‘Quasi’-norm of an arithmetical convolution operator and the order of the Riemann zeta function

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Abstract
In this paper we study Dirichlet convolution with a given arithmetical function $f$ as a linear mapping $\varphi_f$ that sends a sequence $(a_n)$ to $(b_n)$ where $b_n = \sum_{d|n} f(d)a_{n/d}$. We investigate when this is a bounded operator on $l^2$ and find the operator norm. Of particular interest is the case $f(n) = n^{-\alpha}$ for its connection to the Riemann zeta function on the line $\Re s = \alpha$. For $\alpha > 1$, $\varphi_f$ is bounded with $\|\varphi_f\| = \zeta(\alpha)$. For the unbounded case, we show that $\varphi_f : M^2 \rightarrow M^2$ where $M^2$ is the subset of $l^2$ of multiplicative sequences, for many $f \in M^2$. Consequently, we study the ‘quasi’-norm

$$\sup_{\|a\| = T} \frac{\|\varphi_f a\|}{\|a\|}$$

for large $T$, which measures the ‘size’ of $\varphi_f$ on $M^2$. For the $f(n) = n^{-\alpha}$ case, we show this quasi-norm has a striking resemblance to the conjectured maximal order of $|\zeta(\alpha + iT)|$ for $\alpha > \frac{1}{2}$.

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Introduction
Given an arithmetical function $f(n)$, the mapping $\varphi_f$ sends $(a_n)_{n \in \mathbb{N}}$ to $(b_n)_{n \in \mathbb{N}}$, where

$$b_n = \sum_{d|n} f(d)a_{n/d}.$$ (0.1)

Writing $a = (a_n)$, $\varphi_f$ maps $a$ to $f * a$ where $*$ is Dirichlet convolution. This is a ‘matrix’ mapping, where the matrix, say $M(f)$, is of ‘multiplicative Toeplitz’ type; that is,

$$M(f) = (a_{ij})_{i,j \geq 1}$$

where $a_{ij} = f(i/j)$ and $f$ is supported on the natural numbers (see, for example, [6], [7]).

Toeplitz matrices (whose $ij$th-entry is a function of $i - j$) are most usefully studied in terms of a ‘symbol’ (the function whose Fourier coefficients make up the matrix). Analogously, the Multiplicative Toeplitz matrix $M(f)$ has as symbol the Dirichlet series

$$\sum_{n=1}^{\infty} f(n)n^{it}.$$ 

Our particular interest is naturally the case $f(n) = n^{-\alpha}$ when the symbol is $\zeta(\alpha - it)$. We are especially interested how and to what extent properties of the mapping relate to properties of the symbol for $\alpha \leq 1$.

These type of mappings were considered by various authors (for example Wintner [15]) and most notably Toeplitz [13], [14] (although somewhat indirectly, through his investigations of so-called

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“D-forms”). In essence, Toeplitz proved that \( \varphi_f : l^2 \to l^2 \) is bounded if and only if \( \sum_{n=1}^{\infty} f(n) n^{-s} \) is defined and bounded for all \( R > 0 \). In particular, if \( f(n) \geq 0 \) then \( \varphi_f \) is bounded on \( l^2 \) if and only if \( f \in l^1 \); furthermore, the operator norm is \( \| \varphi_f \| = \| f \|_1 \). We prove this in Theorem 1.1 following Toeplitz’s original idea. For example, for \( f(n) = n^{-\alpha} \), \( \varphi_f \) is bounded on \( l^2 \) for \( \alpha > 1 \) with operator norm \( \zeta(\alpha) \). In this special case, the mapping was studied in [7] for \( \alpha \leq 1 \) when it is unbounded on \( l^2 \) by estimating the behaviour of the quantity

\[
\Phi_f(N) = \sup_{\|a\| = 1} \left( \sum_{n=1}^{N} |a_n|^2 \right)^{1/2}
\]

for large \( N \). Approximate formulas for \( \Phi_f(N) \) were obtained and it was shown that, for \( \frac{1}{2} < \alpha \leq 1 \), \( \Phi_f(N) \) is a lower bound for \( \max_{1 \leq \ell \leq T} |\zeta(\alpha + it)| \) with \( N = T^\lambda \) (some \( \lambda > 0 \) depending on \( \alpha \) only). In this way, it was proven that the measure of the set

\[
\left\{ \ell \in [1, T] : |\zeta(\alpha + it)| \geq e^\gamma \log \log T - A \right\}
\]

is at least \( T \exp \{-a \frac{\log T}{\log \log T} \} \) (some \( a > 0 \)) for \( A \) sufficiently large, while for \( \frac{1}{2} < \alpha < 1 \) one has

\[
\max_{1 \leq \ell \leq T} |\zeta(\alpha + it)| \geq \exp \left\{ c (\log T)^{1-\alpha} \frac{\log \log T}{\log T} \right\}
\]

for some \( c > 0 \) depending on \( \alpha \) only, as well providing an estimate for how often \( |\zeta(\alpha + iT)| \) is as large as the right-hand side above. The method is akin to Soundararajan’s ‘resonance’ method and incidentally shows the limitation of this approach for \( \alpha > \frac{1}{2} \) since \( |\zeta(\alpha + iT)| \) is known to be of larger order.

In this paper we study the unbounded case in a different way, by restricting the domain. Thus in section 2, we show that for many multiplicative \( f \), in particular for \( f \) completely multiplicative, \( \varphi_f(\mathcal{M}^2) \subset \mathcal{M}^2 \) even though \( \varphi_f(l^2) \not\subset l^2 \). Here \( \mathcal{M}^2 \) is the set of multiplicative functions in \( l^2 \). As a result we consider, for such \( f \), the ‘quasi’-norm

\[
M_f(T) = \sup_{\|a\| = T} \frac{\|\varphi_f a\|}{\|a\|}
\]

and obtain approximate formulae for large \( T \) (here \( \| \cdot \| \) is the usual \( l^2 \)-norm). We find that for the particular case \( f(n) = n^{-\alpha} \) \((\alpha > \frac{1}{2})\), this quasi-norm has a striking similarity to the conjectured maximal order of \( |\zeta(\alpha + iT)| \). For example, with \( \alpha = 1 \) (i.e. \( f(n) = 1/n \)) we prove

\[
M_f(T) = e^\gamma (\log \log T + \log \log \log T + 2 \log 2 - 1) + o(1),
\]

(0.2)

while for \( \frac{1}{2} < \alpha < 1 \)

\[
\log M_f(T) \sim \frac{B(\frac{1}{2}, 1 - \frac{1}{2\alpha})}{(1 - \alpha)2^\alpha} (\log T)^{1-\alpha} \frac{\log \log T}{\log T}^\alpha,
\]

where \( B(x, y) \) is the Beta function. Writing \( Z_\alpha(T) = \max_{1 \leq \ell \leq T} |\zeta(\alpha + it)| \), Granville and Soundararajan [3] proved that \( Z_1(T) \) is at least as large as \( (0.2) \) minus a log log log \( T \) term for some arbitrarily large \( T \) and they conjectured that it equals \( (0.2) \) (possibly with a different constant term). For \( \frac{1}{2} < \alpha < 1 \), Montgomery [9] found

\[
\log Z_\alpha(T) \geq \frac{\sqrt{\alpha - 1/2}}{20} (\log T)^{1-\alpha} \frac{\log \log T}{\log T}^\alpha
\]

and, using a heuristic argument, conjectured that this is (apart from the constant) the correct order of \( \log Z_\alpha(T) \). Further, in a recent paper (see [8]), Lamzouri suggests \( \log Z_\alpha(T) \sim C(\alpha)(\log T)^{1-\alpha}log \log T)^{-\alpha} \) with some specific constant \( C(\alpha) \) (see also the remark after Theorem 3.1).
Similarly one can study the quantity

$$m_f(T) = \inf_{\varphi \in \mathcal{L}(\mathcal{H}, \mathcal{H})} \frac{\|\varphi f\|}{\|a\|}.$$  

With \(f(n) = n^{-\alpha}\) this is shown to behave like the known and conjectured minimal order of \(|\zeta(\alpha+iT)|\) for \(\alpha > \frac{1}{2}\). It should be stressed here that, unlike the case of \(\Phi_f(N)\) which was shown to be a lower bound for \(Z_n(T)\) in [7], we have not proved any connection between \(\zeta(\alpha+iT)\) and \(M_f(T)\). Even to show \(M_f(T)\) is a lower bound would be very interesting.

Our results, though motivated by the special case \(f(n) = n^{-\alpha}\), extend naturally to completely multiplicative functions \(f\) for which \(f\|_2\) is regularly varying (see section 2 for the definition).

**Addendum.** I would like to thank the anonymous referee for some useful comments and for pointing out an upper bound for \(\|\varphi f\|\), for all \(x \in l^2\). As such, we define the operator norm by

$$\|\varphi\| = \sup_{x \neq 0} \frac{\|\varphi x\|_2}{\|x\|_2} = \sup_{\|x\|_2=1} \|\varphi x\|_2.$$  

We shall assume from now on that \(f(n) \geq 0\) for all \(n \in \mathbb{N}\). We are particularly interested in the case where \(\varphi_f\) acts on \(l^2\). Define the function

$$\Phi_f(N) = \sup_{\|a\|_2=1} \sqrt{\sum_{n \leq N} |b_n|^2},$$  

where \(b_n\) is given in terms of \(a_n\) by (0.1). Note that the supremum will occur when \(a_n \geq 0\) for all \(n\) and when \(\sum_{n \leq N} a_n^2 = 1\).

Suppose now that \(f \in l^1\); i.e. \(\|f\|_1 = \sum_{n=1}^\infty f(n) < \infty\). Then

$$|b_n|^2 = \left| \sum_{d \mid n} \sqrt{f(d)} \cdot \sqrt{f(d)} a_{n/d} \right|^2 \leq \sum_{d \mid n} f(d) \sum_{d \mid n} f(d) |a_{n/d}|^2 \leq \|f\|_1 \sum_{d \mid n} f(d) |a_{n/d}|^2.$$  

Hence

$$\sum_{n \leq N} |b_n|^2 \leq \|f\|_1 \sum_{n \leq N} \sum_{d \mid n} f(d) |a_{n/d}|^2 = \|f\|_1 \sum_{d \leq N} f(d) \sum_{n \leq N/d} |a_n|^2 \leq \|f\|_1^2 \|a\|_2^2,$$

Thus

$$\Phi_f(N) \leq \|f\|_1.$$  

Following Toeplitz [14], we show that this inequality is sharp.

**Theorem 1.1**

Let \(f\) be a non-negative arithmetical function and \(f \in l^1\). Then \(\Phi_f(N) \to \|f\|_1\) as \(N \to \infty\). Thus
\( \varphi_f : l^2 \to l^2 \) is bounded if and only if \( f \in l^1 \), in which case \( \|\varphi_f\| = \|f\|_1 \).

**Proof.** After (1.1), and since \( \Phi_f(N) \) increases with \( N \), we need only provide a lower bound for an infinite sequence of \( N \)s. Let \( a_n = d(N)^{-1/2} \) for \( n \mid N \) and zero otherwise (\( N \) to be chosen later), where \( d(\cdot) \) is the divisor function. Thus \( a_1^2 + \ldots + a_N^2 = 1 \) and

\[
\Phi_f(N) \geq \sum_{n \leq N} a_n b_n = \frac{1}{d(N)} \sum_{n \mid N} \sum_{d \mid n} f(d) = \frac{1}{d(N)} \sum_{d \mid N} f(d) \left( \frac{N}{d} \right),
\]

say. We choose \( N \) such that it has all divisors \( d \) up to some (large) number, and that \( \frac{d(N)/d(n)}{d(N)} \) is close to 1 for each such divisor \( d \) of \( N \). Take \( N \) of the form

\[ N = \prod_{p \leq P} p^{\alpha_p} \text{ where } \alpha_p = \lfloor \frac{\log P}{\log p} \rfloor. \]

Thus every natural number up to \( P \) is a divisor of \( N \). For a divisor \( d = \prod_{p \leq P} p^{\beta_p} \) of \( N \), we have

\[
\frac{d(N)/d(n)}{d(N)} = \prod_{p \leq P} \left(1 - \frac{\beta_p}{\alpha_p + 1}\right).
\]

If we take \( d \leq \sqrt{\log P} \), then \( p^{\beta_p} \leq \sqrt{\log P} \) for every prime divisor \( p \) of \( d \). Hence, for such \( p \), \( \beta_p \leq \frac{\log \log P}{2 \log p} \) and \( \beta_p = 0 \) if \( p > \sqrt{\log P} \). Thus for \( d \leq \sqrt{\log P} \),

\[
\frac{d(N)/d(n)}{d(N)} = \prod_{p \leq \sqrt{\log P}} \left(1 - \frac{\beta_p}{\alpha_p + 1}\right) \geq \prod_{p \leq \sqrt{\log P}} \left(1 - \frac{\log \log P}{2 \log p}\right) = \left(1 - \frac{\log \log P}{2 \log P}\right)^{\pi(\log P)},
\]

where \( \pi(x) \) is the number of primes up to \( x \). Since \( \pi(x) = O\left(\frac{x}{\log x}\right) \), it follows that for all \( P \) sufficiently large, the expression in (1.2) is at least

\[
\left(1 - \frac{A}{\sqrt{\log P}}\right) \sum_{d \leq \sqrt{\log P}} f(d)
\]

for some constant \( A \). The sum can be made as close to \( \|f\|_1 \) as we please by increasing \( P \).

\[ \square \]

2. Unbounded operators on \( l^2 \)

Now we investigate when \( \varphi_f \) is unbounded on \( l^2 \) (i.e. \( f \not\in l^1 \)). In a similar generalisation of Theorem 1.1 of [7], one can readily show that both \( \varphi_f : l^1 \to l^2 \) and \( \varphi_f : l^2 \to l^\infty \) are bounded if and only if \( f \in l^2 \), with \( \|\varphi_f\| = \|f\|_2 \) in either case. So here we shall assume that \( f \in l^2 \setminus l^1 \). In the appendix we see that, for all cases of interest at least, if \( f \not\in l^2 \), then \( \varphi_f a \not\in l^2 \) for all \( a \) except \( a = 0 \).

For unbounded operators, there are different ways of measuring the ‘unboundedness’. One way, which was done in [7] for the case \( f(n) = n^{-\alpha} \), is to restrict the range by looking at a restricted norm; i.e. by considering \( \Phi_f(N) \) for given \( N \). Another way is to restrict the domain to a set \( S \) say, that \( \varphi_f(S) \subset l^2 \) and to consider the size of

\[
\sup_{a \in S, \|a\| = N} \frac{\|\varphi_f a\|}{\|a\|} \text{ for large } N.
\]

For \( f \) completely multiplicative one is naturally led to consider \( S = M^2 \) — the set of square summable multiplicative functions. It is also natural to consider *regularly varying* functions.

**Regular Variation.** A function \( \ell : [A, \infty) \to \mathbb{R} \) is regularly varying of index \( \rho \) if it is measurable and

\[
\ell(\lambda x) \sim \lambda^\rho \ell(x) \quad \text{as } x \to \infty \text{ for every } \lambda > 0
\]
(see [2] for a detailed treatise on the subject). For example, $x^\rho (\log x)\tau$ is regularly-varying of
index $\rho$ for any $\tau$. The Uniform Convergence Theorem says that the above asymptotic formula
is automatically uniform for $\lambda$ in compact subsets of $(0, \infty)$. Note that every regularly varying
function of non-zero index is asymptotic to one which is strictly monotonic and continuous. We
shall make use of Karamata’s Theorem: for $\ell$ regularly varying of index $p$,

$$\int \frac{x^\ell(x)}{\rho + 1} \, dx \sim \frac{x^\ell(x)}{\rho + 1} \quad \text{if } \rho > -1, \quad \int \frac{x^\ell(x)}{\rho + 1} \, dx \sim -\frac{x^\ell(x)}{\rho + 1} \quad \text{if } \rho < -1,$$

while if $\rho = -1$, $\int x^\ell \, dx$ is slowly varying (regularly varying with index 0) and $\int x^\ell \, dx \sim x^\ell(x)$.

**Notation.** Let $\mathcal{M}^2$ and $\mathcal{M}_2^2$ denote the subsets of $l^2$ of multiplicative and completely multiplicative
functions respectively. Further, write $\mathcal{M}^2_+$ for the non-negative members of $\mathcal{M}^2$ and similarly for $\mathcal{M}^2_+$.

### 2.1 The size of $\|\varphi_f\|$ on $\mathcal{M}^2$

Now we consider $\varphi_f$ on the subset $\mathcal{M}^2$ of multiplicative functions in $l^2$. We suppose, as in section 2, that $f \in l^2 \setminus l^1$ so that $\varphi_f$ is unbounded. This implies there exist $a \in l^2$ such that $\varphi_f(a) \not\in l^2$
(by the closed graph theorem). However, if $f$ is multiplicative then, as we shall see, $\varphi_f(\mathcal{M}^2) \subset l^2$
in many cases (and hence $\varphi_f(\mathcal{M}^2_+) \subset l^2$).

**Lemma 2.1**

Let $f, g \in \mathcal{M}^2$ be non-negative. Then $f \ast g \in \mathcal{M}^2$ if and only if

$$\sum_p \sum_{m,n \geq 1} \sum_{k=0}^\infty f(p^m)g(p^n)f(p^{m+k})g(p^{n+k}) \text{ converges.} \quad (2.1)$$

**Proof.** Let $h = f \ast g$. Since $h$ is multiplicative,

$$\sum_{n=1}^\infty h(n)^2 < \infty \iff \sum_p \sum_{k \geq 1} h(p)^2 < \infty.$$

Let $k \geq 1$ and $p$ prime. Then

$$h(p^k) = \sum_{r=0}^k f(p^r)g(p^{k-r}) = f(p^k) + g(p^k) + \sum_{r=1}^{k-1} f(p^r)g(p^{k-r}).$$

Using the inequality $a^2 + b^2 + c^2 \leq (a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ we have

$$\left( \sum_{r=1}^{k-1} f(p^r)g(p^{k-r}) \right)^2 \leq h(p^k)^2 \leq 3f(p^k)^2 + 3g(p^k)^2 + 3\left( \sum_{r=1}^{k-1} f(p^r)g(p^{k-r}) \right)^2.$$

Since $\sum_{p,k \geq 1} f(p^k)^2$ and $\sum_{p,k \geq 1} g(p^k)^2$ converge we find that $\sum_{p,k \geq 1} h(p^k)^2$ converges if and only if

$$\sum_p \sum_{k \geq 2} \left( \sum_{r=1}^{k-1} f(p^r)g(p^{k-r}) \right)^2 \text{ converges.}$$

But

$$\sum_{k=2}^\infty \sum_{r=1}^{k-1} \left( f(p^r)g(p^{k-r}) \right)^2 = \sum_{k=1}^\infty \sum_{r=1}^{k-1} f(p^r)f(p^{k-r+1})g(p^{k-s+1}) \quad (2.2)$$

$$\leq 2 \sum_{k=1}^\infty \sum_{s=1}^k \sum_{r=1}^s f(p^r)f(p^{k-r+1})g(p^{k-s+1})$$

$$= 2 \sum_{r=1}^\infty \sum_{k=1}^\infty \sum_{s=0}^\infty f(p^r)g(p^{s+r})g(p^k)g(p^s).$$
On the other hand, the RHS of (2.2) is greater than
\[ \sum_{k=1}^{\infty} \sum_{s=1}^{k} \sum_{r=1}^{s} f(p^r)f(p^s)g(p^{k-r+1})g(p^{k-s+1}). \]
Hence \( h \in M^2 \) if and only if
\[ \sum_{p} \sum_{m,n \geq 1} \sum_{k=0}^{\infty} f(p^m)g(p^n)f(p^{m+k})g(p^{n+k}) \quad \text{converges.} \]

\[ \square \]

Let \( M_0^2 \) denote the set of \( M^2 \) functions \( f \) for which \( f \ast g \in M^2 \) whenever \( g \in M^2 \); that is,
\[ M_0^2 = \{ f \in M^2 : g \in M^2 \implies f \ast g \in M^2 \}. \]
Thus for \( f \in M_0^2 \), \( \varphi_f(M^2) \subseteq M^2 \). We shall see that it may happen that \( f, g \in M^2 \) but \( f \ast g \notin M^2 \).
So \( M_0^2 \neq M^2 \). The following gives a criterion for multiplicative functions to be in \( M_0^2 \).

**Proposition 2.2**
Let \( f \in M^2 \) be such that \( \sum_{k=1}^{\infty} |f(p^k)| \) converges for every prime \( p \) and that \( \sum_{k=1}^{\infty} |f(p^k)| \leq A \) for some constant \( A \) independent of \( p \). Then \( f \in M_0^2 \).

**Proof.** Without loss of generality we can take \( f \geq 0 \). Let \( g \in M^2 \) (again w.l.o.g. \( g \geq 0 \)) with 
\[ \alpha_p = \sum_{k=1}^{\infty} g(p^k)^2. \]
Thus \( \sum_p \alpha_p \) converges. By the Cauchy-Schwarz inequality,
\[ \left( \sum_{n=1}^{\infty} g(p^n)g(p^{n+k}) \right)^2 \leq \sum_{n=1}^{\infty} g(p^n)^2 \sum_{n=1}^{\infty} g(p^{n+k})^2 \leq \alpha_p \alpha_p = \alpha_p^2. \]
Thus by Lemma 2.1, \( f \ast g \in M^2 \) if
\[ \sum_p \alpha_p \sum_{m=1}^{\infty} f(p^m) \sum_{k=0}^{\infty} f(p^{m+k}) \quad \text{converges.} \]
By assumption, the inner sum over \( k \) is bounded by a constant (independent of \( p \)), and hence so is the sum over \( m \). This implies the convergence of the above. Hence \( f \ast g \in M^2 \).

Now suppose \( \sum_{k=1}^{\infty} f(p_0^k) \) diverges for some prime \( p_0 \). Then with \( g \in M^2 \) and \( g(p_0^k) \) decreasing (to zero) we have
\[ (f \ast g)(p_0^k) = \sum_{r=0}^{k} f(p_0^r)g(p_0^{k-r}) \geq g(p_0^k) \sum_{r=0}^{k} f(p_0^r) = g(p_0^k)c_k, \]
where \( c_k \to \infty \). Thus \( \sum_{k} (f \ast g)(p_0^k)^2 \geq \sum_{k} g(p_0^k)^2 c_k^2 \). But we can always choose \( g(p_0^k) \) decreasing so that \( \sum_{k} g(p_0^k)^2 c_k^2 \) converges while, for the given sequence \( c_k, \sum_{k} g(p_0^k)^2 c_k^2 \) diverges. (Choose \( g(p_0^k)^2 = \frac{1}{k^{2-\epsilon}} \).)
Thus \( f \ast g \notin M^2 \); i.e. \( f \notin M_0^2 \).

\[ \square \]
Thus, in particular, $M_2^c \subset M_2^0$. For $f \in M_2^c$ if and only if $|f(p)| < 1$ for all primes $p$ and
$\sum_p |f(p)|^2 < \infty$. Thus
$$\sum_{k=1}^{\infty} |f(p^k)| = \frac{|f(p)|}{1 - |f(p)|} \leq A,$$

independent of $p$ (since $f(p) \to 0$).

The “quasi-norm” $M_f(T)$

Let $f \in M_2^c$. From above we see that $\varphi_f(M_2) \subset M_2$ but, typically, $\varphi_f$ is not ‘bounded’ on $M_2$ (if $f \notin l^1$) in the sense that $\|\varphi_f a\|/\|a\|$ is not bounded by a constant for all $a \in M_2$. It therefore makes sense to define, for $T \geq 1$,
$$M_f(T) = \sup_{a \in M_2} \frac{\|\varphi_f a\|}{\|a\|}.$$

We aim to find the behaviour of $M_f(T)$ for large $T$.

We shall consider $f$ completely multiplicative and such that $f|_P$ is regularly varying of index $-\alpha$ with $\alpha > 1/2$ in the sense that there exists a regularly varying function $\tilde{f}$ (of index $-\alpha$) with $f(p) = \tilde{f}(p)$ for every prime $p$.

Our main result here is the following:

**Theorem 2.3**

Let $f \in M_2^c$, such that $f \geq 0$ and $f|_P$ is regularly varying of index $-\alpha$ where $\alpha \in (\frac{1}{2}, 1)$. Then
$$\log M_f(T) \sim c(\alpha) \tilde{f}(\log T \log \log T) \log T$$

where $\tilde{f}$ is any regularly varying extension of $f|_P$ and
$$c(\alpha) = \frac{B(\frac{1}{\alpha}, 1 - \frac{1}{2\alpha})}{(1 - \alpha)^2},$$

and $B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} dt$ is the Beta function.

For the proof, we obtain upper and lower bounds for $\log M_f(T)$ which are asymptotic to each other. For the lower bounds, we require a formula for $\|\varphi_f a\|$ when $a \in M_2^c$. This follows from the following rather elegant formula:

**Lemma 2.4**

For $f, g \in M_2^c$,
$$\frac{\|f * g\|}{\|f\| \|g\|} = \frac{|\langle f, g \rangle|}{\|fg\|},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product for $l^2$.

**Proof.** We have
$$\|f * g\|^2 = \sum_{n=1}^{\infty} |(f * g)(n)|^2 = \sum_{n=1}^{\infty} \sum_{c,d | n} f(c)\overline{f(d)} \overline{g\left(\frac{n}{c}\right)} g\left(\frac{n}{d}\right)$$
$$= \sum_{c,d \geq 1} f(c)\overline{f(d)} \sum_{m=1}^{\infty} g\left(\frac{m|c,d|}{c}\right) \overline{g\left(\frac{m|c,d|}{d}\right)}$$
$$= \sum_{m=1}^{\infty} |g(m)|^2 \sum_{c,d \geq 1} f(c)\overline{f(d)} g\left(\frac{m}{c,d}\right) \overline{g\left(\frac{m}{c,d}\right)} g\left(\frac{d}{(c,d)}\right) \overline{g\left(\frac{m}{(c,d)}\right)}.$$
Collecting those terms for which \((c, d) = k\), writing \(c = km, d = kn\), and using complete multiplicativity of \(f\)

\[
\left( \frac{\|f \ast g\|}{\|g\|} \right)^2 = \sum_{k=1}^{\infty} |f(k)|^2 \sum_{m,n \geq 1, (m,n) = 1} f(m)f(n)g(m)g(n).
\]

But

\[
|f, g\rangle^2 = \sum_{m,n \geq 1} f(m)f(n)g(m)g(n) = \sum_{d=1}^{\infty} |f(d)g(d)|^2 \sum_{m,n \geq 1, (m,n) = 1} f(m)f(n)g(m)g(n),
\]

so the result follows. □

Thus for \(f, a \in \mathcal{M}_c^2\),

\[
\frac{\|\varphi f a\|}{\|a\|} = \frac{\|f\| \cdot \|\sum_{n=1}^{\infty} f(n)a_n\|}{(\sum_{n=1}^{\infty} |f(n)a_n|^2)^{1/2}}.
\]

Since \(|a_n| \leq 1\), as a corollary we have:

**Corollary 2.5**

*For \(f, a \in \mathcal{M}_c^2\),*

\[
\left| \sum_{n=1}^{\infty} f(n)a_n \right| \leq \|\varphi f a\| \leq \|f\| \left| \sum_{n=1}^{\infty} f(n)a_n \right|.
\]

Note that by complete multiplicativity,

\[
\sum_{n=1}^{\infty} f(n)a_n = \prod_{p} \frac{1}{1 - f(p)a_p} = \prod_{p} \exp \left\{ f(p)a_p + O(|f(p)a_p|^2) \right\},
\]

and \(\sum_p |f(p)a_p|^2 \leq \sum_p |f(p)|^2 = O(1)\), so that

\[
\log \frac{\|\varphi f a\|}{\|a\|} = \Re \sum_p f(p)a_p + O(1). \tag{2.3}
\]

*Proof of Theorem 2.3.* We consider first upper bounds. The supremum occurs for \(a \geq 0\) which we now assume. Write \(a = (a_n), \varphi f a = b = (b_n)\). Define \(\alpha_p\) and \(\beta_p\) for prime \(p\) by

\[
\alpha_p = \sum_{k=1}^{\infty} a_p^k \quad \text{and} \quad \beta_p = \sum_{k=1}^{\infty} b_p^k.
\]

By multiplicativity of \(a\) and \(b\) we have \(T^2 = \|a\|^2 = \prod_p (1 + \alpha_p)\) and \(\|b\|^2 = \prod_p (1 + \beta_p)\). Thus

\[
\frac{\|\varphi f a\|}{\|a\|} = \prod_p \sqrt{\frac{1 + \beta_p}{1 + \alpha_p}}.
\]

Now for \(k \geq 1\)

\[
b_p^k = \sum_{r=0}^{k} f(p^r)a_{p^{k-r}} = a_p^k + f(p)b_p^{k-1}.
\]

Thus

\[
b_p^2 = a_p^2 + 2f(p)a_p b_{p^{k-1}} + f(p)^2 b_p^{k-1}.
\]

Summing from \(k = 1\) to \(\infty\) and adding 1 to both sides gives

\[
1 + \beta_p = 1 + \alpha_p + 2f(p) \sum_{k=1}^{\infty} a_p^k b_{p^{k-1}} + f(p)^2 (1 + \beta_p). \tag{2.4}
\]
By Cauchy-Schwarz,
\[
\sum_{k=1}^{\infty} a_p b_{p^{-1}} \leq \left( \sum_{k=1}^{\infty} a_p^2 \sum_{k=1}^{\infty} b_{p^{-1}}^2 \right)^{1/2} = \sqrt{\alpha_p (1 + \beta_p)},
\]
so, on rearranging
\[
(1 + \beta_p) - \frac{2f(p)\sqrt{\alpha_p (1 + \beta_p)}}{1 - f(p)^2} \leq 1 + \alpha_p.
\]
Completing the square we find
\[
\left( \sqrt{1 + \beta_p} - \frac{f(p)\sqrt{\alpha_p}}{1 - f(p)^2} \right)^2 \leq \frac{1 + \alpha_p}{(1 - f(p)^2)^2}.
\]
The term on the left inside the square is non-negative for \( p \) sufficiently large since \( f(p) \to 0 \); in fact from (2.4), \( 1 + \beta_p \geq \frac{1 + \alpha_p}{1 - f(p)^2} \) which is greater than \( \frac{f(p)\sqrt{\alpha_p}}{1 - f(p)^2} \) if \( f(p) \leq 1/\sqrt{2} \). Rearranging gives
\[
\sqrt{1 + \beta_p} \leq 1 - f(p)^2 \left( 1 + f(p) \sqrt{\frac{\alpha_p}{1 + \alpha_p}} \right).
\]
Let \( \gamma_p = \sqrt{\frac{\alpha_p}{1 + \alpha_p}} \). Taking the product over all primes \( p \) gives
\[
\| f_a \|_{\| a \|} \leq A\| f \| \prod_p (1 + f(p)\gamma_p) \leq A' \exp \left\{ \sum_p f(p)\gamma_p \right\}
\]
for some constants \( A, A' \) depending only on \( f \). (We can take \( A = 1 \) if \( f(p) \leq 1/\sqrt{2} \).) Note that
\( 0 \leq \gamma_p < 1 \) and \( \prod_p \frac{1}{1 - \gamma_p} = T^2 \).

Let \( \epsilon > 0 \) and put \( P = \log T \log \log T \). We split up the sum on the RHS of (2.5) into \( p \leq aP, aP < p \leq AP \) and \( p > AP \) (for a small and \( A \) large). First
\[
\sum_{p \leq aP} f(p)\gamma_p \leq \sum_{p \leq aP} f(p) \sim \frac{a^{1-\alpha}P\tilde{f}(P)}{(1-\alpha)\log P} < \epsilon f(\log T \log \log T) \log T;
\]
for \( a \) sufficiently small\(^2\). Next, using the fact that \( \log T^2 = \log \prod_p \frac{1}{1-\gamma_p} \) \( \geq \sum_p \gamma_p^2 \), we have (since \( \tilde{f}^2 \) is regularly-varying of index \(-2\alpha\))
\[
\sum_{p > AP} f(p)\gamma_p \leq \left( \sum_{p > AP} f(p)^2 \sum_{p > AP} \gamma_p^2 \right)^{1/2} \leq \left( \frac{2A^{1-2\alpha}P\tilde{f}(P)^2 \log T}{(2\alpha - 1) \log P} \right)^{1/2} \tilde{f}(\log T \log \log T) \log T \log T \log T
\]
for \( A \) sufficiently large. This leaves the range \( aP < p \leq AP \).

Note that the result follows from the case \( f(n) = n^{-\alpha} \). For, by the uniform convergence theorem for regularly varying functions
\[
\left| f(p) - \left( \frac{P}{p} \right)^\alpha \tilde{f}(P) \right| < \epsilon f(p)
\]
for \( aP < p \leq AP \) and \( P \) sufficiently large, depending only on \( \epsilon \). The problem therefore reduces to maximising
\[
\sum_{aP < p \leq AP} \gamma_p.
\]

\(^2\)Using \( \sum_{p \leq x} f(p) \sim \int_2^x \frac{f(t)}{\log t} \, dt \sim \frac{\epsilon f(x)}{(1-\alpha) \log x} \), since \( \tilde{f} \) is regularly-varying of index \(-\alpha\).
subject to $0 \leq \gamma_p < 1$ and $\prod_p \frac{1}{1 - \gamma_p} = T^2$. The maximum clearly occurs for $\gamma_p$ decreasing (if $\gamma_{p'} > \gamma_p$ for primes $p < p'$, then the sum increases in value if we swap $\gamma_p$ and $\gamma_{p'}$). Thus we may assume that $\gamma_p$ is decreasing.

By interpolation we may write $\gamma_p = g(\frac{p}{x})$ where $g : (0, \infty) \to (0, 1)$ is continuously differentiable and decreasing. Of course $g$ will depend on $P$. Let $h = \log \frac{1}{1 - \gamma_p}$, which is also decreasing. Note that

$$2 \log T = \sum_p h \left( \frac{p}{x} \right) \geq \sum_{p \leq aP} h \left( \frac{p}{P} \right) \geq h(a) \pi(aP) \geq c ah(a) \log T,$$

for $P$ sufficiently large, for some constant $c > 0$. Thus $h(a) \leq C_a$ (independent of $T$).

Now, for $F : (0, \infty) \to [0, \infty)$ decreasing,

$$\sum_{ax < p \leq bx} F \left( \frac{p}{x} \right) = \frac{x}{\log x} \int_a^b F + O \left( \frac{xF(a)}{(\log x)^2} \right),$$

where the implied constant is independent of $F$ (and $x$). For, on writing $\pi(x) = li(x) + e(x)$, the LHS is

$$\int_a^b F \left( \frac{t}{x} \right) d\pi(t) = \int_a^b F \left( \frac{t}{x} \right) \frac{d\pi(t)}{\log xt} dt + \int_a^b F(t) e(\pi(t)) dt$$

$$= \frac{x}{\log \theta x} \int_a^b F + \left[ F(t) e(\pi(t)) \right]_a^b - \int_a^b e(\pi(t)) dF(t) \quad (\text{some } \theta \in [a, b])$$

$$= \frac{x}{\log x} \int_a^b F + O \left( \frac{xF(a)}{(\log x)^2} \right),$$

on using $e(x) = O \left( \frac{x}{(\log x)^2} \right)$ and the fact that $F$ is decreasing. Thus by (2.9)

$$2 \log T \geq \sum_{aP < p \leq AP} h \left( \frac{p}{P} \right) \sim \frac{P}{\log P} \int_a^A h \sim (\log T) \int_a^A h.$$

Since $a$ and $A$ are arbitrary, $\int_0^\infty h$ must exist and is at most 2. Also, by (2.9)

$$\sum_{aP < p \leq AP} \gamma_p = \frac{1}{P^\alpha} \sum_{aP < p \leq AP} \frac{g(p)}{p} \left( \frac{p}{P} \right)^{-\alpha} \sim \frac{P^{1-\alpha}}{\log P} \int_a^A \frac{g(u)}{u^\alpha} du.$$

Hence by (2.8),

$$\sum_{aP < p \leq AP} f(p) \gamma_p \sim \tilde{f}(P) P^\alpha \sum_{aP < p \leq AP} \gamma_p = \frac{P \tilde{f}(P)}{\log P} \int_a^A \frac{g(u)}{u^\alpha} du.$$

As $a, A$ are arbitrary, it follows from above and (2.5), (2.6), (2.7) that

$$\log \frac{\| f a \|}{\| a \|} \leq \left( \int_0^\infty \frac{g(u)}{u^\alpha} du + o(1) \right) \tilde{f}(\log T \log \log T) \log T.$$

Thus we need to maximize $\int_0^\infty g(u) u^{-\alpha} du$ subject to $\int_a^\infty h \leq 2$ over all decreasing $g : (0, \infty) \to (0, 1)$. Since $h$ is decreasing,

$$\frac{1}{2} x^2 h(x) \leq \int_{x/2}^x h.$$

The RHS can be made as small as we please for $x$ sufficiently small or large (as $\int_0^\infty h$ converges).

In particular, $x h(x) \to 0$ as $x \to \infty$ and as $x \to 0^+$. In fact, for the supremum, we can consider just those $g$ (and $h$) which are continuously differentiable and strictly decreasing, since we can
approximate arbitrarily closely with such functions. On writing \( g = s \circ h \) where \( s(x) = \sqrt{1 - e^{-x}} \), we have

\[
\int_0^\infty \frac{g(u)}{u^\alpha} \, du = \left[ \frac{g(u)u^{1-\alpha}}{1-\alpha} \right]_0^\infty - \frac{1}{1-\alpha} \int_0^\infty g'(u) u^{1-\alpha} \, du
\]

\[
= -\frac{1}{1-\alpha} \int_0^\infty s'(h(u)) h'(u) u^{1-\alpha} \, du = \frac{1}{1-\alpha} \int_0^{h(0^+)} s'(x) l(x)^{1-\alpha} \, dx,
\]

where \( l = h^{-1} \), since \( \sqrt{a} g(u) \to 0 \) as \( u \to \infty \). The final integral is, by Hölder’s inequality at most

\[
\left( \int_0^{h(0^+)} s^{1/\alpha} \right)^{\alpha} \left( \int_0^{h(0^+)} l \right)^{1-\alpha}.
\]  

(2.10)

But \( f_0^{h(0^+)} l = -\int_0^\infty uh'(u) du = \int_0^\infty h \leq 2 \), so

\[
\int_0^\infty \frac{g(u)}{u^\alpha} \, du \leq \frac{2^{1-\alpha}}{1-\alpha} \left( \int_0^\infty s^{1/\alpha} \right)^{\alpha}.
\]

A direct calculation shows that\(^3 \int_0^\infty (s')^{1/\alpha} = 2^{-1/\alpha} B \left( \frac{1}{\alpha}, 1 - \frac{1}{2\alpha} \right) \). This gives the upper bound.

The proof of the upper bound leads to the optimum choice for \( g \) and the lower bound. We note that we have equality in (2.10) if \( l/(s')^{1/\alpha} \) is constant; i.e. \( l(x) = cs'(x)^{1/\alpha} \) for some constant \( c > 0 \) — chosen so that \( \int_0^\infty l = 2 \). This means we take

\[
h(x) = (s')^{-1} \left( \left( \frac{x}{c} \right)^{\alpha} \right) = \log \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \left( \frac{c}{x} \right)^{2\alpha}} \right).
\]

from which we can calculate \( g \). In fact, we show that we get the required lower bound by just considering \( a_n \) completely multiplicative. To this end we use (2.3), and define \( a_p \) by:

\[
a_p = g_0 \left( \frac{p}{T} \right),
\]

where \( T = \log T \log \log T \) and \( g_0 \) is the function

\[
g_0(x) = \sqrt{1 - \frac{2}{1 + \sqrt{1 + \left( \frac{c}{x} \right)^{2\alpha}}}},
\]

with \( c = 2^{1+1/\alpha}/B \left( \frac{1}{\alpha}, 1 - \frac{1}{2\alpha} \right) \). As such, by the same methods as before, we have \( \|a\| = T^{1+o(1)} \) and

\[
\log \|\varphi_a\| \|a\| = \sum_p f(p) g_0 \left( \frac{p}{T} \right) + O(1) \sim \frac{p f(P)}{\log P} \int_0^\infty \frac{g_0(u)}{u^\alpha} \, du.
\]

By the choice of \( g_0 \), the integral on the right is \( \frac{B \left( \frac{1}{1-\alpha} \right)}{(1-\alpha)^2} \), as required.

\[\square\]

**Remark.** From the above proof, we see that the supremum (of \( \|\varphi_f a\|/\|a\| \)) over \( M_2^2 \) is roughly the same size as the supremum over \( M^2 \); i.e. they are log-asymptotic to each other. Is it true that these respective suprema are closer still; eg. are they asymptotic to each other for \( \frac{1}{2} < \alpha < 1 \)?

3. The special case \( f(n) = n^{-\alpha} \).

In this case we can take \( \tilde{f}(x) = x^{-\alpha} \) which is regularly varying of index \(-\alpha\). Here we shall write \( \varphi_n \) for \( \varphi_f \) and \( M_n \) for \( M_f \).
In a recent paper (see [8]), Lamzouri suggests log
for the remaining range
for all large enough
But
the product into the ranges
decreasing. Let
ourselves with
we show in the appendix, we cannot do this by restricting
since, for
here are just larger than what is known about the lower bounds for
the (conjectured) maximal order of
Remark. As noted in the introduction, these asymptotic formulae bear a strong resemblance to
the analogy — if such exists — between
Mlamzouri has
C(α) = G1(α)α−2α(1−α)α−1, where
G1(x) = ∫0∞ u−1−1/2 log(∑∞n=0 u/2n)(2n)−1du.
by (2.9). Thus
\[
\frac{\|\varphi_1 a\|}{\|a\|} \leq e^\gamma \left( \log \log_2 T + \log_3 T + \int_0^A \frac{g(u)}{u} \, du - \int_0^1 \frac{1}{u} \, du + \frac{2}{\sqrt{A}} + o(1) \right)
\]
for all $A > 1 > a > 0$. We need to minimise the constant term. Since $g(u) < 1$, the minimum occurs for $a$ arbitrarily small. On the other hand $\int_0^A \frac{g(u)}{u} \, du \leq \left( \frac{1}{A} \int_0^\infty g^2 \right)^{1/2} = o(1/\sqrt{A})$, so the constant is minimised for arbitrarily large $A$; i.e. it is at most $\int_1^\infty \frac{a(u)}{u} \, du - \int_0^1 \frac{1-g(u)}{u} \, du$. Thus
\[
M_1(T) \leq e^\gamma \left( \log \log T + \log \log T + \kappa + o(1) \right) \quad \text{where} \quad \kappa = \sup \{ L(g) : g \in G \}.
\]
Here $L(g) = \int_0^\infty \frac{g(u)}{u} \, du - \int_0^1 \frac{1-g(u)}{u} \, du$ and $G$ is the set of all decreasing $g : (0, \infty) \to (0, 1)$ for which $\int_0^\infty \log \frac{1}{1-g} \leq 2$. As in the proof of Theorem 2.3, let $h = \log \frac{1}{1-g}$ so that $g = s \circ h$ where $s(x) = \sqrt{1-e^{-x}}$. Now we show $\kappa = 2 \log 2 - 1$. Trivially, by Cauchy-Schwarz, we have
\[
L(g) \leq \sqrt{\int_0^\infty \frac{1}{u^2} \, du \int_0^\infty g(u)^2 \, du} \leq \sqrt{\int_0^\infty h \leq \sqrt{2}},
\]
so $\kappa \leq \sqrt{2}$.

Note that the supremum is achieved for $\int_0^\infty h = 2$. For if $\int_0^\infty h < 2$, then we can always increase $g$ by a small amount while keeping it less than 1 and decreasing, while $\int h$ is increased by a prescribed amount – just take $g_1 = k \circ g$ where $k : (0, 1) \to (0, 1)$ is increasing and $k(x) > x$. With $k(x) < x$ sufficiently small, $\int h_1 \leq 2$ while $L(g_1) > L(g)$.

Further, we may use the decreasing property of $g$ for which $g$ is continuously differentiable and strictly decreasing, since they can approximate functions in $G$ arbitrarily closely.

Now, for $L(g)$ to be finite (i.e. $> -\infty$) we need $\int_0^1 \frac{1-g(u)}{u} \, du$ to converge. For $x \in (0, 1)$,
\[
\int_x^\infty \frac{1-g(u)}{u} \, du \geq (1-g(x)) \int_x^\infty \frac{1}{u} \, du = \frac{1}{2} (1-g(x)) \log \frac{1}{x}.
\]

The LHS tends to 0 as $x \to 0^+$, so we must have
\[
(1-g(x)) \log x \to 0 \quad \text{as} \quad x \to 0^+.
\]
In particular, $g(x) \to 1$ as $x \to 0^+$ (so $h(x) \to \infty$ as $x \to 0^+$). Also, as in Theorem 2.3, $xh(x) \to 0$ as $x \to \infty$. Now, with $g = s \circ h$,
\[
\int_1^\infty \frac{g(u)}{u} \, du = [g(u) \log u]_1^\infty - \int_1^\infty s'(h(u)) h'(u) \log u \, du = \int_0^{h(1)} s'(y) \log l(y) \, dy,
\]
where $l = h^{-1}$ is the inverse function of $h$. Also,
\[
\int_0^1 \frac{1-g(u)}{u} \, du = [(1-g(u)) \log u]_0^1 + \int_0^1 s'(h(u)) h'(u) \log u \, du = - \int_{h(1)}^\infty s'(y) \log l(y) \, dy.
\]
Hence $L(g) = \int_0^\infty s' \log l$ and $\int_0^\infty l = 2$.

Now, using Jensen’s inequality $\int \log f \, d\mu \leq \log(\int f \, d\mu)$ for $\mu$ a probability measure ([11], p.62), we have
\[
\int_0^\infty s' \log l \leq \log 2 + \int_0^\infty s' \log s' = \log 2 + \int_0^1 \log \left( \frac{1-u^2}{2u} \right) \, du = 2 \log 2 - 1.
\]

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after some calculation.

The proof of the upper bound leads to the optimum choice for \( g \) and the lower bound. We note that we have equality in (3.3) if \( l/s' \) is constant; i.e. \( l(x) = c s'(x) \) for some constant \( c > 0 \) — chosen so that \( \int_0^\infty l = 2 \) (i.e. we take \( c = 2 \)). Thus, actually \( \kappa = 2 \log 2 - 1 \) and the supremum is achieved for the function \( g_0 \), where

\[
g_0(x) = \left( 1 - \frac{2}{1 + \sqrt{1 + \left( \frac{2}{x} \right)^2}} \right).
\]

In fact, we show that we get the required lower bound by just considering \( a \) completely multiplicative. To this end we use Corollary 2.5, and define \( a_p \) by:

\[
a_p = g_0 \left( \frac{p}{P} \right),
\]

where \( P = \log T \log \log T \). As such, by the same methods as before, we have \( \|a\| = T^{1+o(1)} \). Let \( a > 0 \) and \( P = \log T \log \log T \). By Corollary 2.5

\[
\|\varphi_1 a\| \geq \prod_p \frac{1 - a_p}{1 - \frac{1}{p}} \prod_{p \leq aP} \frac{1}{1 + \frac{1 - g_0}{p^{-1} - aP}} \prod_{p > aP} \frac{1}{1 - \frac{1}{p}}. \tag{3.4}
\]

Using Merten’s Theorem, the first product on the right is \( e^\gamma (\log aP + o(1)) \), while the second product is greater than

\[
\exp \left\{ - \sum_{p \leq aP} \frac{1 - a_p}{p - 1} \right\} \geq 1 - 2 \sum_{p \leq aP} \frac{1 - g_0(p/P)}{p}.
\]

The sum is asymptotic to \( a \int_0^a \frac{1 - g_0(u) du}{u} < \frac{\varepsilon}{\log P} \), for any given \( \varepsilon > 0 \), for sufficiently small \( a \).

The third product in (3.4) is greater than

\[
\exp \left\{ \sum_{p > aP} \frac{a_p}{p} \right\} = \exp \left\{ \frac{(1 + o(1))}{\log P} \int_a^\infty \frac{g_0(u)}{u} du \right\}
\]

by (2.9). Thus

\[
\|\varphi_1 a\| \geq e^{\gamma} \left( \log P + \int_a^\infty \frac{g_0(u)}{u} du + \log a - \varepsilon \right) \geq e^{\gamma} \left( \log P + L(g_0) - \varepsilon \right)
\]

for \( a \) sufficiently small. As \( L(g_0) = 2 \log 2 - 1 \) and \( \varepsilon \) arbitrary, this gives the required lower bound. \( \square \)

**Lower bounds for \( \varphi_\alpha \) and some further speculations**

We can study lower bounds of \( \varphi_\alpha \) via the function

\[
m_\alpha(T) = \inf_{a \in M_\alpha^2} \frac{\|\varphi_\alpha a\|}{\|a\|}.
\]

Using very similar techniques, one obtains analogous results to Theorem 3.1:

\[
\frac{1}{m_1(T)} = \frac{6 e^{\gamma}}{\pi^2} (\log \log T + \log \log \log T + 2 \log 2 - 1 + o(1))
\]

and

\[
\log \frac{1}{m_\alpha(T)} \sim \frac{B(\frac{1}{\alpha}, 1 - \frac{1}{\alpha}^2)}{(1 - \alpha)^{2+\alpha} (\log \log T)^{\alpha}} \quad \text{for} \quad \frac{1}{2} < \alpha < 1.
\]

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We see that $m_\alpha(T)$ corresponds closely to the conjectured minimal order of $|\zeta(\alpha + iT)|$ (see [3] and [9]). We omit the proofs, but just point out that for an upper bound (for $1/m_\alpha(T)$) we use

$$
\frac{||a||}{\|\varphi_\alpha a\|} \leq \prod_p \left(1 + \frac{\gamma_p}{p^\alpha}\right),
$$

which can be obtained in much the same way as (2.5). For the lower bound, we choose $a_p$ as $-1$ times the choice in Theorem 3.1 and use Corollary 2.5.

The above formulae suggest that the supremum (respectively infimum) of $\|\varphi_\alpha a\|/\|a\|$ with $a \in \mathcal{M}^2$ and $\|a\| = T$ are close to the supremum (resp. infimum) of $|\zeta_\alpha|$ on $[1, T]$. One could therefore speculate further that there is a close connection between $\|\varphi_\alpha a\|/\|a\|$ (for such $a$) and $|\zeta(\alpha + iT)|$, and hence between $Z_\alpha(T)$ and $M_\alpha(T)$. Recent papers by Gonek [4] and Gonek and Keating [5] suggest this may be possible, or at least that $M_\alpha$ is a lower bound for $Z_\alpha$. On the Riemann Hypothesis, it was shown in [4] (Theorem 3.5) that $\zeta(s)$ may be approximated for $\sigma > \frac{1}{2}$ up to height $T$ by the truncated Euler product

$$
\prod_{p \leq P} \frac{1}{1 - p^{-s}} \quad \text{for } P \ll T.
$$

Thus one might expect that, with $a \in \mathcal{M}^2_+$ maximizing $\|\varphi_\alpha a\|/\|a\|$ subject to $\|a\| = T$, and $A(s) = \prod_{p \leq P} \frac{1}{1 - a_p s}$ (with $P \ll T$),

$$
\int_{-T}^T |\zeta(\alpha - it)|^2 |A(it)|^2 \, dt \sim \int_{-T}^T \left|1 - \frac{p^it}{p^\alpha}\right|^{2} \left|1 - a_pp^it\right|^{-2} \, dt = \int_{-T}^T \sum_{p \leq P} |B_p(it)|^2 \, dt
$$

where $B_p(s) = \sum_{k \geq 0} b_{k,p} p^{-ks}$. The heuristics of Gonek and Keating now suggests this is asymptotic to

$$
2T \sum_{p \leq P} \sum_{k \geq 0} b_{k,p}^2 \sim 2T \|\varphi_\alpha a\|^2
$$

if $P \gg T \log \log T$ (for the last step). Thus it would follow that

$$
Z_\alpha(T)^2 \geq \frac{\int_{-T}^T |\zeta(\alpha - it)|^2 |A(it)|^2 \, dt}{\int_{-T}^T |A(it)|^2 \, dt} \sim \frac{2T \|\varphi_\alpha a\|^2}{2T \|a\|^2} \sim M_\alpha(T)^2
$$

and hence $Z_\alpha(T) \geq M_\alpha(T)$.

As mentioned before, this would contradict Lamzouri’s suggestion (that $\log Z_\alpha(T) \sim C(\alpha)(\log T)^{1-\alpha}(\log \log T)^{-\alpha}$) since $C(\alpha) < c(\alpha)$ (notation from Theorem 2.3) for $\alpha$ sufficiently close to $\frac{1}{2}$ at least. It is unclear to the author which possibility is more likely.

References

Appendix

Here we show that if \( f \notin l^2 \), we cannot hope to ‘capture’ \( \varphi_f \) by considering the mapping on some non-trivial subset of \( l^2 \).

**Proposition A1**

Suppose \( \sum_p |f(p)|^2 \) diverges, where \( p \) ranges over the primes. Then \( \varphi_f a \in l^2 \) for \( a \in l^2 \) if and only if \( a = 0 \).

**Proof.** Suppose there exists \( a \in l^2 \) with \( a \neq 0 \) such that \( \varphi_f a \in l^2 \). Let \( a_m \) be the first non-zero coordinate for \( a \). Let \( b = (b_n) = \varphi_f a \in l^2 \). Consider \( b_{pm} \) for \( p \) prime such that \( p \mid m \). We have

\[
b_{pm} = \sum_{d \mid pm} f(d)a_{pm/d} = a_m f(p) + k(p),
\]

where \( k(p) = \sum_{d \mid m} f(d)a_{pm/d} \). Since

\[
\sum_p |k(p)|^2 \leq \sum_p \left( \sum_{d \mid m} |f(d)|^2 \sum_{d \mid m} |a_{pm/d}|^2 \right) \leq A \sum_{d \mid m} \sum_p |a_{pm/d}|^2 < \infty,
\]

and \( \sum_p |b_{pm}|^2 \) converges, we must have

\[
|a_m|^2 \sum_p |f(p)|^2 < \infty.
\]

This is a contradiction. \( \square \)