Ω-results for Beurling’s zeta function and lower bounds for the generalised Dirichlet divisor problem

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Abstract

In this paper we study generalised prime systems for which the integer counting function $N_P(x)$ is asymptotically well-behaved, in the sense that $N_P(x) = \rho x + O(x^\beta)$, where \( \rho \) is a positive constant and $\beta < \frac{1}{2}$. For such systems, the associated zeta function $\zeta_P(s)$ is holomorphic for $\Re s > \beta$. We prove that for $\beta < \sigma < \frac{1}{2}$,\( \int_0^T |\zeta_P(\sigma + it)|^2 dt = \Omega(T^{2-2\sigma-\varepsilon}) \) for any $\varepsilon > 0$, and also for $\varepsilon = 0$ for all such $\sigma$ except possibly one value.

The Dirichlet divisor problem for generalised integers concerns the size of the error term $N_kP(x) - \text{Res}_{s=1}(\zeta_P(s)x^s/s)$, which is $O(x^\theta)$ for some $\theta < 1$. Letting $\alpha_k$ denote the infimum of such $\theta$, we show that $\alpha_k \geq 1/2 - 1/k$.

Keywords: Beurling’s generalised primes, Dirichlet divisor problem.

1. Introduction

A generalised prime system (or $g$-prime system) $P$ is a sequence of positive reals $p_1, p_2, p_3, \ldots$ satisfying

$$1 < p_1 \leq p_2 \leq \cdots \leq p_n \leq \cdots$$

and for which $p_n \to \infty$ as $n \to \infty$. From these can be formed the system $\mathcal{N}$ of generalised integers or Beurling integers; that is, the numbers of the form

$$p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

where $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \mathbb{N}_0$. Such systems were first introduced by Beurling [2] and have been studied by many authors since then (see in particular [1]). Define the g-integer counting function $N_P(x)$ and the associated Beurling zeta function, respectively, by

$$N_P(x) = \sum_{n \in \mathcal{N}, n \leq x} 1, \quad \zeta_P(s) = \sum_{n \in \mathcal{N}} \frac{1}{n^s},$$

(Here, $\sum_{n \in \mathcal{N}}$ means a sum over all the g-integers, counting multiplicities.) In this paper, we shall be concerned with g-prime systems for which

$$N_P(x) = \rho x + O(x^\beta), \quad (1.1)$$

for some $\beta < \frac{1}{2}$ and $\rho > 0$. Then $\zeta_P(s)$ is defined and holomorphic for $\Re s > 1$, and has an analytic continuation to the half-plane $\Re s > \beta$ except for a simple pole at $s = 1$ with residue $\rho$. Furthermore, $\zeta_P(s)$ has finite order for $\Re s > \beta$; i.e. $\zeta_P(\sigma + it) = O(|t|^\lambda)$ for some $\lambda$ for $\sigma > \beta$. Let $\mu_P(\sigma)$ denote the infimum of all such $\lambda$. It is well-known that $\mu_P(\sigma)$ is non-negative,

\[2\]Here, $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{P} = \{2, 3, 5, \ldots\}$ — the set of primes.
decreasing, and convex (and hence continuous) (see, for example, [5]). For \( P = \mathbb{P} \) (so that \( \mathcal{N} = \mathbb{N} \)), the Lindelöf Hypothesis is the conjecture that \( \mu_P(\sigma) = \mu_0(\sigma) \) for all \( \sigma \), where

\[
\mu_0(\sigma) = \begin{cases} 
\frac{1}{2} - \sigma & \text{if } \sigma < \frac{1}{2} \\
0 & \text{if } \sigma \geq \frac{1}{2} . 
\end{cases}
\]

In [4], it was proven that for all g-prime systems satisfying (1.1), \( \mu_P(\sigma) \) must be at least as large as \( \mu_0(\sigma) \): i.e. \( \mu_P(\sigma) \geq \frac{1}{2} - \sigma \) for \( \sigma \in (\beta, \frac{1}{2}) \). In this paper we prove a stronger result by considering the mean square behaviour of \( \zeta_P(\sigma + it) \). For \( \sigma > \beta \), define \( \nu_P(\sigma) \) to be the infimum of numbers \( \lambda \) such that

\[
\int_1^T |\zeta_P(\sigma + it)|^2 \, dt = O(T^{1+2\lambda}).
\]

As in the case of \( \mu_P(\sigma) \), \( \nu_P(\sigma) \) is non-negative and convex decreasing (cf. [6], §7.8). Trivially, \( \nu_P(\sigma) \leq \mu_P(\sigma) \). We show here that \( \nu_P(\sigma) \geq \mu_0(\sigma) \). In fact we prove slightly more.

**Theorem 1**

Let \( P \) be a g-prime system for which (1.1) holds for some \( \beta < \frac{1}{2} \) and \( \rho > 0 \). Then \( \nu_P(\sigma) \geq \mu_0(\sigma) \) for \( \sigma \in (\beta, \frac{1}{2}) \). Furthermore,

\[
\int_0^T |\zeta_P(\sigma + it)|^2 \, dt = o(T^{2-2\sigma}) \tag{1.2}
\]

can hold for at most one value of \( \sigma \) in this range. In this case \( T^{2\sigma-2} \int_0^T |\zeta_P(\sigma + it)|^2 \, dt \) is unbounded for all other values of \( \sigma \).

**Remark.** For \( P = \mathbb{P} \), we have \( \nu_P(\sigma) = \mu_0(\sigma) \), which shows the first part of Theorem 1 is best possible. However, in this case we have the asymptotic formula

\[
\int_1^T |\zeta(\sigma + it)|^2 \, dt \sim \frac{\zeta(2-2\sigma)}{(2\pi)^{1-2\sigma}(2-2\sigma)}T^{2-2\sigma}
\]

for \( 0 < \sigma < \frac{1}{2} \), showing that the exceptional value need not exist. In fact it seems unlikely an exceptional value exists and hence that \( \int_0^T |\zeta_P(\sigma + it)|^2 \, dt = \Omega(T^{2-2\sigma}) \) for all \( \sigma \in (\beta, \frac{1}{2}) \), but we cannot quite show this. Furthermore it seems plausible that we should have \( \int_0^T |\zeta_P(\sigma + it)|^2 \, dt \geq C_\sigma T^{2-2\sigma} \) for some \( C_\sigma > 0 \).

**2. Dirichlet divisor problems for g-primes**

For a g-prime system satisfying (1.1) (with \( \beta < 1 \)), we can study the equivalent of the Dirichlet divisor problem concerning the error term in the asymptotic formula for the average of the ‘generalised divisor’ function. For \( k \in \mathbb{N} \), let \( kP \) denote the g-prime system obtained from \( P \) by letting every g-prime from \( P \) be counted \( k \) times. (If an original g-prime has multiplicity \( m \), then in the new system it will have multiplicity \( km \).) The Beurling zeta function of \( kP \) is

\[
\zeta_{kP}(s) = \zeta_P(s)^k.
\]

By standard methods using Perron’s formula,

\[
N_{kP}(x) = \text{Res}_{s=1} \left\{ \frac{\zeta_P(s)^k}{s} x^s \right\} + \Delta_{P,k}(x) = xP_{k-1}(\log x) + \Delta_{P,k}(x),
\]

2
where $P_{k-1}(\cdot)$ is a polynomial of degree $k - 1$ and $\Delta_{P,k}(x) = O(x^\theta)$ for some $\theta < 1$, depending on $k$. Let $\alpha_k$ denote the infimum of such $\theta$. The generalised Dirichlet divisor problem is the problem of determining $\alpha_k$. Also let $\beta_k$ denote the infimum of $\nu$ for which
\[
\int_0^x \Delta_{P,k}(y)^2 \, dy = O(x^{1+2\phi}).
\]
Trivially, $\beta_k \leq \alpha_k$.

For $\mathcal{P}$, it is known that
\[
\alpha_k \geq \beta_k \geq \frac{1}{2} - \frac{1}{2k}
\]
and it is conjectured that there is equality throughout (actually $\beta_k = \frac{1}{2} - \frac{1}{2k}$ for all $k$ is equivalent to the Lindelöf Hypothesis — see [6], Theorem 13.4). We use Theorem 1 to show that (2.1) remains true for $\mathcal{P}$ satisfying (1.1). In fact we have the following two corollaries:

**Corollary 2**
Let $\mathcal{P}$ satisfy (1.1) for some $\beta < \frac{1}{2}$. Then for $\sigma \in (\beta, \frac{1}{2} - \frac{1}{2k})$, 
\[
\int_{-\infty}^{\infty} \frac{\zeta_p(\sigma + it)^{2k}}{|\sigma + it|^2} \, dt
\]
diverges. Further, if $\frac{1}{2} - \frac{1}{2k}$ is not the exceptional value in (1.2), then the integral also diverges for $\sigma = \frac{1}{2} - \frac{1}{2k}$.

**Corollary 3**
Let $\mathcal{P}$ satisfy (1.1) for some $\beta < \frac{1}{2}$. With $\alpha_k$ and $\beta_k$ as above, $\alpha_k \geq \beta_k \geq \max\{\beta, \frac{1}{2} - \frac{1}{2k}\}$.

3. **Proofs**

**Proof of Theorem 1.** If $\nu_p(\sigma') < \frac{1}{2} - \sigma'$ for some $\sigma' \in (\beta, \frac{1}{2})$ then, by continuity of $\nu_p(\cdot)$, $\nu_p(\sigma) < \frac{1}{2} - \sigma$ throughout some interval around $\sigma'$ and (1.2) holds for all such $\sigma$; in particular for two such values. We shall show that this is impossible.

Suppose, for a contradiction, that (1.2) holds for $\sigma = \sigma_0, \sigma_1$ where $\beta < \sigma_0 < \sigma_1 < \frac{1}{2}$.

For $N \geq 1$ let $\zeta_{N,p}(s) = \sum_{n \leq N} n^{-s}$, where the sum ranges over $n \in \mathcal{N}$. As was stated in [4] (and shown in [3]), for $\sigma < \frac{1}{2}$ there exist constants $c_1, c_2 > 0$ such that for $R \geq c_1 N$, 
\[
\sum_{r=1}^{R} \int_0^{2r-1} |\zeta_{N,p}(\sigma + it)|^2 \, dt \geq c_2 R^2 N^{1 - 2\sigma}.
\]
(3.1)

Also, writing $s = \sigma + it$, and following the arguments in [3], we have
\[
\zeta_{N,p}(s) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta_p(s + w) N^w}{w} \, dw + O\left(\frac{N^c}{T(c + \sigma - 1)}\right) + O\left(\frac{N^{1-\sigma}}{T}\sum_{\frac{N}{2} < n < 2N} \frac{1}{|n - N|}\right),
\]
(3.2)

for $|t| < T$, $c > 1 - \sigma$ and $N \notin \mathcal{N}$. We shall put $c = 1 - \sigma + \frac{1}{\log N}$ and choose $N$ in such a way that $(N - \alpha, N + \alpha) \cap \mathcal{N} = \emptyset$. (As was shown in [4], this is possible for arbitrarily large $N$ if
$0 < \alpha < \frac{1}{4\pi}$.) With this choice of $N$, the final sum in (3.2) was shown to be $O(\sqrt{N})$. As such (3.2) becomes

$$\zeta_{N,P}(s) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta_P(s+w)N^w}{w} \, dw + O\left(\frac{N^{\frac{3}{2} - \sigma}}{T}\right). \quad (3.3)$$

Now put $\sigma = \sigma_1$ and push the contour in the integral to the left as far as $\Re w = \sigma_0 - \sigma_1 < 0$, picking up the residues at $w = 0$ and $w = 1 - s$ (since $|t| < T$).

The contribution along the horizontal line $[\sigma_0 - \sigma_1 + iT, c + iT]$ is, in modulus, less than

$$\frac{1}{2\pi} \int_{\sigma_0 - \sigma_1}^{\epsilon} N^y|\zeta_P(\sigma_1 + y + i(t + T))| \, dy.$$ 

Using the uniform bound $|\zeta_P(\sigma + it)| = O(t^{1-\beta+\varepsilon})$, this is at most a constant times

$$\frac{1}{T} \int_{\sigma_0 - \sigma_1}^{1-\sigma_1} T^{-\beta+\varepsilon}N^y \, dy + \frac{1}{T} \int_{1-\sigma_1}^{1-\sigma_1 + \frac{1}{2\pi N}} T^\sigma N^y \, dy = O(T^{1-\beta+\varepsilon} N^{\sigma_0 - \sigma_1}) + O(T^{\varepsilon - 1} N^{1 - \sigma_1}). \quad (3.4)$$

Similarly on $[\sigma_0 - \sigma_1 - iT, c - iT]$.

The integral along $\Re w = \sigma_0 - \sigma_1$ is at most

$$\frac{N^{\sigma_0 - \sigma_1}}{2\pi} \int_{-T}^{T} \frac{|\zeta_P(\sigma_0 + i(t + y))|}{\sqrt{(\sigma_0 - \sigma_1)^2 + y^2}} \, dy = O\left(N^{\sigma_0 - \sigma_1} \int_{1}^{2T} \frac{|\zeta_P(\sigma_0 + iy)|}{y} \, dy\right)$$

$$= o\left(N^{\sigma_0 - \sigma_1} T^{\frac{3}{2} - \sigma_0}\right), \quad (3.5)$$

using\(^3\) the hypothetical bound $\int_{0}^{T} |\zeta_P(\sigma_0 + it)|^2 \, dt = o(T^{2 - 2\sigma_0})$.

The residues at $w = 0$ and $w = 1 - s$ are, respectively, $\zeta_P(s)$ and $\rho N^{1-s}/(1-s) = O(\frac{N^{1-s}}{|t| + 1})$. Putting (3.3), (3.4), and (3.5) together gives

$$\zeta_{N,P}(\sigma_1 + it) = \zeta_P(\sigma_1 + it) + O\left(\frac{N^{1-s}}{|t| + 1}\right) + O(N^{1-s} T^{\varepsilon - 1}) + o(N^{\sigma_0 - \sigma_1} T^{\frac{3}{2} - \sigma_0}) + o\left(N^{\frac{3 - \sigma_1}{2}}\right),$$

for $|t| < T$. (Note that the first $O$-term in (3.4) is superfluous since $\frac{\beta - \sigma_0}{1 - \beta} < \frac{1}{2} - \sigma_0$.) Hence, using $(a + b + c + d + e)^2 \leq 5(a^2 + b^2 + c^2 + d^2 + e^2)$, we have

$$|\zeta_{N,P}(\sigma_1+it)|^2 \leq 5|\zeta_P(\sigma_1+it)|^2 + O\left(\frac{N^{2-2\sigma_1}}{t^2+1}\right) + O(N^{2-2\sigma_1} T^{2\varepsilon - 2}) + o(N^{2\sigma_0 - 2\sigma_1} T^{1 - 2\sigma_0}) + o\left(\frac{N^{3-2\sigma_1}}{T^2}\right).$$

Now apply $\sum_{r=1}^{R} \int_{0}^{2r-1} \ldots dt$ to both sides to give (for $2R-1 < T$)

$$\sum_{r=1}^{R} \int_{0}^{2r-1} |\zeta_{N,P}(\sigma_1+it)|^2 \, dt = O\left(\sum_{r=1}^{R} \int_{0}^{2r-1} |\zeta_P(\sigma_1+it)|^2 \, dt\right) + \sum_{r=1}^{R} \int_{0}^{2r-1} N^{2-2\sigma_1} (t+1)^2 \, dt$$

$$= O(R^3 N^{2-2\sigma_1} T^{2\varepsilon - 2}) + O\left(\frac{R^2 N^{3-2\sigma_1}}{T^2}\right) + o(R^2 N^{2(\sigma_0-\sigma_1)} T^{1-2\sigma_0})$$

$$= o(R^3 - 2\sigma_1) + O(R N^{2-2\sigma_1}) + O(R^2 N^{2-2\sigma_1} T^{2\varepsilon - 2}) + O\left(\frac{R^2 N^{3-2\sigma_1}}{T^2}\right) + o(R^2 N^{2(\sigma_0-\sigma_1)} T^{1-2\sigma_0})$$

\(^3\)If $f \geq 0$ and $\int_{0}^{T} f^2 \, dy = o(T^\lambda)$ (some $\lambda > 1$), then $\int_{T/2}^{T} f(y)^2 \, dy \leq \frac{2}{T} \int_{0}^{T} f^2 \leq \frac{2}{T} \sqrt{T} \int_{0}^{T} f^2 = o(T^{\frac{\lambda - 1}{2}})$, and $\int_{T/2}^{T} f(y)^2 \, dy = o(T^{\frac{\lambda - 1}{2}})$ follows.
using (1.2) for $\sigma_1$. Let $T = 2R$. The left-hand side above is at least $c_2 R^2 N^{1 - 2\sigma_1}$ by (3.1) if $R \geq c_1 N$. Dividing both sides through by $R^2 N^{1 - 2\sigma_1}$ gives

$$c_2 \leq o\left( \left( \frac{R}{N} \right)^{1 - 2\sigma_1} \right) + O\left( \frac{N}{R} \right) + O(NR^{2\varepsilon - 2}) + O\left( \frac{N^2}{R^2} \right) + o\left( \left( \frac{R}{N} \right)^{1 - 2\sigma_0} \right).$$

(3.6)

Put $R = KN$ where $K \geq c_1$ is a fixed, but arbitrary, constant. Letting $N \to \infty$, the $o$-terms both tend to zero as does the middle $O$-term. Hence

$$c_2 \leq \frac{A}{K} + \frac{B}{K^2}$$

for some absolute constants $A, B$. But $K$ can be made arbitrarily large, so this gives a contradiction.

For the final part, suppose (1.2) holds for $\sigma = \sigma_0$ say. If $\int_0^T |\zeta_p(\sigma + it)|^2 \, dt = O(T^{2 - 2\sigma'})$ for some $\sigma' \in (\beta, \frac{1}{2})$ with $\sigma' \neq \sigma_0$, then (1.2) actually holds for all $\sigma$ between $\sigma_0$ and $\sigma'$. (This follows from the Phragmen-Lindelöf Theorem for a strip (see [6], §7.8, with $\varepsilon$ in the place of $C$)). This was shown to be impossible, and hence $T^{2\sigma - 2} \int_0^T |\zeta_p(\sigma + it)|^2 \, dt$ must be unbounded for all $\sigma \neq \sigma_0$.

Now we apply Theorem 1 to find lower bounds in the Dirichlet divisor problem. Note that Theorem 1 actually shows that given $\varepsilon > 0$,

$$\int_{T/2}^T |\zeta_p(\sigma + it)|^2 \, dt = \Omega(T^{2 - 2\sigma - \varepsilon}),$$

for if it was $o(T^{2 - 2\sigma - \varepsilon})$, then by telescoping it would follow that $\int_0^T |\zeta_p(\sigma + it)|^2 \, dt = o(T^{2 - 2\sigma - \varepsilon})$ which is false.

**Proofs of Corollaries 2 and 3.** By Hölder’s inequality,

$$\int_{T/2}^T |\zeta_p(\sigma + it)|^{2k} \, dt \geq \frac{2^{k-1}}{T^{k-1}} \left( \int_{T/2}^T |\zeta_p(\sigma + it)|^2 \, dt \right)^k,$$

for every $k \in \mathbb{N}$. By Theorem 1, given $\varepsilon > 0$, $\int_{T/2}^T |\zeta_p(\sigma + it)|^2 \, dt \geq aT^{2 - 2\sigma - \varepsilon}$ for some $a > 0$ and some arbitrarily large $T$. Hence for such $T$,

$$\int_{T/2}^T |\zeta_p(\sigma + it)|^{2k} \, dt \geq a^k T^{k(1 - 2\sigma) + 1 - \varepsilon k}.$$

It follows that

$$\int_{T/2}^T \frac{|\zeta_p(\sigma + it)|^{2k}}{|\sigma + it|^2} \, dt \geq a'T^{k(1 - 2\sigma) - 1 - \varepsilon k}$$

for some $a' > 0$. But for $\sigma < \frac{1}{2} - \frac{1}{2k}$, we have $k(1 - 2\sigma) - 1 > 0$. Hence for $\varepsilon$ sufficiently small, $k(1 - 2\sigma) - 1 - \varepsilon k > 0$ also, and so $\int_{T/2}^T \frac{|\zeta_p(\sigma + it)|^{2k}}{|\sigma + it|^2} \, dt \not\to 0$ as $T \to \infty$, and Corollary 2 follows.

Of course, if $\frac{1}{2} - \frac{1}{2k}$ is not the exceptional value in Theorem 1, then we can take $\varepsilon = 0$ in the above and the result also holds for $\sigma = \frac{1}{2} - \frac{1}{2k}$.  

5
Let $\gamma_k$ be the infimum of $\sigma$ (with $\sigma > \beta$) for which $\int_{-\infty}^{\infty} \frac{|\zeta_p(\sigma+it)|^{2k}}{|\sigma+it|^2} dt$ converges. By Corollary 2, $\gamma_k \geq \frac{1}{2} - \frac{1}{2k}$. An identical argument as in the $P = \mathbb{P}$ case (see [6], Theorem 12.5) shows that $\gamma_k = \beta_k$. (The argument is simply based upon Parseval’s formula for Mellin transforms, which in this case is the identity

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\zeta_p(\sigma+it)|^{2k}}{|\sigma+it|^2} dt = \int_0^\infty \frac{\Delta_{\mathbb{P},k}(x)^2}{x^{1+2\sigma}} dx$$

for $\sigma$ in some interval $(\theta, 1)$ with $\theta < 1$.) Hence $\beta_k \geq \frac{1}{2} - \frac{1}{2k}$.

\[\square\]

4. On the line $\sigma = \frac{1}{2}$

In this article, we have considered the mean-value along vertical lines $\Re s = \sigma$ with $\sigma < \frac{1}{2}$. This raises the question of what happens on the line $\sigma = \frac{1}{2}$. For $P = \mathbb{P}$, we have $\int_0^T |\zeta_p(\frac{1}{2}+it)|^2 dt \sim T \log T$, so do we have $\int_0^T |\zeta_p(\frac{1}{2}+it)|^2 dt = \Omega(T \log T)$ in general? As in the $\sigma < \frac{1}{2}$ case, we relate the behaviour of the mean-square value at $\sigma = \frac{1}{2}$ to the behaviour of the mean-square for some $\sigma = \sigma_0 < \frac{1}{2}$.

Theorem 4

Let $P$ be a g-prime system for which (1.1) holds. If $\int_1^T |\zeta_p(\sigma+it)| \frac{dt}{t} = o(T \log T)^{\frac{1}{2}-\sigma}$ for some $\sigma \in (\beta, \frac{1}{2})$, then $\int_0^T |\zeta_p(\frac{1}{2}+it)|^2 dt = \Omega(T \log T)$.

Note that the assumption is implied by $\int_1^T |\zeta_p(\sigma+it)|^2 dt = o(T^{2-2\sigma} (\log T)^{1-2\sigma})$.

Sketch of Proof. We follow the proof of Theorem 1 as much as possible, this time taking $\sigma_1 = \frac{1}{2}$.

Using the argument in [3] for $\sigma = \frac{1}{2}$, (3.1) becomes: there exist constants $c_1, c_2 > 0$ such that for $R \geq c_1 N / \log N$,

$$\sum_{r=1}^R \int_0^{2r-1} |\zeta_{\mathbb{P},P}(\frac{1}{2}+it)|^2 dt \geq c_2 R^2 \log N. \tag{4.1}$$

To see this, note that we have

$$\int_0^T |\zeta_{\mathbb{P},P}(\frac{1}{2}+it)|^2 dt = T \sum_{n \leq N} \frac{1}{n} + 2 \sum_{n \leq N} \frac{1}{\sqrt{n}} \sum_{m \leq n} \frac{S_m(n)}{\sqrt{m}},$$

where $S_m(n) = \frac{\sin(T \log(n/m))}{\log(n/m)}$. (Here $m, n \in \mathbb{N}$ and the * indicates that any multiplicities must be squared.) In any case, we have $S_m(n) \leq 1 / \log 2$, so this part of the double sum is $O(\sum_{n \leq N} \frac{1}{\sqrt{n}}) \sum_{m \leq n/2} \frac{1}{\sqrt{m}} = O(N)$. Thus, for some positive constants $k_1, k_2, k_3$, independent of $T$ and $N$,

$$\int_0^T |\zeta_{\mathbb{P},P}(\frac{1}{2}+it)|^2 dt \geq k_1 T \log N + 2 \sum_{n \leq N} \frac{1}{\sqrt{n}} \sum_{n \leq n/2} \frac{S_m(n)}{\sqrt{m}} - k_2 N.$$

Putting $T = 2r - 1$ for $r = 1, 2, \ldots, R$, and summing both sides gives, on noticing that

$$\sum_{r=1}^R \sin((2r-1) \log \frac{n}{m}) = \frac{\sin^2((2r-1) \log n/m)}{\sin((2r-1) \log n/m)} \geq 0$$

since $0 < \log n/m < \log 2$,

$$\sum_{r=1}^R \int_0^{2r-1} |\zeta_{\mathbb{P},P}(\frac{1}{2}+it)|^2 dt \geq k_3 R^2 \log N - k_2 R N,$$

\[4\]This follows readily from $N_{\mathbb{P}}(x) \sim \rho x$. 6
and (4.1) follows.

In (3.2), we need a better estimate for the final sum. Let $M \in \mathbb{N}$. Then, with $N$ such that $(N - \alpha, N + \alpha) \cap \mathbb{N} = \emptyset$,

$$
\sum_{\frac{N}{2} < n < 2N} \frac{1}{|n - N|} = \sum_{m=1}^{M} \frac{1}{\alpha N^{m-1}} \sum_{|n-N|<\alpha N} \frac{1}{|n-N|} + O(1)
$$

$$
\leq \frac{1}{\alpha} \sum_{m=1}^{M} \frac{1}{N^{m-1}} \left( N(N + \alpha N^{m/M}) - N(N - \alpha N^{m/M}) \right) + O(1)
$$

$$
= O(N^{1/M}) + O(N^{\beta}),
$$

using (1.1). Since $M$ is arbitrary, this is $O(N^{\beta + \varepsilon})$ for every $\varepsilon > 0$ in any case. Thus (3.3) becomes

$$
\zeta_{N,P}(s) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta(s + w)N^w}{w} dw + O\left(\frac{N^{1+\beta+\varepsilon}}{T}\right).
$$

The analysis up to (3.5) remains the same (with $\sigma_0 = \sigma$ and $\sigma_1 = \frac{1}{2}$) but in (3.5) we use the bound assumed in the statement to give $o(N^{\sigma - \frac{1}{2}}(T \log T)^{\frac{1}{2} - \sigma})$. The arguments following (3.5) remain valid and we put $T = 2R$ again, but this time we divide through by $R^2 \log N$. On assuming $\int_0^T |\zeta_P(\frac{1}{2} + it)|^2 dt = o(T \log T)$, (3.6) now becomes

$$
c_2 \leq o\left(\frac{\log R}{\log N}\right) + O\left(\frac{N}{R \log N}\right) + O\left(\frac{NR^{2\varepsilon-2}}{\log N}\right) + O\left(\frac{N^{1+2\beta+2\varepsilon}}{R^2}\right) + o\left(\left(\frac{R \log R}{N}\right)^{1-2\sigma} \frac{1}{\log N}\right).
$$

Put $R = KN/\log N$ where $K \geq c_1$ is a fixed, but arbitrary, constant. Letting $N \to \infty$, all the terms tend to zero except the first $O$-term. Hence

$$
c_2 \leq \frac{A}{K}
$$

for some absolute constant $A$. As $K$ can be made arbitrarily large, this gives a contradiction. Hence $\int_0^T |\zeta_P(\frac{1}{2} + it)|^2 dt = \Omega(T \log T)$.

□

References
