Extremal index, hitting time statistics and periodicity

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joint work with Jorge Freitas and Mike Todd
Consider a stationary stochastic process $X_0, X_1, X_2, \ldots$ with marginal d.f. $F$.

Let $\bar{F} = 1 - F$ and $u_F = \sup\{x : F(x) < 1\}$.

The main goal of the Extreme Value Theory (EVT) is the study of the distributional properties of the maximum

$$M_n = \max\{X_0, \ldots, X_{n-1}\}$$

as $n \to \infty$. 

(CMUP & FEP)
Definition

We say that we have an Extreme value law (EVL) for $M_n$ if there is a non-degenerate d.f. $H : \mathbb{R} \to [0, 1]$ (with $H(0) = 0$) and for all $\tau > 0$, there exists a sequence of levels $u_n = u_n(\tau)$ such that

$$nP(X_0 > u_n) \to \tau \quad \text{as} \quad n \to \infty,$$

(2)

and for which the following holds:

$$P(M_n \leq u_n) \to \bar{H}(\tau) \quad \text{as} \quad n \to \infty.$$  

(3)
The independent case

In the case $X_0, X_1, X_2, \ldots$ are i.i.d. r.v. then since

$$P(M_n \leq u_n) = (F(u_n))^n$$

we have that if (2) holds, then (3) holds with $\bar{H}(\tau) = e^{-\tau}$:

$$P(M_n \leq u_n) = (1 - P(X_0 > u_n))^n \sim \left(1 - \frac{\tau}{n}\right)^n \rightarrow e^{-\tau} \text{ as } n \rightarrow \infty,$$

and vice-versa.

When $X_0, X_1, X_2, \ldots$ are not i.i.d. but satisfy some mixing condition $D(u_n)$ introduced by Leadbetter then something can still be said about $H$. 
Condition $D(u_n)$ from Leadbetter

Let $F_{i_1,...,i_n}$ denote the joint d.f. of $X_{i_1}, \ldots, X_{i_n}$, and set $F_{i_1,...,i_n}(u) = F_{i_1,...,i_n}(u, \ldots, u)$.

**Condition ($D(u_n)$)**

We say that $D(u_n)$ holds for the sequence $X_0, X_1, \ldots$ if for any integers $i_1 < \ldots < i_p$ and $j_1 < \ldots < j_k$ for which $j_1 - i_p > t$, and any large $n \in \mathbb{N}$,

$$
|F_{i_1,...,i_p,j_1,...,j_k}(u_n) - F_{i_1,...,i_p}(u_n)F_{j_1,...,j_k}(u_n)| \leq \gamma(n, t),
$$

where $\gamma(n, t_n) \xrightarrow{n \to \infty} 0$, for some sequence $t_n = o(n)$. 
**Theorem (Leadbetter)**

If \( D(u_n) \) holds for \( X_0, X_1, \ldots \) and the limit (3) exists for some \( \tau > 0 \) then there exists \( 0 \leq \theta \leq 1 \) such that \( \bar{H}(\tau) = e^{-\theta \tau} \) for all \( \tau > 0 \).

**Definition**

We say that \( X_0, X_1, \ldots \) has an *Extremal Index* (EI) \( 0 \leq \theta \leq 1 \) if we have an EVL for \( M_n \) with \( \bar{H}(\tau) = e^{-\theta \tau} \) for all \( \tau > 0 \).
Linear normalising sequences

The sequences of real numbers $u_n = u_n(\tau), \ n = 1, 2, \ldots$, are usually taken to be one parameter linear families such as $u_n = a_n y + b_n$, where $y \in \mathbb{R}$ and $a_n > 0$, for all $n \in \mathbb{N}$.

Observe that $\tau$ depends on $y$ through $u_n$ and, in fact, depending on the tail of the marginal d.f. $F$, we have that $\tau = \tau(y)$ is of one of the following 3 types (for some $\alpha > 0$):

Type 1: $\tau_1(y) = e^{-y}$ for $y \in \mathbb{R}$,
Type 2: $\tau_2(y) = y^{-\alpha}$ for $y > 0$,
Type 3: $\tau_3(y) = (-y)^\alpha$ for $y \leq 0$.
Characterization of the three types

**Theorem (Gnedenko)**

*Necessary and sufficient conditions for $\tau$ to be of one of the three types are:*

*Type 1:* There exists some strictly positive function $g$ such that, for all real $y$,

$$
\lim_{t \uparrow u_F} \frac{1 - F(t + yg(t))}{1 - F(t)} = e^{-y}.
$$

*Type 2:* $u_F = \infty$ and $\lim_{t \to \infty} \frac{1 - F(ty)}{1 - F(t)} = y^{-\alpha}, \alpha > 0$, for each $y > 0$.

*Type 3:* $u_F < \infty$ and

$$
\lim_{h \downarrow 0} \frac{1 - F(u_F - yh)}{1 - F(u_F - h)} = y^\alpha, \alpha > 0, \text{ for each } y > 0.
$$
Corollary

The constants $a_n$ and $b_n$ may be taken as follows:

Type 1: $a_n = g(\gamma_n), b_n = \gamma_n$;

Type 2: $a_n = \gamma_n, b_n = 0$;

Type 3: $a_n = u_F - \gamma_n, b_n = u_F$,

where $\gamma_n = F^{-1}(1 - 1/n) = \inf\{x : F(x) \geq 1 - 1/n\}$. 
Examples

1. If $F(x) = 1 - e^{-x}$ then $\tau$ is of type 1.

2. If $F(x) = 1 - kx^{-\alpha}$, $\alpha > 0$, $K > 0$, $x \geq K^{1/\alpha}$, then $\tau$ is of type 2.

3. If $F(x) = x$, $0 \leq x \leq 1$, then $\tau$ is of type 3.
Hitting Times and Kac’s Lemma

Consider the system \((\mathcal{X}, \mathcal{B}, \mu, f)\), where \(\mathcal{X}\) is a topological space, \(\mathcal{B}\) is the Borel \(\sigma\)-algebra, \(f : \mathcal{X} \to \mathcal{X}\) is a measurable map and \(\mu\) is an \(f\)-invariant probability measure, i.e., \(\mu(f^{-1}(B)) = \mu(B)\), for all \(B \in \mathcal{B}\).

For a set \(A \subset \mathcal{X}\) let \(r_A(x)\) the first hitting time to \(A\) of the point \(x\), i.e. \(r_A(x) = \min\{j \in \mathbb{N} : f^j(x) \in A\}\).

Let \(\mu_A\) denote the conditional measure on \(A\), i.e. \(\mu_A := \frac{\mu|_A}{\mu(A)}\).

By Kac’s Lemma, the expected value of \(r_A\) with respect to \(\mu_A\) is

\[
\int_A r_A \, d\mu_A = \frac{1}{\mu(A)}.
\]
Hitting Time Statistics and Return Time Statistics

Definition

Given a sequence of sets \((U_n)_{n \in \mathbb{N}}\) so that \(\mu(U_n) \to 0\), the system has RTS \(\tilde{G}\) for \((U_n)_{n \in \mathbb{N}}\) if for all \(t \geq 0\)

\[
\mu_{U_n}\left(r_{U_n} \leq \frac{t}{\mu(U_n)}\right) \to \tilde{G}(t) \text{ as } n \to \infty. \tag{4}
\]

and the system has HTS \(G\) for \((U_n)_{n \in \mathbb{N}}\) if for all \(t \geq 0\)

\[
\mu\left(r_{U_n} \leq \frac{t}{\mu(U_n)}\right) \to G(t) \text{ as } n \to \infty, \tag{5}
\]

For systems with ‘good mixing properties’, \( G(t) = \tilde{G}(t) = 1 - e^{-t} \), in which case we say that we have exponential HTS/RTS.
Consider a discrete dynamical system

\[(\mathcal{X}, \mathcal{B}, \mu, f),\]

where
- \(\mathcal{X}\) is a \(d\)-dimensional Riemannian manifold,
- \(\mathcal{B}\) is the Borel \(\sigma\)-algebra,
- \(f : \mathcal{X} \rightarrow \mathcal{X}\) is a map,
- \(\mu\) is an \(f\)-invariant probability measure, absolutely continuous with respect to Lebesgue (acip).
In this context, we consider the stochastic process \( X_0, X_1, \ldots \) given by

\[
X_n = \varphi \circ f^n, \quad \text{for each } n \in \mathbb{N},
\]

(6)

where \( \varphi : \mathcal{X} \to \mathbb{R} \cup \{\pm \infty\} \) is an observable (achieving a global maximum at \( \xi \in \mathcal{X} \)) of the form

\[
\varphi(x) = g(\text{dist}(x, \xi)),
\]

where \( \xi \in \mathcal{X} \), “dist” denotes a Riemannian metric in \( \mathcal{X} \) and the function \( g : [0, +\infty) \to \mathbb{R} \cup \{+\infty\} \) has a global maximum at 0 and is a strictly decreasing bijection for a neighborhood \( V \) of 0.
Observe that if at time $j \in \mathbb{N}$ we have an exceedance of the level $u$ sufficiently large, i.e. $X_j(x) > u$, then we have an entrance of the orbit of $x$ in the ball of radius $g^{-1}(u)$ around $\xi$, at time $j$.

$$\{X_0 > u\} = \{g(\text{dist}(x, \xi)) > u\} = \{\text{dist}(x, \xi) < g^{-1}(u)\} = B_{g^{-1}(u)}(\xi).$$

and

$$1 - F(u) = \mu \left( B_{g^{-1}(u)}(\xi) \right).$$

Based on the characterisation of the 3 types in terms of the tail of $F$, we may assume that $g$ is of one of the following 3 types:
**Type 1:** there exists some strictly positive function $p : W \rightarrow \mathbb{R}$ such that for all $y \in \mathbb{R}$

$$\lim_{s \rightarrow g_1(0)} \frac{g_1^{-1}(s + yp(s))}{g_1^{-1}(s)} = e^{-y}; \quad \text{Example: } g_1(y) = -\log y$$

**Type 2:** $g_2(0) = +\infty$ and there exists $\beta > 0$ such that for all $y > 0$

$$\lim_{s \rightarrow +\infty} \frac{g_2^{-1}(sy)}{g_2^{-1}(s)} = y^{-\beta}; \quad \text{Example: } g_2(y) = y^{-1/\alpha}$$

**Type 3:** $g_3(0) = D < +\infty$ and there exists $\gamma > 0$ such that for all $y > 0$

$$\lim_{s \rightarrow 0} \frac{g_3^{-1}(D - sy)}{g_3^{-1}(D - s)} = y^{\gamma}; \quad \text{Example: } g_3(y) = D - y^{1/\alpha}$$
Connection between EVL and HTS

\((\mathcal{X}, \mathcal{B}, \mu, f)\) is a dynamical system where \(\mu\) is an acip.
\(\xi \in \mathcal{X}\) are points for which the Lebesgue’s Differentiation Theorem holds.
Motivated by Collet’s work, [C01], we obtained:

**Theorem (F,F,Todd (2010))**

- If we have HTS \(G\) for balls centred on \(\xi \in \mathcal{X}\), then we have an EVL for \(M_n\) with \(H = G\).

**Theorem (F,F,Todd (2010))**

- If we have an EVL \(H\) for \(M_n\), then we have HTS \(G = H\) for balls centred on \(\xi\).
Idea of the proof:

\[ \{ x : M_n(x) \leq u_n \} = \bigcap_{j=0}^{n-1} \{ x : X_j(x) \leq u_n \} \]

\[ = \bigcap_{j=0}^{n-1} \{ x : g(\text{dist}(f^j(x), \xi)) \leq u_n \} \]

\[ = \bigcap_{j=0}^{n-1} \{ x : \text{dist}(f^j(x), \xi) \geq g^{-1}(u_n) \} = \{ x : r_{B_{g^{-1}(u_n)}(\xi)}(x) \geq n \} \]

Thus,

\[ \mu \{ x : M_n(x) \leq u_n \} = \mu \{ x : r_{B_{g^{-1}(u_n)}(\xi)}(x) \geq n \} \]
Note that

\[
\frac{\tau}{n} \sim 1 - F(u_n) = \mu \left( B_{g^{-1}(u_n)}(\xi) \right) \Leftrightarrow n \sim \frac{\tau}{\mu \left( B_{g^{-1}(u_n)}(\xi) \right)}
\]

and so

\[
\mu \{ x : M_n(x) \leq u_n \} \sim \mu \left\{ x : r_{B_{g^{-1}(u_n)}(\xi)}(x) \geq \frac{\tau}{\mu \left( B_{g^{-1}(u_n)}(\xi) \right)} \right\} \rightarrow 1 - G(\tau)
\]
Consider now a sequence $\delta_n \to 0$. We want to study

$$
\mu \left( \left\{ x : r_{B\delta_n}(\xi)(x) < \frac{t}{\mu(B\delta_n(\xi))} \right\} \right)
$$

Choose $\ell_n$ such that $g^{-1}(u_{\ell_n}) \sim \delta_n$. We have that

$$
\{ x : M_{\ell_n}(x) \leq u_{\ell_n} \} = \bigcap_{j=0}^{\ell_n-1} \{ x : X_j(x) \leq u_{\ell_n} \}
$$

$$
= \bigcap_{j=0}^{\ell_n-1} \{ x : g(\text{dist}(f^j(x), \xi)) \leq u_{\ell_n} \}
$$

$$
= \bigcap_{j=0}^{\ell_n-1} \{ x : \text{dist}(f^j(x), \xi) \geq g^{-1}(u_{\ell_n}) \} = \{ x : r_{B_{g^{-1}(u_{\ell_n})}(\xi)}(x) \geq \ell_n \}
As before,

\[ \frac{\tau}{\ell_n} \sim 1 - F(u_{\ell_n}) = \mu \left( B_{\delta_n}(\xi) \right) \sim \mu \left( B_{g^{-1}(u_{\ell_n})}(\xi) \right) \iff \ell_n \sim \frac{\tau}{\mu \left( B_{\delta_n}(\xi) \right)}. \]

In this way,

\[ \mu \left\{ x : r_{B_{\delta_n}(\xi)}(x) < \frac{\tau}{\mu(B_{\delta_n}(\xi))} \right\} \sim 1 - \mu \left\{ x : M_{\ell_n}(x) \leq u_{\ell_n} \right\} \rightarrow H(\tau) \]
EVLs for the partial maximum have been studied in [C01, FF08, FF08a, VHF09, FFT10, FFT11, GHN11, HNT12, FFT12, FFT12a, HVRSB12, FLTV11, FLTV11a, FLTV11b, LFW12, K12, AFV12, FFLTV12].

The dynamical systems covered in these papers include:

- non-uniformly hyperbolic 1-dimensional maps (in all of them),
- higher dimensional non-uniformly expanding maps in [FFT10],
- suspension flows in [HNT12],
- billiards and Lozi maps in [GHN11]
- Hénon maps in [FLTV11b, HVRSB12]
Incomplete list of papers where a standard exponential HTS/RTS law has been proved around almost every point:

- Markov chains in [P91],
- Axiom A diffeomorphisms in [H93],
- uniformly expanding maps of the interval in [C96],
- 1-dimensional non-uniformly expanding maps in [HSV99, BSTV03, BV03, BT09]..., 
- partially hyperbolic dynamical systems in [D04],
- toral automorphisms in [DGS04],
- higher dimensional non-uniformly hyperbolic systems (including Hénon maps) in [CC10].
Assuming $D(u_n)$ holds, let $(k_n)_{n \in \mathbb{N}}$ be a sequence of integers such that

$$k_n \to \infty \quad \text{and} \quad k_n t_n = o(n). \quad (7)$$

**Condition ($D'(u_n)$)**

We say that $D'(u_n)$ holds for the sequence $X_0, X_1, \ldots$ if

$$\limsup_{n \to \infty} n \sum_{j=1}^{[n/k]} P\{X_0 > u_n \text{ and } X_j > u_n\} = 0. \quad (8)$$

**Theorem (Leadbetter)**

Let $\{u_n\}$ be such that $n(1 - F(u_n)) \to \tau$, as $n \to \infty$, for some $\tau \geq 0$. Assume that conditions $D(u_n)$ and $D'(u_n)$ hold. Then

$$P(M_n \leq u_n) \to e^{-\tau} \quad \text{as} \quad n \to \infty.$$
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Let $\{u_n\}$ be such that $n(1 - F(u_n)) \to \tau$, as $n \to \infty$, for some $\tau \geq 0$. Assume that conditions $D(u_n)$ and $D'(u_n)$ hold. Then

$$P(M_n \leq u_n) \to e^{-\tau} \quad \text{as } n \to \infty.$$
Motivated by the work of Collet (2001) we introduced:

**Condition \((D_2(\mathbf{u}_n))\)**

We say that \(D_2(\mathbf{u}_n)\) holds for the sequence \(X_0, X_1, \ldots\) if for any integers \(\ell, t\) and \(n\)

\[
|P \left\{ X_0 > \mathbf{u}_n \cap \max \{ X_t, \ldots, X_{t+\ell-1} \leq \mathbf{u}_n \} \right\} - 
P \{ X_0 > \mathbf{u}_n \} \cdot P \{ M_{\ell} \leq \mathbf{u}_n \} | \leq \gamma(n, t),
\]

where \(\gamma(n, t)\) is nonincreasing in \(t\) for each \(n\) and \(n \gamma(n, t_n) \rightarrow 0\) as \(n \rightarrow \infty\) for some sequence \(t_n = o(n)\).

**Theorem (F,F (2008))**

Let \(\{ \mathbf{u}_n \}\) be such that \(n(1 - F(\mathbf{u}_n)) \rightarrow \tau\), as \(n \rightarrow \infty\), for some \(\tau \geq 0\). Assume that conditions \(D_2(\mathbf{u}_n)\) and \(D'(\mathbf{u}_n)\) hold. Then

\[
P(M_n \leq \mathbf{u}_n) \rightarrow e^{-\tau} \quad \text{as} \quad n \rightarrow \infty.
\]
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**Condition \((D_2(u_n))\)**

We say that \(D_2(u_n)\) holds for the sequence \(X_0, X_1, \ldots\) if for any integers \(\ell, t\) and \(n\)

\[
|P \{X_0 > u_n \cap \max \{X_t, \ldots, X_{t+\ell-1} \leq u_n\}\} - P\{X_0 > u_n\} P\{M_\ell \leq u_n\}| \leq \gamma(n, t),
\]

where \(\gamma(n, t)\) is nonincreasing in \(t\) for each \(n\) and \(n\gamma(n, t_n) \to 0\) as \(n \to \infty\) for some sequence \(t_n = o(n)\).

**Theorem (F,F (2008))**

Let \(\{u_n\}\) be such that \(n(1 - F(u_n)) \to \tau\), as \(n \to \infty\), for some \(\tau \geq 0\). Assume that conditions \(D_2(u_n)\) and \(D'(u_n)\) hold. Then

\[
P(M_n \leq u_n) \to e^{-\tau} \quad \text{as} \quad n \to \infty.
\]
Decay of correlations implies $D_2(u_n)$

Suppose that there exists a nonincreasing function $\gamma : \mathbb{N} \to \mathbb{R}$ such that for all $\phi : \mathcal{X} \to \mathbb{R}$ with bounded variation and $\psi : \mathcal{X} \to \mathbb{R} \in L^\infty$, there is $C > 0$ independent of $\phi, \psi$ and $n$ such that

$$\left| \int \phi \cdot (\psi \circ f^t) d\mu - \int \phi d\mu \int \psi d\mu \right| \leq C \text{Var}(\phi) \|\psi\|_\infty \gamma(t), \quad \forall n \geq 0,$$

where $\text{Var}(\phi)$ denotes the total variation of $\phi$ and $n\gamma(t_n) \to 0$, as $n \to \infty$ for some sequence $t_n = o(n)$.

Taking $\phi = 1_{\{X > u_n\}}$ and $\psi = 1_{\{M_\ell \leq u_n\}}$, then

$$(9) \Rightarrow D_2(u_n),$$

(with $\gamma(n, t) = C \text{Var}(1_{\{X > u_n\}}) 1_{\{M_\ell \leq u_n\}} \|\psi\|_\infty \gamma(t) \leq C' \gamma(t)$ and for the sequence $\{t_n\}$ such that $t_n/n \to 0$ and $n\gamma(t_n) \to 0$ as $n \to \infty$).
Periodic points

From here on $\xi$ is a repelling $p$-periodic point, which implies that $f^p(\xi) = \xi$, $f^p$ is differentiable at $\xi$ and $0 < |\det D(f^{-p})(\xi)| < 1$.

Moreover, we assume $\xi \in X$ is a point for which the Lebesgue’s Differentiation Theorem holds.

Note that $\{X_0 > u\} \cap \{X_p > u\} \neq \emptyset$ and for all $u$ sufficiently large

$$P(\{X_0 > u\} \cap \{X_p > u\}) \sim |\det D(f^{-p})(\xi)| P(X_0 > u).$$

Consequently, $D'(u_n)$ does not hold since

$$n \left\lceil \frac{n}{k_n} \right\rceil \sum_{j=1}^{[n/k_n]} P(X_0 > u_n, X_j > u_n) \geq nP(X_0 > u_n, X_p > u_n) \to |\det D(f^{-p})(\xi)| \tau$$
For repelling periodic points as above, let $\theta = 1 - |\det D(f^{-p})(\xi)|$. Then the following condition holds:

**Condition (MP$_{p,\theta}(u_n)$)**

We say that $X_0, X_1, X_2, \ldots$ satisfies the condition MP$_{p,\theta}(u_n)$ for $p \in \mathbb{N}$ and $\theta \in [0, 1]$ if

$$\lim_{n \to \infty} \sup_{1 \leq j < p} P(X_j > u_n | X_0 > u_n) = 0, \quad \lim_{n \to \infty} P(X_p > u_n | X_0 > u_n) = (1 - \theta)$$

and

$$\lim_{n \to \infty} \sup_i \frac{P(X_p > u_n, X_{2p} > u_n, \ldots, X_{ip} > u_n | X_0 > u_n)}{(1 - \theta)^i} = 1.$$
Define the event $Q_{p,0}(u) := \{X_0 > u, X_p \leq u\}$.

Observe that for $u$ sufficiently large, $Q_{p,0}(u)$ corresponds to an annulus centred at $\xi$.

Define the events: $Q_{p,i}(u) := \{X_i > u, X_{i+p} \leq u\}$,

$Q^*_{p,i}(u) := \{X_i > u\} \setminus Q_{p,i}(u)$ and $Q_{p,s,\ell}(u) = \cap_{i=s}^{s+\ell-1} Q^c_{p,i}(u)$.
Theorem (F, F, Todd (2012))

Let \((u_n)_{n \in \mathbb{N}}\) be such that \(nP(X_0 > u_n) \to \tau\), for some \(\tau \geq 0\). Suppose \(X_0, X_1, \ldots\) is as in (6) and satisfies \(\text{MP}_{p, \theta}(u_n)\) for \(p \in \mathbb{N}\), and \(\theta \in (0, 1)\). Then

\[
\lim_{n \to \infty} P(M_n \leq u_n) = \lim_{n \to \infty} P(Q_{p,0,n}(u_n))
\]

(10)

1. First observe that \(\{M_n \leq u_n\} \subset Q_{p,0,n}(u_n)\).
2. Moreover, \(Q_{p,0,n}(u_n) \setminus \{M_n \leq u_n\} \subset \bigcup_{i=0}^{n-1} \{X_i > u_n, X_{i+p} > u_n, \ldots, X_{i+s_ip} > u_n\}\), where \(s_i = \left\lfloor \frac{n-1-i}{p} \right\rfloor\).
3. It follows by \(\text{MP}_{p, \theta}(u_n)\) and stationarity that

\[
P(Q_{p,0,n}(u_n) \setminus \{M_n \leq u_n\}) \leq \sum_{i=0}^{n-1} P(X_i > u_n, X_{i+p} > u_n, \ldots, X_{i+s_ip} > u_n)
\]

\[
\leq p \sum_{i=0}^{[n/p]} P(X_0 > u_n, X_p > u_n, X_{2p} > u_n, \ldots, X_{ip} > u_n) \xrightarrow{n \to \infty} 0.
\]
Condition \((D^p(u_n))\)

We say that \(D^p(u_n)\) holds for \(X_0, X_1, \ldots\) if for any \(\ell, t\) and \(n\)

\[
\left| P\left( Q_{p,0}(u_n) \cap Q_{p,t,\ell}(u_n) \right) - P(Q_{p,0}(u_n))P(Q_{p,0,\ell}(u_n)) \right| \leq \gamma(n, t),
\]

where \(\gamma(n, t)\) is nonincreasing in \(t\) for each \(n\) and \(n\gamma(n, t_n) \to 0\) as \(n \to \infty\) for some sequence \(t_n = o(n)\).

Let \((k_n)_{n \in \mathbb{N}}\) be a sequence of integers such that \(k_n \to \infty\) and \(k_n t_n = o(n)\).

Condition \((D'_p(u_n))\)

We say that \(D'_p(u_n)\) holds for the sequence \(X_0, X_1, X_2, \ldots\) if there exists a sequence \(\{k_n\}_{n \in \mathbb{N}}\) satisfying (7) and such that

\[
\lim_{n \to \infty} n \sum_{j=1}^{[n/k_n]} P(Q_{p,0}(u_n) \cap Q_{p,j}(u_n)) = 0. \quad (11)
\]
Theorem (F, F, Todd (2012))

Let \((u_n)_{n \in \mathbb{N}}\) be such that \(nP(X > u_n) \to \tau\), as \(n \to \infty\) for some \(\tau \geq 0\). Consider a stationary stochastic process \(X_0, X_1, X_2, \ldots\) satisfying \(MP_{p,\theta}(u_n)\) for some \(p \in \mathbb{N}\), and \(\theta \in (0, 1)\). Assume further that conditions \(D_p(u_n)\) and \(D'_p(u_n)\) hold. Then

\[
\lim_{n \to \infty} P(M_n \leq u_n) = \lim_{n \to \infty} P(Q_{p,0,n}(u_n)) = e^{-\theta \tau}. \tag{12}
\]

Note that

\[
P(Q_{p,0}(u)) = P(X_0 > u, X_p \leq u) =
\]

\[
= P(X_0 > u) - P(X_0 > u, X_p > u) =
\]

\[
\sim P(X_0 > u) - (1 - \theta)P(X_0 > u) = \theta P(X_0 > u),
\]

and so

\[
\theta \sim \frac{P(Q_{p,0}(u))}{P(X_0 > u)}.
\]
Applications to specific systems

Examples of systems for which we can prove the existence of an EI $0 < \theta < 1$ at repelling periodic points

- Systems with decay of correlations against $L^1$ observables:
  - Uniformly expanding maps on the circle ([H93])
  - Conformal repellers and Markov maps ([FP12])
  - Piecewise expanding maps of the interval like Rychlik maps ([FFT12, K12])
  - Higher dimensional piecewise expanding maps like in [S00] ([FFT12, K12])

- Non-uniformly hyperbolic systems admitting “nice” induced first return time maps:
  - Maps with indifferent fixed points like Manneville-Pomeau or Liverani-Saussol-Vaienti maps ([FFT12a])

- Non-uniformly hyperbolic systems with critical points:
  - Benedicks-Carleson quadratic maps ([FFT12a])


Suppose that there exists a nonincreasing function $\gamma : \mathbb{N} \to \mathbb{R}$ such that for all $\phi : \mathcal{X} \to \mathbb{R}$ with bounded variation and $\psi : \mathcal{X} \to \mathbb{R} \in L^1$, there is $C > 0$ independent of $\phi, \psi$ and $n$ such that

$$\left| \int \phi \cdot (\psi \circ f^t) d\mu - \int \phi d\mu \int \psi d\mu \right| \leq C \text{Var}(\phi) \|\psi\|_1 \gamma(t), \quad \forall n \geq 0,$$

where $\text{Var}(\phi)$ denotes the total variation of $\phi$ and $n \gamma(t_n) \to 0$, as $n \to \infty$ for some sequence $t_n = o(n)$.

Taking $\phi = 1_{Q_p(u_n)}$ and $\psi = 1_{Q_p(u_n)}$, then

$$(13) \Rightarrow D'_p(u_n),$$

$$\quad P(Q_{p,0}(u_n) \cap Q_{p,j}(u_n)) \leq P(Q_{p,0}(u_n))^2 + C' P(Q_{p,0}(u_n)) \gamma(j) \lesssim (\tau/n)^2 + C'(\tau/n) \gamma(j).$$
Doubling map
Rychlik map
Intermittent map
Benedicks-Carleson maps