probably conjures up in your mind numbers like 1/2, 3/4 and 7/3. All these fractions are **ratios** of two whole numbers which can be rewritten as a single number by dividing the number on top (the **numerator**) by the one on the bottom (the **denominator**) to find their quotient. But in physics you also have to deal with fractions that are ratios of two algebraic expressions, such as $Q/C$, $mv^2/r$, and $q_1q_2/(4\pi\varepsilon_0r^2)$. Algebraic fractions can be manipulated in exactly the same ways as arithmetic ones, and the terms numerator and denominator are still used for the expressions on top and underneath respectively.

Any number may be expressed as a fraction of any other number; 2, for instance, is 1/2 of 4, and 6 is 3/2 of 4. This observation provides the basis of the system of **percentages** in which one number is expressed as so many hundreths of another number. For example

$$2 = \frac{50}{100} \times 4 \quad \text{so we may say 2 is 50\% (read as 50 per cent) of 4}$$

More generally, if we want to express $a$ as a percentage of $b$ that percentage is given by $(a/b) \times 100$.

## *Simplifying fractions by cancelling common factors*

If the numerator and denominator of a fraction have a common factor, then that factor both multiplies and divides the rest of the fraction — and those two operations cancel each other out. **Cancelling** therefore allows common factors to be removed from both the numerator and denominator of a fraction without having any net effect. For example, in the case of 6/10, the numerator and denominator each have 2 as a factor and 6/10 can therefore be written as 3/5. The same principle applies to algebraic fractions such as $\pi r^2/(2\pi r)$, the ratio of the area of a circle to its circumference. The numerical constant π ☞ and the radius $r$ are both common factors of the numerator and the denominator, so they may be cancelled from both, leaving $r/2$.

**Question T6**

Simplify these fractions by cancelling the common factors in their numerators and denominators:

(a) $\dfrac{7}{42}$ (b) $\dfrac{6x}{2x^2}$ (c) $\dfrac{8(x+3)}{2x}$ ❑

**Question T7**

By first using brackets to group terms in the numerator and/or denominator, simplify the following fractions by cancelling common factors:

(a) $(x+2)(x+3)\big/\left(3x^2y+6xy\right)$, (b) $\left(3xy+3y^2\right)\big/\left(4x^2y+4x^3\right)$. ❑

Notice that in Question T7 you were able to cancel factors that were not just single numbers or symbols, and that using brackets to write the various expressions as products made these factors much easier to spot.

## *Arithmetic operations with fractions*

All the standard mathematical operations can be applied to fractions. *Multiplication* and *division* are especially easy — provided the problem is written out clearly in the first place. In general algebraic terms;

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \qquad (4)$$

and $$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} \qquad (5)$$

So, dividing by $(c/d)$ is the same as multiplying by $(d/c)$.

❖ Simplify the following fractions: (a) $(3/5) \div (5/2)$, (b) $(mv^2/2)/(r/2)$.

*Adding* and *subtracting* fractions is also straightforward provided they all have the same denominator — a so-called **common denominator**. Expressions such as 4/5 + 7/5, or 8/3 − 5/3 can be rewritten as (4 + 7)/5 or (8 − 5)/3. But if fractions that are to be added or subtracted do not share a common denominator, they can—and must — be rewritten in such a way as to give them one. For example, to add the fractions 4/5 and 8/3, you can write

$$\frac{4}{5}+\frac{8}{3}=\frac{(4\times 3)}{(5\times 3)}+\frac{(8\times 5)}{(3\times 5)}$$

Notice that the value of neither fraction has been changed, since all that has been done is to introduce a common factor into the numerator and denominator of each, also notice that the denominator of each fraction was used to rewrite the other. Now the addition can be written as

$$\frac{4}{5}+\frac{8}{3}=\frac{(4\times 3)+(8\times 5)}{(5\times 3)}$$

This technique can, of course, also be used when fractions are to be subtracted, and algebraic fractions are treated in exactly the same way. Thus, in general algebraic terms:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd} \qquad (6)$$

and $$\frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd} \qquad (7)$$

✥ Rewrite the fractions $3x/5$ and $2/y$ so that they have a common denominator and then write $(3x/5) - (2/y)$ as a single fraction.

## *Avoiding common mistakes*

The rules for manipulating fractions are fairly straightforward but under pressure some students panic when faced with fractions and make serious mistakes. Here is a short catalogue of common fallacies that you should try to avoid.

***Forgetting the common denominator*** It is not uncommon to see totally bogus additions such as:

$$\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}$$ it looks plausible, but it's *WRONG*.

***Incomplete cancelling*** It is also quite common to see half-hearted cancelling along the following lines:

$$\frac{1+ax}{x}=1+a$$ this too is *WRONG*.

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***Wrongly ordered divisions*** If asked to divide $(2ax/b)$ by $(2a/b)$ most students will find the correct answer $(x)$ without any difficulty. But different ways of presenting the same calculation can lead to mistakes. The following are all equal (if not equally attractive).

$$(2ax/b)/(2a/b) = \frac{2ax/b}{2a/b} = \frac{\dfrac{2ax}{b}}{\dfrac{2a}{b}} = \frac{2ax}{b(2a/b)} = \frac{2axb}{2ab} = x$$

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**Question T8**

Simplify the following, if possible:

(a) $\dfrac{2}{3} + \dfrac{3}{5}$, (b) $\dfrac{x}{3} + \dfrac{x}{2}$, (c) $(1/u) + (1/v)$, (d) $\dfrac{1+a}{3+a}$, (e) $\dfrac{2(1+x)}{(1+x)^2}$,

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(f) $\dfrac{5/7}{2/3}$, (g) $\dfrac{3x}{x/3}$, (h) $\dfrac{3}{a} + \dfrac{7}{2a}$, (i) $(-7x^2) \div (-3x)$. ❏

## 2.4 Powers, roots and reciprocals

A **power** (often called an **index**) indicates repeated multiplication as in:

| | | |
|---|---|---|
| squares | $a^2 = a \times a$ | (read as '$a$ squared') |
| cubes | $a^3 = a \times a \times a$ | (read as '$a$ cubed') |

or, generally

| | | |
|---|---|---|
| $n$th powers | $a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ factors}}$ | (read as '$a$ to the $n$') |

(Note the use of the three dots (…), known as an **ellipsis**, to indicate that the sequence continues in a similar fashion.)

Powers (or indices) enable many expressions to be written neatly and compactly. For instance, Kepler's third law, which relates the time $T$ that a planet requires to orbit the Sun to the mean distance $R$ between that planet and the Sun, can be written

$$T^2 = kR^3 \qquad (8)$$

where $k$ is a constant. Relationships of this kind, that relate one quantity to a power of another are often called **power laws**.

The use of powers is simple and straightforward as long as the power is a positive whole number (1, 2, 3, etc.), but you will encounter many situations in which the power is neither positive nor a whole number. Fortunately, the general rules for manipulating powers are simply extensions of the rules that apply to positive whole number powers, so let's start with those.

It follows directly from the definition of $a^n$ that if $m$ and $n$ are positive whole numbers

$$a^m \times a^n = \underbrace{a \times a \times \ldots \times a}_{m \text{ factors}} \times \underbrace{a \times a \times \ldots \times a}_{n \text{ factors}}$$

Now, the right-hand side of this equation consists of $m + n$ factors of $a$, so it may be written $a^{m+n}$. Thus,

$$a^m \times a^n = a^{m+n} \qquad (9)$$

Similarly, raising $a^m$ to the power $n$: gives

$$(a^m)^n = \underbrace{\overbrace{(a \times a \times \ldots \times a)}^{m \text{ factors of } a} \times (a \times a \times \ldots \times a) \times \ldots \times (a \times a \times \ldots \times a)}_{n \text{ factors of } a^m}$$

The right-hand side now contains $m \times n$ factors of $a$. So,

$$(a^m)^n = a^{m \times n} = a^{mn} \qquad (10)$$

**Question T9**

Using the boxed equations given above, simplify and then evaluate the following expressions:

(a) $2^2 \times 2^3$, (b) $2^3 \times 2^4 \times 2^5$, (c) $(-2)^4 \times (-2)^2$, (d) $(2^3)^2$, (e) $((-0.2)^2)^3$,

(f) $((-1)^{17})^{23}$. ❑

## *Negative powers and reciprocals*

We can make sense of negative powers by extending the pattern of behaviour that we find for a sequence of terms such as $a^2$, $a^3$, $a^4$ … and so on. For the sake of having a concrete example we will assume that $a = 5$, but the general results that we deduce will be independent of this particular choice and would be equally true for any value of $a$ other than zero. Given our chosen value of $a$, the first few terms in the sequence $a^2$, $a^3$, $a^4$… are

$$5 \times 5 \qquad 5 \times 5 \times 5 \qquad 5 \times 5 \times 5 \times 5 \qquad \dots$$

To go from one term to the next in the sequence $a^2$, $a^3$, $a^4$… you simply *raise* the power by 1, that's to say you *multiply* the previous term by $a$, or 5 in this case.

✦ Now suppose that the sequence is written in reverse order: … $a^4$, $a^3$, $a^2$. What must you now do to go from one term to the next, working from left to right?

It is useful to continue this sequence of decreasing powers for a few steps:

$$\ldots\ a^3, a^2, a^1, a^0, a^{-1}, a^{-2}\ \ldots$$

and to evaluate some of the terms using our chosen value, $a = 5$.

Dividing $5^2$ by 5 gives 5, so $5^1 = 5$.

Dividing $5^1$ by 5 gives 1, so $5^0 = 1$.

Dividing $5^0$ by 5 gives $\dfrac{1}{5}$, so $5^{-1} = \dfrac{1}{5} = 0.2$.

Dividing $5^{-1}$ by 5 gives $\dfrac{1}{5^2}$, so $5^{-2} = \dfrac{1}{5^2} = 0.04$.

You might like to repeat this process using some other value of $a$ (especially a negative value such as $a = -2$) but as long as $a$ is not zero the pattern should always be the same and you should always find the following:

$$a^1 = a, \qquad a^0 = 1, \qquad a^{-1} = 1/a, \qquad a^{-2} = 1/a^2$$

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Given any non-zero quantity $a$, the quantity $1/a$ is called the **reciprocal** of $a$. So our deductions have shown that the reciprocal of a quantity $a$ can be written $a^{-1}$.

✦ Write down the reciprocal of $a^2$ in two different ways.

If the power notation is to make sense we must require these two alternative ways of writing the reciprocal to be equal. But we already know that $1/a^2$ may be written as $a^{-2}$, so

$$(a^2)^{-1} = 1/a^2 = a^{-2}.$$

This is an example of a more general result that works for any value of $n$.

$$(a^n)^{-1} = 1/a^n = a^{-n} \qquad (11)$$

## Question T10

Evaluate the following: (a) $2^{-4}$ (b) $10^{-3}$ (c) $\left(\dfrac{1}{2}\right)^{-3}$ (d) $(0.2)^{-2}$ ❏

The use of negative powers to indicate reciprocals is consistent with the two general rules that were introduced earlier (Equations 9 and 10).

$$a^m \times a^n = a^{m+n} \qquad \text{(Eqn 9)}$$

$$(a^m)^n = a^{m \times n} = a^{mn} \qquad \text{(Eqn 10)}$$

For example, we know that if $m$ and $n$ are positive whole numbers

$$a^m \times a^n = a^{m+n}$$

but even if $m$ or $n$ (or both) are negative, the rule still works

$$2^3 \times 2^{-5} = (2 \times 2 \times 2) \times \frac{1}{2 \times 2 \times 2 \times 2 \times 2} = \frac{1}{2 \times 2} = \frac{1}{2^2} = 2^{-2}$$

which is exactly the result we would expect from the general rule (Equation 9).

Similarly, for the second rule, $(a^m)^n = a^{mn}$

$(2^2)^{-4} = \dfrac{1}{(2^2)^4} = \dfrac{1}{2^8} = 2^{-8}$, which is also the result we would expect from the general rule (Equation 10).

**Question T11**

Express each of the following in the form $a^n$: (a) $5^2 \times 5^{-5}$

(b) $(5^2)^{-3}$ (c) $(2^4)/(2^6)$ (d) $\left(3^{-1}\right)^{-1}$ (e) $\left(\dfrac{1}{2}\right)^{-3}$ (f) $\left(\dfrac{2^5}{2^6}\right)^{-1} \times 2 \times 2^3$ ❑

Negative powers, or powers appearing in the denominator of an expression, are often referred to as **inverse powers,** so Newton's law of gravitation, which asserts that the strength of the gravitational force attracting a mass $m_1$ towards another mass $m_2$ a distance $r$ away is

$$F = \frac{Gm_1m_2}{r^2}$$

is often described as an *inverse square law*.

## *Fractional powers and roots*

Although Kepler's third law ($T^2 = kR^3$) is a valuable result, it is often even more valuable to know the way in which $T$ itself (rather than $T^2$) is related to $k$ and $R$. Such a relation is easily obtained from Kepler's law by taking *square roots*:

$$T = \sqrt{kR^3}$$

The root symbol $(\sqrt{\ })$ can be used to indicate a variety of **roots** such as

square roots: $\sqrt{a}\sqrt{a} = a$

cube roots: $\sqrt[3]{a}\sqrt[3]{a}\sqrt[3]{a} = a$

or, generally, *n*th roots: $\underbrace{\sqrt[n]{a}\sqrt[n]{a}\ldots\sqrt[n]{a}}_{n\text{ factors}} = a$

In fact, it is somewhat unusual to see cube (or higher) roots written in this form since most physicists prefer to make use of another notation based on fractional powers. According to this alternative notation

$$\sqrt[n]{a} = a^{1/n} \tag{12}$$

Once again this new notation is consistent with the rules given earlier. For example, we know that $8^{1/3} = 2$ since $2 \times 2 \times 2 = 8$, but the second of our rules for powers ($(a^m)^n = a^{mn}$) would lead us to expect that

$$(8^{1/3})^3 = 8^{(1/3)\times 3} = 8^1 = 8$$

which is indeed the case since $8^{1/3} = 2$ and $2^3 = 8$.

All this may seem fairly straightforward, but, in fact, a certain degree of care is needed when dealing with roots and fractional powers. In particular, the following points should be noted.

### *(i) Positive numbers have two square roots*

It is true that $4 = 2 \times 2$, but it is equally true that $4 = (-2) \times (-2)$. So, both 2 *and* –2 are square roots of 4. Thus we may write $4^{1/2} = 2$ or $4^{1/2} = -2$. These two statements may be combined in the form

$$4^{1/2} = \pm 2$$

where the symbol ± is to be read as '+ or –'. Some authors introduce a convention whereby $a^{1/2}$ or $\sqrt{a}$ or both are always used to represent the *positive* square root of $a$ — *FLAP* does not use such a convention, though negative square roots will be ignored when there is a physical reason for doing so. (For instance, we know physically that quantities such as mass and length cannot be negative.)

The need to remember that $a^{1/n}$ may be positive or negative is not restricted to the case of square roots where $n = 2$ — it applies to all cases where $n$ is an even whole number.

### *(ii) Negative numbers do not have square roots*

Some negative numbers certainly have roots for instance:

$$-27 = (-3) \times (-3) \times (-3) \qquad \text{So } (-27)^{1/3} = -3$$

But it is not possible to find any number $a$, positive or negative, such that

$$a^2 = -4 \qquad \text{or} \qquad a^2 = -0.1$$

Indeed, it is generally said that negative quantities do not have square roots. Nor, in the same spirit, is it possible to find the $n$th root of any negative quantity unless $n$ is an odd whole number. ☞

**Question T12**

Evaluate the following: (a) $9^{1/2}$ (b) $16^{1/2}$ (c) $16^{1/4}$

(d) $27^{1/3}$ (e) $\sqrt{64}$ (f) $\sqrt[4]{49}$ ❏

**Question T13**

Write down the following roots, or explain why they cannot be found:

(a) $(-8)^{1/3}$   (b) $\left(-\dfrac{1}{16}\right)^{1/2}$   (c) $(-2)^{1/2}$ ❑

## *Working with general powers*

We have seen that, subject to certain cautionary notes, the rules for manipulating negative and fractional powers are similar to those for powers that are positive whole numbers. In fact the general rules for manipulating powers are:

$$a^p \times a^q = a^{p+q} \qquad (13)$$

and $$(a^p)^q = a^{pq} \qquad (14)$$

✦ Use the above rules to derive a similar rule expressing the *quotient* $a^p/a^q$ as a power of $a$.

✦ How would you interpret $a^{p/q}$ in words?

**Question T14**

Evaluate the following:

(a) $7^{4/9}$ (b) $(2.7)^3/(2.7)^2$ (c) $\sqrt{2} \times \sqrt[3]{2}$ (d) $\dfrac{\sqrt{3}}{\sqrt[5]{3}}$ ❏

Quite apart from numerical evaluations, the rules for powers allow algebraic relations and physical laws to be written in convenient forms. For example, if you are dealing with a formula that involves a term $r\sqrt{r}$ you might well find it more convenient (and elegant) to rewrite the term as $r^{3/2}$ or even $r^{1.5}$. Such rewriting might also make the term easier to evaluate since many calculators have an $x^y$ button but none are likely to have an $r\sqrt{r}$ button.

### Question T15

Rewrite the following in the form $r^p$:

(a) $1/\sqrt{r}$ (b) $r^3\sqrt{r}$ (c) $r^5/\sqrt{r}$ ❏

### Question T16

Simplify the following expressions:

(a) $4x^2y^2z^4$ (b) $\dfrac{h^2n^2}{8m}\left(\sqrt[3]{L^3}\right)^{-2}$ (c) $\left(\dfrac{8\pi^2 Z\sqrt{l^5}}{3\sqrt[3]{Z}}\right)^{1/2}$ ❏

# 3 Equations and inequalities

In Section 2 we dealt with simplifying and rewriting expressions, but an expression on its own is not usually very useful. You are much more likely to deal with an **equation** — a statement that two expressions are equal, such as

$$s = ut + \frac{1}{2}at^2 \qquad (15)$$

It is worth stressing that the = sign means 'has the same value as', so, while the expressions on either side of the equation may look very different, they actually represent *exactly the same* number. This holds true even for algebraic equations—the letters in such an equation simply stand for the numbers that make the two sides equal.

Equations generally involve **constants** (i.e. quantities with a fixed and pre-determined value) and **variables** (i.e. quantities that may take on any one of a range of values). Often, an equation will relate the value of one variable to the value (or values) of one or more other variables. Thus, in Equation 15, if the values of $u$, $a$ and $t$ are specified then the value of $s$ may be determined. In this way, Equation 15 provides both a general relationship between the variables $s$, $u$, $a$ and $t$, and a recipe for calculating the value of $s$ when $u$, $a$ and $t$ have particular known values. In the case of Equation 15, $s$ is said to be the **subject** of the equation since it is the single variable that is expressed in terms of the others.

## 3.1 Simplifying equations

In Section 2, you met various ways of using brackets and common factors to simplify expressions. These same techniques can be used to simplify equations, as the following example illustrates.

When a drop of water falls vertically through air three forces act on it: a downward gravitational force of strength $F_{\rm g}$, an upward buoyancy force of strength $F_{\rm b}$ (due to air displacement), and another upward force (due to air resistance — or more properly viscosity) of strength $F_{\rm v}$. The strengths of these forces are determined by the following equations:

$$F_{\rm g} = \frac{4}{3}\pi r^3 \rho g \qquad F_{\rm b} = \frac{4}{3}\pi r^3 \sigma g \qquad F_{\rm v} = 6\pi\eta r \upsilon$$ ☞

where $r$ is the radius of the water drop, $\rho$ is its density, $g$ is a constant known as (the magnitude of) the *acceleration due to gravity*, $\sigma$ is the *density* of air, $\eta$ is a constant known as the *viscosity* of air and $\upsilon$ is the speed of the drop.

✦ If the directions of the forces are taken into account, the strength of the total force acting on the falling drop is $F = F_{\rm g} - F_{\rm b} - F_{\rm v}$. Express $F$ in terms of $r$, $g$, $\rho$, $\sigma$, $\eta$ and $\upsilon$, and then simplify it as much as possible.

## Question T17

Show (justifying each step) that

$$\phi = \frac{b^2 Vr\theta}{b^2 - a^2} - \frac{b^2 Va^2\theta}{r\left(b^2 - a^2\right)} - \frac{a^2 Ur\theta}{b^2 - a^2} + \frac{a^2 Ub^2\theta}{r\left(b^2 - a^2\right)}$$

may be simplified to give

$$\phi = \left[b^2 V\left(r - \frac{a^2}{r}\right) - a^2 U\left(r - \frac{b^2}{r}\right)\right]\frac{\theta}{b^2 - a^2}.$$ ❏

## 3.2 Rearranging equations

You will often need to rearrange equations, perhaps to simplify them, or to ensure that they express relationships in a more illuminating way, or simply because you want to calculate the value of a particular variable that is not already the subject of the equation. Whatever the reason, the **rearrangement** should provide a different but equivalent relationship between the relevant variables.

A given equation can usually be rearranged in a variety of ways. The important principle to remember while performing the rearrangement is that *both sides of the equation represent the same number*. So, if you add or subtract a term to one side of an equation you must add or subtract an identical term to the other side of the equation. The same principle applies to multiplication and division and to other operations such as squaring or taking square roots. (In the latter case you must remember that a positive number has both a positive and a negative square root.) When rearranging equations you should bear in mind two important points:

1. When one side of an equation consists of the sum of several terms, each of those terms must be treated in the same way. (It’s no good just multiplying the first term on one side of an equation by some factor if you forget to do the same to all the other terms on that side.)
2. When dividing both sides of an equation by some divisor you must ensure that divisor is not equal to zero. (You wouldn’t expect to get a meaningful result if you divided a number by zero, so you shouldn’t try to do it to equations either.)

✧ The following 'proof' shows that 1 = 2. It is, of course, quite wrong. Spot the error.

Let $a$ and $b$ represent two numbers and suppose $a = b$.

Multiply both sides by $a$, $\quad a^2 = ab$

Subtract $b^2$ from both sides, $\quad a^2 - b^2 = ab - b^2$

Extract a common factor of $(a - b)$ from both sides,

$$(a - b)(a + b) = b(a - b)$$

Divide both sides by $(a - b)$, $\quad (a + b) = b$

But, since $a = b$, this implies that $\quad 2b = b$

Finally, divide both sides by $b$, $\quad 2 = 1$

In general, if both sides of an equation are divided by an algebraic expression such as $(a - b)$ then the result of that division (and anything deduced from that result) will only be valid if $a \neq b$.

The following questions will give you some practice in rearranging equations. The first two make further use of the example of the falling water drop that was introduced in Subsection 3.1.

**Question T18**

Initially, as a drop of water falls through air its speed $\upsilon$ will increase and $F_v$ will become larger. Consequently the value of $F$ $(= F_g - F_b - F_v)$ will decrease until $\upsilon$ becomes so large that $F = 0$. The speed at which this happens is called the *terminal speed* and is denoted $\upsilon_t$. Find as simple an expression as possible for $\upsilon_t$ in terms of $r$, $g$, $\rho$, $\sigma$ and $\eta$. ❏

**Question T19**

The relative importance of $F_g$, $F_b$ and $F_v$ under various circumstances can be determined by studying the ratios $F_b/F_g$, $F_v/F_g$ and $F_b/F_v$.

(a) Work out fully simplified equations relating each of these ratios to $r$, $g$, $\rho$, $\sigma$, $\eta$ and $\upsilon$.

(b) If $F_v$ is ten times greater than $F_b$ when the drop reaches its terminal speed $(\upsilon = \upsilon_t)$ find an equation that expresses the radius of the drop in terms of $\sigma$, $g$, $\eta$ and $\upsilon_t$.

(c) If $F_b$ is negligibly small when $\upsilon = \upsilon_t$ then $F_g = F_v$. Under these circumstances, find an equation that expresses $r$ in terms of $\rho$, $g$, $\eta$ and $\upsilon_t$. ❏

## Question T20

Rearrange the following equations to make $x$ the subject:

(a) $y = 2ax + b^2$

(b) $(y - b) + (x - a) = 5xh^2 + 3$

(c) $\dfrac{1}{x-a} + \dfrac{1}{y-b} = 3t^2$

(d) $t + a = \sqrt{\dfrac{3b}{x-y}}$ ❏

Once you have completed the rearrangement of an equation it is often a good idea to check that the result you have obtained is consistent with the original equation. This can often be done quite easily by substituting some simple numbers for the various algebraic quantities that enter the equation. For instance, suppose you start out with the equation

$$v^2 = u^2 + 2as$$

and you rearrange it to find

$$s = \frac{v^2 - u^2}{2a}$$

As a simple check of the correctness of this rearrangement pick some simple values for $u$, $a$ and $s$ in the original equation and substitute them into the equation to find $v^2$.

Note that these values don't have to be realistic nor do they involve units of measurement — we're just checking for mathematical consistency. Letting $u = 2$ (i.e. $u^2 = 4$), $a = 3$ and $s = 4$, for example, gives $v^2 = 2^2 + 2 \times 3 \times 4 = 28$. Now, if these values are substituted into the rearranged equation we get $s = (28 - 4)/(2 \times 3) = 4$, which was exactly the value of $s$ we chose. By showing that the same set of values for $u^2$, $v^2$, $a$ and $s$ satisfy both of the equations we have not *proved* that the rearrangement is correct but at least we haven't found any inconsistency, so there's a good chance that the rearrangement is right.

✥ Use the method outlined above to show that $v^2 = u^2 + 2as$ may not be rearranged to give

$$\frac{v^2}{2s} = u^2 + a.$$

**Question T21**

Rearrange the equation $F = Gm_1m_2/r^2$ to make $r$ the subject, and check your result using small whole numbers before looking at the answer. ❑

## 3.3 Equations and proportionality

Many important relationships in physics involve just two variables. For example, an object of fixed mass $m$ travelling at speed $\upsilon$ has momentum of magnitude $p$ given by

$$p = m\upsilon \qquad (17)$$

Similarly, light travelling at fixed speed $c$ through a vacuum is characterized by a frequency $f$ and a wavelength $\lambda$ that are related by

$$f\lambda = c \qquad (18)$$

In these two examples $m$ and $c$ are constants, so only two variables are involved in each case.

In the case of Equation 17, the variables $p$ and $\upsilon$ are said to be **directly proportional** (or just **proportional**). This indicates that if $\upsilon$ is multiplied by some factor then $p$ must be multiplied by the same factor. Or, to put it another way, the *ratio* $p/\upsilon$ is constant.

In Equation 18 the relationship between $f$ and $\lambda$ is very different. Now, it is the *product* $f\lambda$ that is constant with the consequence that if $f$ is multiplied by a given factor then $\lambda$ must be divided by that same factor. This situation is described by saying that $f$ and $\lambda$ are **inversely proportional**.

Direct proportionality and inverse proportionality are closely related, as can be seen by dividing both sides of Equation 18 by $\lambda$ to obtain

$$f = \frac{c}{\lambda}$$

This shows that if $f$ and $\lambda$ are inversely proportional then $f$ and $(1/\lambda)$ are directly proportional.

The symbol $\propto$ is used to mean 'is directly proportional to' ☞ so the proportionalities embodied in Equations 17 and 18 can be written as

$$p \propto v \quad \text{and} \quad f \propto \frac{1}{\lambda}$$

Conversely, a proportional relationship may always be represented by an equation, as follows:

$$\text{If } y \propto x \quad \text{then} \quad y = Kx \quad \text{where } K \text{ is a constant} \qquad (19)$$

A constant introduced in this way is generally referred to as a **constant of proportionality**. In Equation 17, $m$ is the constant of proportionality relating $p$ to $v$, while in Equation 18 $c$ is the constant of proportionality relating $f$ to $1/\lambda$.

Of course, physical relationships often involve more than two variables. For example, the pressure $P$, volume $V$ and temperature $T$ of a low-density sample of gas obey (at least approximately) an equation of the form

$$PV = NkT$$

where $N$ is the number of particles in the sample and $k$ is a constant ☞. Despite the relative complexity of this relationship the idea of proportionality is still very useful in this context, provided we keep the full relationship in mind. For instance, it is quite correct to say that $V$ is proportional to $T$ *provided $P$ and $N$ are kept constant*, and it is equally correct to say that $P$ and $V$ are inversely proportional *provided $N$ and $T$ are kept constant.*

**Question T22**

Write down equations that embody the following relationships. (You will have to introduce constants of proportionality for yourself; choose an appropriate letter and make a note to remind yourself that the newly introduced letter represents a constant.) ☞

(a) $V \propto I$

(b) $R \propto L$ provided $r$ is constant, and $R \propto 1/r^2$ provided $L$ is constant.

(c) $(E + \phi) \propto f$

(d) $F_{\rm grav} \propto 1/r^2$ provided $m_1$ and $m_2$ are constant, $F_{\rm grav} \propto m_1$ provided $r$ and $m_2$ are constant and $F_{\rm grav} \propto m_2$ provided $r$ and $m_1$ are constant. ❏

## 3.4 Inequalities

As you have seen, equations are often used to specify the precise value of a quantity, but it is sometimes useful to be able to express the *range* of values that a variable might take. For example, you might have a variable power-supply unit incorporating a circuit breaker that trips if the current reaches a certain maximum value $I_{max}$. The condition that the current must be less than $I_{max}$ can be written as $I < I_{max}$: the symbol < means 'is less than'. The symbol ≤ means 'is less than or equal to'.

✦ What do you think the symbols > and ≥ mean?

A statement that uses any of the symbols <, ≤, > or ≥ to compare two values is called an **inequality**. Note that whichever symbol is used, the greater of the two quantities is always at the wider end of the symbol. So, you might correctly write 3 < 6 or 6 > 3 but *not* 6 < 3 or 3 > 6.

When using an inequality it is important to remember that any negative number is considered to be less than any positive number. In fact, numbers are treated as if they are laid out along a line, sometimes called the **number line**, with increasingly large positive numbers further and further to the right of zero, and increasingly large negative numbers further and further to the left of zero. Any given point on the number line then represents a number that is greater than the numbers represented by points to the left of the given point.

| lesser | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 | greater |
|---|---|---|---|---|---|---|---|---|---|---|

**Question T23**

Which of the following inequalities are correct?

(a) $3 > -4$ (b) $-4 > -16$ (c) $(-4)^2 > 8$ (d) $-10 \le -8$ (e) $\dfrac{1}{2} \le 0.5$

(f) $\dfrac{1}{2} < \dfrac{1}{8}$ (g) $\sqrt{2} > 1.40$ (h) $-1 > 0$ ❏

Inequality symbols can also be used to express both ends of a range of values. For example the statement $3 < x < 10$ is read as '$x$ is greater than 3 *and* less than 10', in other words '$x$ is in the range 3 to 10 — *excluding* the 'endpoints', 3 and 10.'

✥ Use inequality symbols to represent the statement ‘$x$ is in the range 3 to 10 — *including* the endpoint values 3 and 10’.

## *Rearranging inequalities*

In practice, purely arithmetic inequalities such as $2 > 1$ are rarely written down since they are ‘obvious’. It is much more likely that you will have to deal with algebraic inequalities such as $x > b$ or $2x \geq b - y$. It is quite likely that you will be called on to rearrange such inequalities, either to simplify them or possibly to change their subject. The procedure is broadly similar to that for rearranging equations, but much greater care is needed when dealing with inequalities.

The basic rules for manipulating inequalities that involve the symbol $>$ are given below. Similar rules concerning the other inequality symbols may be obtained by replacing the $>$ by $\geq$, $<$ or $\leq$ throughout and simultaneously replacing the $<$ by $\leq$, $>$ or $\geq$ throughout.

*Rules for manipulating inequalities*:

1 If $x > y$ and $a$ is any number, then $x + a > y + a$.

2 If $x > y$ and $k$ is a positive number, then $kx > ky$.

3 If $x > y$ and $k$ is a negative number, then $kx < ky$.

Pay particular attention to Rule 3: if both sides of an inequality are multiplied by the same *negative* quantity then the inequality must be reversed. (So, if $2 < 3$ then $(-2)(2) > (-2)(3)$, i.e. $-4 > -6$, which is correct.)

❖ What problems might arise from the following actions?

(a) Divide both sides of an inequality by $(1 - a)$ when $a = 1$.

(b) Multiply both sides of an inequality by $(1 - a)$ when $a = 1$.

(c) Square both sides of an inequality.

(d) Take square roots of both sides of an inequality.

Obviously, the most important thing to do when manipulating inequalities is to *think carefully* about each step. Here are some questions that require you to do just that.

**Question T24**

Make $x$ (alone) the subject of each of the following inequalities:

(a) $2x + 4 > 6$

(b) $2(a - x) \le 7$

(c) $2(a - x) \le 3x + b$. ❏

**Question T25**

The electrical resistance $R$ of a piece of wire of length $l$ and cross-sectional area $A$, made from a material of resistivity $\rho$ is given by the expression $\rho l/A$. Suppose you need a sample of wire with resistance $R$ such that $R > R_0$. Write down inequalities expressing, (a) the range of acceptable lengths for a sample of given $\rho$ and $A$, and (b) the range of acceptable areas, for a sample of given $l$ and $\rho$. ❏

# 4 Closing items

## 4.1 Module summary

1 The overall order of priority for *operations* is

- first deal with anything in brackets
- then deal with powers (including roots and reciprocals)
- then multiply and divide
- then add and subtract.

2 *Expressions* containing a product of bracketed factors can be *expanded* by multiplying each term inside one pair of brackets by all the terms in the other pair of brackets, e.g.

$(a + b)(c + d) = ac + ad + bc + bd$ $\qquad$ $(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)(a - b) = a^2 - b^2$ $\qquad$ $(a - b)^2 = a^2 - 2ab + b^2$

3 *Expressions* can be *simplified* by using brackets to group together terms with a *common factor*, e.g. $ab + ac = a(b + c)$.

4 *Fractions* may be multiplied and divided according to the following rules:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \qquad \text{(Eqn 5)}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} \qquad \text{(Eqn 6)}$$

5 *Fractions* may be added or subtracted by rewriting them so that they have a *common denominator*:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd} \qquad \text{(Eqn 7)}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd} \qquad \text{(Eqn 8)}$$

6 *Powers* may be used to express *roots* and *reciprocals*. In particular, if $x$ is greater than zero: $x^1 = x$, $x^0 = 1$, $x^{-1} = 1/x$ and $x^{1/n} = \sqrt[n]{x}$.

7 The rules for manipulating powers are:

$$a^p \times a^q = a^{p+q} \qquad \text{(Eqn 13)}$$

$$(a^p)^q = a^{pq} \qquad \text{(Eqn 14)}$$

$$a^p / a^q = a^{p-q}$$

$$a^p b^p = (ab)^p$$

8 An *equation* can be *rearranged* by performing exactly the same operations on the whole of both sides of the equation.

9 Two *variables* $x$ and $y$ are *directly proportional* ($x \propto y$) if their *ratio* can be expressed wholly in terms that do not include $x$ and $y$. They are *inversely proportional* ($x \propto 1/y$) if their product can be expressed wholly in terms that do not include $x$ and $y$.

If $x \propto y$ then $x = Ky$ where $K$ is a *constant of proportionality.*

10 *Inequality* symbols $<$, $\leq$, $>$ and $\geq$ may be used to express the range of values of a variable that fulfil particular conditions.

11 *Inequalities* can be manipulated in ways similar to equations, but more care is needed and the direction of an inequality is reversed if both sides are multiplied by the same negative quantity.

## 4.2 Achievements

Having completed this module, you should be able to:

A1 Define the terms that are emboldened and flagged in the margins of the module.

A2 Decide the correct order of priority for operations in an arithmetic or algebraic expression and hence evaluate such expressions (using a calculator where appropriate).

A3 Use powers to express roots and reciprocals, and, where appropriate, combine powers in arithmetic and algebraic expressions.

A4 Manipulate and simplify fractions, both arithmetic and algebraic, including cases where there is initially no common denominator.

A5 Expand expressions that include brackets, and use brackets to simplify expressions.

A6 Rearrange equations in order to simplify them or to change their subject.

A7 Recognize examples of direct and inverse proportionality, and combine two or more proportional relationships.

A8 Rewrite proportional relationships as equations by introducing appropriate constants of proportionality.

A9 Interpret, use and rearrange inequalities.

***Study comment*** You may now wish to take the *Exit test* for this module which tests these Achievements. If you prefer to study the module further before taking this test then return to the *Module contents* to review some of the topics.

## 4.3 Exit test

***Study comment*** Having completed this module, you should be able to answer the following questions each of which tests one or more of the *Achievements*.

### Question E1

(*A2*) Write down the operations you would need to carry out, in the correct order, to evaluate the following expressions, e.g. for $(a - b)/3$ you subtract $b$ from $a$ and then divide by 3. (a) $5(x + 3y^2)$, (b) $E/(1 + r/R)$.

### Question E2

(*A3* and *A5*) Expand the following expressions and write them in as simple a form as possible:

(a) $(x + 5)(x + 7)$, (b) $x(x + 2)(x^2 - 1) - x^2(2x - 1)$.

## Question E3

(*A4*) Rewrite the following expression as a single fraction expressed in as simple a form as possible: $2p/q + 7pr/3$.

## Question E4

(*A6*) The equation $\upsilon = \sqrt{E/\rho}$ relates the speed, $\upsilon$, of sound in a solid to the elastic modulus $E$ and density $\rho$. Rearrange the equation so that $\rho$ is the subject.

## Question E5

(*A4* and *A6*) The equation $(1/u) + (1/\upsilon) = 1/f$ describes a relationship between the distance $u$ of an object from a lens of focal length $f$ and the distance $\upsilon$ from the lens to the image. Rearrange the equation so that $f$ is the subject.

## Question E6

(*A4*, *A5* and *A6*) The kinetic energy $E$ of an object of mass $m$ and speed $\upsilon$ is given by $E = m\upsilon^2/2$. If the object increases its speed from an initial value $\upsilon_1$ to a final value $\upsilon_2$, its kinetic energy increases from $E_1$ to $E_2$. Derive expressions in terms of $m$, $\upsilon_1$ and $\upsilon_2$ for (a) the ratio of the final to the initial kinetic energy $E_2/E_1$, (b) the increase in kinetic energy $E_2 - E_1$, and (c) the fractional increase in kinetic energy $(E_2 - E_1)/E_1$. In each case, write the expression in as simple a form as possible.

## Question E7

(*A3*) Simplify the following expressions:

(a) $\left(l\sqrt{l}\right)/l^2$, (b) $z\left[\left(\sqrt[5]{z^{-1}}\,x^{0.5}y^{0.1}\right)\Big/\left(x^{-1/2}y^{1/5}\right)^2\right]^{1/3}$

### Question E8

(*A7*) The strength $F_{\rm el}$ of the electrostatic force between two objects with positive charges $q_1$ and $q_2$, separated by a distance $r$, is given by $F_{\rm el} = (q_1q_2)/(4\pi\varepsilon_0r^2)$ where $\varepsilon_0$ is a constant known as the permittivity of free space. Express the relationships between the following variables as proportionalities (a) $F_{\rm el}$ and $q_1$, (b) $F_{\rm el}$ and $r$.

### Question E9

(*A8*) A pendulum of length $l$, swinging under the influence of gravity, completes a full swing in a time $T$. Observations in many different places, each characterized by a local value for $g$ (the magnitude of the acceleration due to gravity), show that $T \propto \sqrt{l}$ for fixed $g$ and $T \propto 1\big/\sqrt{g}$ for fixed $l$. Write down an equation relating $T$, $g$ and $l$.

### Question E10

(*A9*) (a) Express in words the statement $3 < x \leq 20$.

(b) If $a^2x^2 - a^2x^2h \geq 2y$, find the condition under which $x^2 \leq \dfrac{2y}{a^2(1-h)}$.

***Study comment*** This is the final *Exit test* question. When you have completed the *Exit test* go back to Subsection 1.2 and try the *Fast track questions* if you have not already done so.

If you have completed **both** the *Fast track questions* and the *Exit test*, then you have finished the module and may leave it here.