Real indeterminacy and the timing of money in open Economies

by

Stephen McKnight
Real Indeterminacy and the Timing of Money in Open Economies

Stephen McKnight*
University of Reading
July 2007†

Abstract
This paper investigates the conditions under which interest-rate rules induce real equilibrium indeterminacy in a two-country, sticky-price, monetary model. Using a discrete-time framework, we employ the two most commonly used timing assumptions on which money balances enter into the utility function. This paper shows that the timing equivalence result derived for a closed-economy no longer holds for open economies. This arises because modifications in the trading environment impact on the behavior of the real exchange rate. Consequently this helps explain the seemingly contradictory findings in the literature on real indeterminacy in open economies. Furthermore it challenges the belief that domestic inflation targeting is superior to consumer price inflation targeting, in minimizing aggregate instability.

JEL Classification Number: E32; E43; E53; E58; F41

Keywords: Real indeterminacy; Open economy macroeconomics; Interest rate rules; Monetary Policy.

*Correspondence address: Stephen McKnight, Department of Economics, University of Reading Business School, Whiteknights, PO Box 219, Reading, RG6 6AW, UK. E-mail: Stephen.McKnight@reading.ac.uk.
†I would like to thank Roy Bailey, Ruth Baldry, Ken Burnet, João Miguel Ejarque, Aditya Goenka, Mark Guzman, Alexander Mihailov, Kerry Patterson and Helmut Rainer for helpful comments and advice. Financial assistance was gratefully provided by the Economic and Social Research Council (ESRC). As always, the usual disclaimer applies.
1 Introduction

The purpose of this paper is to investigate the real indeterminacy implications of designing interest rate rules in a world comprised of two symmetric countries characterized by complete asset markets and nominal price rigidities. Using different targets of inflation, it is shown that the conditions needed to guarantee real equilibrium determinacy depend not only on the monetary policy stance of the monetary authority (i.e. active vs passive) but can also depend on the degree of trade openness and the relative size of the elasticity of substitution between cross-country tradeable goods and the intertemporal substitution elasticity of consumption. However, the interaction of the above conditions for real determinacy crucially depend on the timing assumption specified on how money enters the utility function. In stark contrast to the closed-economy literature, there is no equivalence between the policy-rule targeted and the way money is modeled, because alternative timing assumptions on money have differing behavioral implications for the real exchange rate.

A standard result that emerges from the monetary literature is that in order to rule out aggregate instability the monetary authority should follow an active policy stance, that is, a policy that aggressively targets either expected inflation (e.g. Bernanke and Woodford (1997), Clarida et al. (2000)) or current inflation (e.g. Kerr and King (1996)) by raising the nominal interest rate by proportionally more than the increase in inflation. These results are obtained using a money-in-the-utility function (MIUF), labor-only, closed economy model where prices are sticky and preferences are separable with a zero-cross partial between consumption and real money balances. In addition, Carlstrom and Fuerst (2001, 2005) show that the design of the interest-rate rule crucially depends on the timing assumption specified when using the popular MIUF approach. While the monetary authority can aggressively target expected inflation under the traditional “cash-when-I’m-done” (CWID) timing convention, the adoption of such a policy under “cash-in-advance” (CIA) timing,

---

1 Our focus is on real indeterminacy instead of price-level (or nominal) indeterminacy. By real indeterminacy we simply mean that there exists a continuum of equilibrium paths, starting from the same initial conditions, which converge to the steady state. Price-level indeterminacy on the other hand, is where for any equilibrium sequence there exists an infinite number of initial price levels consistent with a perfect-foresight equilibrium. As discussed by Carlstrom and Fuerst (2000) price-level indeterminacy is of no consequence in and of itself, but is only important if it leads to real indeterminacy.

2 Throughout we use the terms aggregate instability, multiple equilibria and (real) indeterminacy interchangeably. Our focus of attention rests solely with the consideration of local stability (i.e. local determinacy) as opposed to global stability (i.e. global determinacy)
typically results in multiple equilibria.\(^3\) This arises because the essential difference between CWID and CIA timing assumptions is that the nominal interest rate in the latter is scrolled forward one period. Consequently Carlstrom and Fuerst (2001) find the following *timing equivalence* result: a current-looking (backward-looking) rule with CIA-timing has the same determinacy properties as a forward-looking (current-looking) rule with CWID-timing. Our aim is to investigate these timing differences in an international setting and re-evaluate this equivalence result.

In this paper we develop a two-country MIUF model in the spirit of Benigno (2001), Bergin *et al.* (2006) and Kollman (2003). Financial markets are assumed to be complete in the sense that agents in both countries have access to a complete set of contingent claims. Price stickiness is introduced following Calvo (1983). The Aoki (1981) decomposition approach is employed to analyze the determinacy properties of the model. The conditions for real equilibrium determinacy are analyzed for forward, current and backward-looking versions of the interest rate rule. In addition, two alternative price indexes, which can be chosen as the policy indicator, are considered: domestic price inflation and consumer price inflation. The main results from the analysis can be summarized as follows. With CWID-timing, the adoption of a current-looking rule introduces no additional restrictions for achieving equilibrium determinacy in open economies. However, under a forward-looking rule, the potential range of indeterminacy is greater under consumer price inflation targeting when compared with domestic inflation targeting, which is exacerbated in the former as the degree of trade openness increases. With CIA-timing, indeterminacy can only occur under an active current-looking rule if domestic inflation is targeted. However, under a backward-looking rule the range of determinacy generally becomes smaller as the degree of trade openness increases, for both targets of inflation.

Therefore, regardless of the price index targeted, the timing equivalence result obtained from the closed-economy literature does not hold for open economies. Under a forward-looking rule with CWID-timing, the range of indeterminacy is always greater when compared to a current-looking rule with CIA-timing.\(^4\) However under a current-looking

\(^3\)As discussed by Carlstrom and Fuerst (2001) the traditional MIUF approach assumes that end-of-period money balances enter the utility functional, whereas under cash-in-advance timing the money one has left over after engaging in asset transactions but before entering the goods market enters the functional.

\(^4\)This is only true for domestic inflation targeting provided that the size of the elasticity of substitution between cross-country tradeable goods is greater than the intertemporal substitution elasticity of consumption.
rule with CWID-timing the range of determinacy is always greater when compared to a backward-looking rule with CIA-timing. The explanation behind this breakdown of the timing equivalence result for open economies arises from the fact that alternative assumptions on how money balances enter the utility function impact on risk-sharing between households in the two countries. The risk-sharing condition under CWID-timing equates the real exchange rate with the marginal utilities of consumption. However, under CIA-timing, the marginal utilities of money additionally feature in this risk-sharing condition. For instance, suppose the central banks targets consumer price inflation. Under an active forward-looking rule with CWID-timing, an inflationary belief leads to an increase in the real interest rate. This not only lowers marginal cost, putting downward pressure on inflation, but also leads to an improvement in the terms of trade, putting upward pressure on inflation. If the latter effect is strong enough, which is determined by the degree of trade openness, then the initial inflationary belief can be validated. However under an active current-looking rule with CIA-timing (the closed-economy equivalent) the liquidity effect of an increase in the nominal interest rate dampens the terms of trade effect, thereby preventing the validation of the initial inflationary belief.

The breakdown of this timing equivalence helps explain the contradictory results currently found in the literature on policy induced dynamic (in)stability in open economies. Developing a small open economy version of the Cooley and Hansen (1989) model, De Fiore and Liu (2005) find that the range of determinacy increases the higher the degree of trade openness under a forward-looking rule. This is in contrast to the findings of Linnemann and Schabert (2002) and Zanna (2003), which employ a cashless, continuous time approach and conclude that that relatively open economies are more prone to aggregate instability than relatively closed economies. Linnemann and Schabert (2006), using a cashless, discrete time approach, show that the range of determinacy decreases the higher the degree of trade openness under a forward-looking rule. As this paper shows, these different findings regarding the impact of trade openness for equilibrium determinacy can easily be explained by alternative timing assumptions regarding money. Furthermore, this paper also suggests that caution is required when making conclusions when only one timing assumption is con-

---

5De Fiore and Liu (2005) employ a strict cash-in-advance constraint to introduce money into the economy. Linnemann and Schabert (2002, 2006) and Zanna (2003) assume a cashless economy, which is isomorphic to the traditional MIUF approach under CWID-timing, provided the utility function is separable between consumption and real money balances.
sidered. For example, Linnemann and Schabert (2002, 2006), Zanna (2003) and Batini et al. (2004) conclude that domestic inflation targeting is superior to consumer inflation targeting, as it reduces the potential range of aggregate instability. Our analysis suggests that this conclusion is a by-product of adopting CWID-timing and is not robust if alternative timing assumptions on money are employed.

Overall our analysis suggests that the timing assumption specified on money balances crucially affects the relationship between the degree of trade openness and the range of aggregate instability under both consumer and domestic inflation targeting. The basic assumptions of monetary models can thus have serious implications for interest-rate rules in both closed and open economy environments, by affecting the conditions for equilibrium determinacy. However, the timing convention affects the closed economy and open economy in different ways. In the former, the assumption imposed results in different pricing equations for the nominal interest rate, whereas in the latter, the assumption imposed also affects the behavior of the real exchange rate.

The remainder of the paper is organized as follows. Section 2 develops the two-country model. Section 3 discusses the conditions for real equilibrium determinacy for different interest-rate rule specifications under CWID-timing. Section 4 considers the impact on the determinacy conditions when CIA-timing is adopted. Finally Section 5 concludes.

2 The Model

Consider a global economy that consists of two-countries denoted home and foreign, where an asterisk denotes foreign variables. Within each country there exists a representative infinitely-lived agent, a representative final good producer, a continuum of intermediate good producing firms, and a monetary authority. The representative agent owns all domestic intermediate good producing firms and supplies labor to the production process. Intermediate firms operate under monopolistic competition and use domestic labor as inputs to produce tradeable goods which are sold to the home and foreign final good producers. The labor market is assumed to be competitive. Each representative final good producer is a

---

6Indeed, Linnemann and Schabert (2006) conclude that the particular price index chosen as the policy indicator is irrelevant as long as the policy is not forward-looking. Our analysis suggests that this conclusion needs to be qualified for a monetary economy, since the timing assumption specified on how money enters the utility function can generate different results.
competitive firm that bundles domestic and imported intermediate goods into non-tradeable final goods which are consumed by the domestic agent. Preferences and technologies are symmetric across the two countries. The following presents the features of the model for the home country on the understanding that the foreign case can be analogously derived.

2.1 Final Good Producers

The home final good \((Z)\) is produced by a competitive firm that uses \(Z_H\) and \(Z_F\) as inputs according to the following CES aggregation technology index:

\[
Z_t = \left[ a^\theta Z_{H,t}^{\frac{\theta - 1}{\theta}} + (1 - a)^\theta Z_{F,t}^{\frac{\theta - 1}{\theta}} \right]^{\frac{\theta}{\theta - 1}},
\]

where the relative share of domestic and imported intermediate inputs used in the production process is \(0 < a < 1\) and the constant elasticity of substitution between aggregate home and foreign intermediate goods is \(\theta > 0\). The inputs \(Z_H\) and \(Z_F\) are defined as the quantity indices of domestic and imported intermediate goods respectively:

\[
Z_{H,t} = \left[ \int_0^1 z_{H,t}(i)^{\frac{\lambda - 1}{\lambda}} di \right]^{\frac{\lambda}{\lambda - 1}}, \quad Z_{F,t} = \left[ \int_0^1 z_{F,t}(j)^{\frac{\lambda - 1}{\lambda}} dj \right]^{\frac{\lambda}{\lambda - 1}},
\]

where the elasticity of substitution across domestic (foreign) intermediate goods is \(\lambda > 1\), and \(z_{H}(i)\) and \(z_{F}(j)\) are the respective quantities of the domestic and imported type \(i\) and \(j\) intermediate goods. Let \(p_H(i)\) and \(p_F(j)\) represent the respective prices of these goods in home currency. Cost minimization in final good production yields the aggregate demand conditions for home and foreign goods:

\[
Z_{H,t} = a \left( \frac{P_{H,t}}{P_t} \right)^{-\theta} Z_t, \quad Z_{F,t} = (1 - a) \left( \frac{P_{F,t}}{P_t} \right)^{-\theta} Z_t,
\]

where the demand for individual goods is given by

\[
z_{H,t}(i) = \left( \frac{p_{H,t}(i)}{P_{H,t}} \right)^{-\lambda} Z_{H,t}, \quad z_{F,t}(j) = \left( \frac{p_{F,t}(j)}{P_{F,t}} \right)^{-\lambda} Z_{F,t}.
\]
Furthermore, since the final good producer is competitive it sets its price equal to marginal cost

\[ P_t = \left[ a P_{H,t}^{1-\theta} + (1-a) P_{F,t}^{1-\theta} \right]^{\frac{1}{1-\theta}}, \tag{4} \]

where \( P \) is the consumer price index and \( P_H \) and \( P_F \) are the respective price indices of home and foreign intermediate goods, all denominated in the home currency:

\[ P_{H,t} = \left[ \int_0^1 p_{H,t}(i)^{1-\lambda} di \right]^{\frac{1}{1-\lambda}}, \quad P_{F,t} = \left[ \int_0^1 p_{F,t}(j)^{1-\lambda} dj \right]^{\frac{1}{1-\lambda}}. \]

We assume that there are no costs to trade between the two countries and the law of one price holds, which implies that

\[ P_{H,t} = e_t P_{H,t}^*, \quad P_{F,t}^* = \frac{P_{F,t}}{e_t}, \tag{5} \]

where \( e \) denotes the nominal exchange rate. Letting \( Q = \frac{e_{P^*}}{P^*} \) denote the real exchange rate, under the law of one price, the CPI index (4) and its foreign equivalent imply:

\[ \left( \frac{1}{Q_t} \right)^{1-\theta} = \left( \frac{P_t}{e_t P_t^*} \right)^{1-\theta} = \frac{a P_{H,t}^{1-\theta} + (1-a) \left( e_t P_{F,t}^* \right)^{1-\theta}}{a \left( e_t P_{F,t}^* \right)^{1-\theta} + (1-a) P_{H,t}^{1-\theta}} \tag{6} \]

and hence the purchasing power parity condition is satisfied only in the absence of any bias between home and foreign intermediate goods (i.e. \( a = 0.5 \)). The relative price \( T \), the terms of trade, is defined as \( T = \frac{e_{P^*}}{P^*_H} \).

### 2.2 Intermediate Goods Producers

Intermediate firms hire labor to produce output given a (real) wage rate \( w_t \). A firm of type \( i \) has a linear production technology

\[ y_t(i) = L_t(i). \tag{7} \]

Given competitive prices of labor, cost minimization yields

\[ mc_t = w_t \frac{P_t}{P_{H,t}}, \tag{8} \]
where \( mc_t = \frac{MC_{t+s}}{P_{H,t}} \) is real marginal cost.

Firms set prices according to Calvo (1983), where in each period there is a constant probability \( 1 - \phi \) that a firm will be randomly selected to adjust its price, which is drawn independently of past history. A domestic firm \( i \), faced with changing its price at time \( t \), has to choose \( p_{H,t}(i) \) to maximize its discounted value of profits, taking as given the indexes \( P, P_H, P_F, Z \) and \( Z^* \):

\[
\max_{p_{H,t}(i)} E_t \sum_{s=0}^{\infty} (\beta \phi)^s X_{t,t+s} \left\{ [p_{H,t}(i) - MC_{t+s}(i)] \left[ z_{H,t+s}(i) + z^*_{H,t+s}(i) \right] \right\}, \tag{9}
\]

where

\[
z_{H,t+s}(i) + z^*_{H,t+s}(i) = \left( \frac{p_{H,t}(i)}{P_{H,t+s}} \right)^{-\lambda} [Z_{H,t+s} + Z^*_{H,t+s}]
\]

and the firm’s discount factor is \( \beta \phi^s X_{t,t+s} = (C_{t+s}/C_t)^s (P_t/P_{t+s}) \).\(^8\) Firms that are given the opportunity to change their price, at a particular time, all behave in an identical manner. The first-order condition to the firm’s maximization problem yields

\[
\tilde{P}_{H,t} = \frac{\lambda}{\lambda - 1} E_t \sum_{s=0}^{\infty} q_{t,t+s} MC_{t+s}. \tag{10}
\]

The optimal price set by a domestic home firm \( \tilde{P}_{H,t} \) is a mark-up \( \frac{\lambda}{\lambda - 1} \) over a weighted average of future nominal marginal costs, where the weight \( q_{t,t+s} \) is given by

\[
q_{t,t+s} = \frac{(\beta \phi)^s X_{t,t+s} P_{H,t+s}^\lambda (Z_{H,t+s} + Z^*_{H,t+s})}{E_t \sum_{s=0}^{\infty} (\beta \phi)^s X_{t,t+s} P_{H,t+s}^\lambda (Z_{H,t+s} + Z^*_{H,t+s})}.
\]

Since all prices have the same probability of being changed, with a large number of firms, the evolution of the price sub-indexes is given by

\[
P_{H,t}^{1-\lambda} = \phi P_{H,t-1}^{1-\lambda} + (1 - \phi) \tilde{P}_{H,t}^{1-\lambda} \tag{11}
\]

since the law of large numbers implies that \( 1 - \phi \) is also the proportion of firms that adjust their price each period.

\(^7\)While the demand for a firm’s good is affected by its pricing decision \( p_{H,t}(i) \), each producer is small with respect to the overall market.

\(^8\)Under the assumption that all firms are owned by the representative agent, this implies that the firm’s discount factor is equivalent to the individual’s intertemporal marginal rate of substitution.
2.3 Representative Agent

The representative agent chooses consumption $C$, domestic real money balances $A/P$, and labor $L$, to maximize utility:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U \left( C_t, \frac{A_t}{P_t}, L_t \right)$$

where the discount factor is $0 < \beta < 1$, subject to the period budget constraint

$$E_t \Gamma_{t,t+1} B_{t+1} + M_t + P_t C_t \leq B_t + M_{t-1} + P_t w_t L_t + \int_0^1 \Pi_t d(h) - \Upsilon_t. \quad (13)$$

The agent carries $M_{t-1}$ units of money, and $B_t$ nominal bonds into period $t$. Before proceeding to the goods market, the agent visits the financial market where a state contingent nominal bond $B_{t+1}$ can be purchased that pays one unit of domestic currency in period $t+1$ when a specific state is realized at a period $t$ price $\Gamma_{t,t+1}$. During period $t$ the agent supplies labor to the intermediate good producing firms, receiving real income from wages $w_t$, nominal profits from the ownership of domestic intermediate firms $\Pi_t$ and a lump-sum nominal transfer $\Upsilon_t$ from the monetary authority. The agent then uses these resources to purchase the final good.

Following Carlstrom and Fuerst (2001), we will consider two alternative measures of money which may appear in the utility function: the traditional *cash-when-i’m-done* (CWID)-timing and the alternative *cash-in-advance* (CIA)-timing. Under CWID-timing, end of period money balances enter into the utility function:

$$A_t = M_t. \quad (14)$$

Here the stock of money that yields utility to the representative agent is the amount of money he leaves the goods market with. However, under CIA-timing, the stock of money that yields utility is the value of money holdings after bonds have been purchased in the financial markets, but before income has been received or final goods have been purchased:

$$A_t = M_{t-1} - \Upsilon_t + B_t - E_t \Gamma_{t,t+1} B_{t+1}. \quad (15)$$
The period utility function is assumed to be separable among the three arguments

\[ U(C, A/P, L) = U(C) + V(A/P) - H(L). \]

The first-order conditions from the home agent’s maximization problem yield:

\[
\beta R_{t+i} E_t \left\{ \frac{U_c(C_{t+1})}{U_c(C_t)} \frac{P_t}{P_{t+1}} \right\} = 1 \quad (16)
\]

\[
\frac{H_L(L_t)}{U_c(C_t)} = w_t \quad (17)
\]

\[
\frac{V_m(m_t)}{U_c(C_t)} = \frac{R_t - 1}{R_t} \quad (18)
\]

\[
\frac{V_m(m_t)}{U_c(C_t)} = R_t - 1 \quad (19)
\]

where \( R_t \) denotes the gross nominal yield on a one-period discount bond defined as \( R_t^{-1} = E_t \{ \Gamma_{t,t+1} \} \). Equation (16) is the consumption Euler equation for the holdings of domestic bonds where \( i = 0 \) represents CWID-timing and \( i = 1 \) corresponds to CIA-timing, with the respective money demand equation given by equations (18) and (19). Thus, the first key difference between the two timing assumptions is that under CIA-timing the nominal interest rate is scrolled forward one period. Changes in real holdings of money directly influence the real interest rate under CIA-timing, whereas they only have an indirect effect on the real interest rate under CWID-timing. The labor supply decision is determined by equation (17). Optimizing behavior implies that the budget constraint (13) holds with equality in each period and the appropriate transversality condition is satisfied. Analogous conditions apply to the foreign agent.

From the first-order conditions for the home and foreign agent, the following risk-sharing conditions can be derived:

\[
R_t = R_t^* E_t \left\{ \frac{\epsilon_{t+1}}{\epsilon_t} \right\} \quad (20)
\]

\[
Q_t = q_0 \frac{U_c(C_t^*)}{U_c(C_t)} \quad (21)
\]

\[
Q_t = q_0 \frac{U_c(C_t^*) + V_m(m_t^*)}{U_c(C_t) + V_m(m_t)} \quad (22)
\]
where the constants \( q_0 = Q_0 \left[ \frac{U_c(C_0)}{U_c(C_0)} \right] \) and \( q_0^* = Q_0 \left[ \frac{U_c(C_0) + V_m(m_0)}{U_c(C_0) + V_m(m_0)} \right] \). Equation (20) is the standard uncovered interest rate parity condition, whereas equations (21) and (22) follow from the assumption of complete asset markets, under CWID and CIA-timing respectively. Hence, the second key difference between the timing assumptions relates to the risk sharing condition which equates the real exchange rate \( Q \) with the marginal utilities of consumption. Under CIA-timing, the marginal utilities of money are also included in (22), reflecting the fact that under CIA-timing a bond sale for consumption purposes, increases the utility from current consumption and current liquidity.

### 2.4 Monetary Authority

The monetary authority can adjust the nominal interest rate in response to changes in domestic price inflation \( \pi_{t+\nu}^h \) or to changes in consumer price inflation \( \pi_{t+\nu} \), according to the simple rules:

\[
R_t = \mu (\pi^h_{t+\nu}) = \overline{R} \left( \frac{\pi^h_{t+\nu}}{\pi} \right)^\mu, \tag{23}
\]

\[
R_t = \mu (\pi_{t+\nu}) = \overline{R} \left( \frac{\pi_{t+\nu}}{\pi} \right)^\mu, \tag{24}
\]

where \( \overline{R} > 1 \) and the timing-index \( \nu \) represents the inflation-targeting behavior of the monetary authority. If \( \nu = 0 \), the monetary authority targets current inflation. If \( \nu = -1 \) the policy rule is backward-looking, whereas \( \nu = 1 \) corresponds to forward-looking inflation targeting. The parameter \( \mu \) determines whether monetary policy is active or passive. An active monetary policy corresponds to \( \mu > 1 \), where the real interest rate rises in response to higher inflation, as the monetary authority increases the nominal interest rate by more than the increase in inflation. A passive monetary policy on the other hand corresponds to \( 0 \leq \mu < 1 \), where the real interest rate falls in response to higher inflation.

### 2.5 Market Clearing and Equilibrium

Market clearing for the home goods market requires

\[
Z_{H,t} + Z_{H,t}^* = Y_t. \tag{25}
\]
Total home demand must equal the supply of the final good,

\[ Z_t = C_t, \quad (26) \]

and the labor, money and bond markets all clear:

\[ \Upsilon_t = M_t - M_{t-1} \quad B_t + B^*_t = 0. \quad (27) \]

**Definition 1** (Rational Expectations Equilibrium): Given an initial allocation of \( B_{t_0}, B^*_{t_0}, \) and \( M_{t_0-1}, M^*_{t_0-1}, \) a rational expectations equilibrium is a set of sequences \( \{C_t, C^*_t, M_t, M^*_t, L_t, L_t^*, L_{t^*}, B_t, B_t^*, R_t, R^*_t, MC_t, MC^*_t, w_t, w^*_t, Y_t, Y^*_t, e_t, Q_t, P_t, P^*_t, P_{H,t}, P_{H^*}, \}

along with the foreign counterparts for (i)-(vii) and the conditions, (5), (6), (20) and either (21) if CWID-timing is adopted or (22) if CIA-timing is adopted.

### 2.6 Local Equilibrium Dynamics

In order to analyze the equilibrium dynamics of the model, a first-order Taylor approximation is taken around a steady state to replace the non-linear equilibrium system with an approximation which is linear. In what follows, a variable \( \tilde{X}_t \) denotes the percentage deviation of \( X_t \) with respect to its steady state value \( \bar{X} \) (i.e. \( \tilde{X}_t = \frac{X_t - \bar{X}}{\bar{X}} \)). To be precise
the model is linearized around a symmetric steady state in which prices in the two countries are equal and constant ($\bar{P}_H = \bar{P}_F = \bar{P} = \bar{P}^* = \bar{P}_H^* = \bar{P}_F^*$). Then by definition inflation is zero ($\pi = \pi^* = 0$), and the steady state terms of trade and nominal and real exchange rate are $\bar{T} = \bar{r} = \bar{Q} = 1$. For convenience, the complete linearized system of equations is summarized in Table 1 where the parameters are: $\sigma > 0$ measures the intertemporal substitution elasticity of consumption; $\phi > 0$ measures the inverse of the Frisch labor supply elasticity; $\theta > 0$ measures the elasticity of substitution between aggregate home and foreign goods; $\Lambda_1 \equiv \frac{(1-\psi)(1-\beta\psi)}{\psi} > 0$ is the degree of monopolistic competition in the intermediate firm sector where $0 < \beta < 1$ is the discount factor and $0 < \psi < 1$ is the degree of price stickiness; and $a \in \{(0,0.5) \cup (0.5,1)\}$ is the degree of trade openness measured by the relative share of intermediate imports used in final good production $(1-a)$.

Since the two countries are symmetric, we employ the sum and difference approach in order to analyze the determinacy properties of the model.\textsuperscript{9} Thus, we solve both for cross-country differences $X^R \equiv \hat{X} - \hat{X}^*$ and worldwide aggregates $X^W \equiv \hat{X} + \hat{X}^*$. Then, given solutions for $X^R$ and $X^W$, one can recover solutions for $\hat{X}$ and $\hat{X}^*$ from:

$$\hat{X} = X^W + \frac{X^R}{2}$$

$$\hat{X}^* = X^W - \frac{X^R}{2}$$

The Aoki decomposition decomposes the the two-country model into two decoupled dynamic systems: the aggregate system that captures the properties of the closed world economy and the difference system that portrays the open-economy dimension. Consequently for the equilibrium to be determinate it must be the case that there is a unique solution both for cross-country differences and world aggregates. Therefore, from a policy perspective, interest rate setting in both countries must ensure determinacy of both relative variables and the aggregate world economy.\textsuperscript{10}

\textsuperscript{9}This is the standard approach to solving two-country models since Aoki (1981).

\textsuperscript{10}In terms of the Aoki decomposition of the model, the choice of which measure of inflation each monetary authority targets is irrelevant in terms of the determinacy conditions for the aggregate system. This follows since world aggregate inflation ($\pi^W$) is given by

$$\pi^W = \frac{\pi + \pi^*}{2} = \frac{\pi^h + \pi^f}{2}.$$  

Therefore the determinacy conditions can be affected by each monetary authorities choice of responding to consumer or domestic price inflation, through influencing the eigenvalues only of the difference system.
### Cross-Country Differences

<table>
<thead>
<tr>
<th>Equation</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_t { \hat{Z}<em>{t+1}^R } = \hat{Z}<em>t^R + \sigma \hat{R}</em>{t+1}^R - \sigma E_t { \hat{\pi}</em>{t+1}^R }$</td>
<td>IS$^R$</td>
</tr>
<tr>
<td>$\hat{R}<em>t^R = E_t { \Delta \hat{e}</em>{t+1} } + \beta E_t { \hat{\pi}_{t+1}^R }$</td>
<td>UIP</td>
</tr>
<tr>
<td>$\hat{\pi}_t^R = \Lambda_1 2(1-a)[1 + \phi \theta a] \hat{T}_t + \Lambda_1 [\phi(2a - 1) + \frac{1}{a}] \hat{Z}_t$</td>
<td>AS$^R$</td>
</tr>
<tr>
<td>$\hat{\pi}<em>{t+1}^R = \mu E_t { \hat{\pi}</em>{t+1}^R }$ or $\hat{R}<em>t^R = \mu E_t { \hat{\pi}</em>{t+1}^R }$</td>
<td>Taylor rule</td>
</tr>
<tr>
<td>$\hat{Q}_t = \frac{1}{2} \hat{Z}_t^R = (2a - 1) \hat{T}_t$</td>
<td>RER(CWID-timing)</td>
</tr>
<tr>
<td>$\hat{Q}_t = \frac{1}{2} \hat{Z}_t^R - \hat{R}_t^R = (2a - 1) \hat{T}_t$</td>
<td>RER(CIA-timing)</td>
</tr>
</tbody>
</table>

### World Aggregates

<table>
<thead>
<tr>
<th>Equation</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_t { \hat{Z}<em>{t+1}^W } = \hat{Z}<em>t^W + \sigma \hat{R}</em>{t+1}^W - \sigma E_t { \hat{\pi}</em>{t+1}^W }$</td>
<td>IS$^W$</td>
</tr>
<tr>
<td>$\beta E_t { \hat{\pi}_{t+1}^W } = \hat{\pi}_t^W - \Lambda_1 [\phi + \frac{1}{a}] \hat{Z}_t^W$</td>
<td>AS$^W$</td>
</tr>
<tr>
<td>$\hat{R}<em>t^W = \mu E_t { \hat{\pi}</em>{t+1}^W }$</td>
<td>Taylor rule</td>
</tr>
</tbody>
</table>

**Notes:** The index $R$ refers to the difference between home and foreign variables e.g. $\hat{C}_t^R \equiv (\hat{C}_t - \hat{C}_t^*)$, $\hat{\pi}_t^R(h-f^*) \equiv (\hat{\pi}_t - \hat{\pi}_t^*)$. The index $W$ refers to world aggregates where $\pi^W = \frac{\pi + \pi^*}{2} = \frac{\pi^h + \pi^f}{2}$ and $\Delta \hat{e}_t \equiv \hat{e}_t - \hat{e}_{t-1}$.

The set of linear equations summarized in Table 1 can be reduced to a system of linear difference equations given by

$$
\mathbf{A}_t^R \mathbf{x}_{t+1}^R = \mathbf{Z}_t^R \mathbf{x}_t^R
$$

$$
\mathbf{A}_t^W \mathbf{x}_{t+1}^W = \mathbf{Z}_t^W \mathbf{x}_t^W
$$

where $\mathbf{A}$ and $\mathbf{Z}$ are coefficient matrices and $\mathbf{x}$ is the column vector containing the endogenous variables. Provided $\mathbf{A}$ is non-singular, then the dynamic behavior of the system is governed by the eigenvalues of the reduced form coefficient matrix $\mathbf{A}^{-1} \mathbf{Z}$. For linear rational expectation models, Blanchard and Kahn (1980) provide the local conditions for existence and uniqueness of equilibrium. A general condition for determinacy is that the number of eigenvalues of the matrix $\mathbf{A}^{-1} \mathbf{Z}$ outside the unit circle (i.e. eigenvalues of modulus greater than 1) has to be equal to the number of non-predetermined variables for a unique solution. If however the number of eigenvalues outside the unit circle is less than the number of non-predetermined variables the equilibrium is locally indeterminate.
3 Equilibrium Determinacy

This section examines the conditions for equilibrium determinacy under CWID-timing when monetary policy is characterized by either a forward or current-looking rule and both domestic and consumer price inflation are possible inflation targets.

3.1 Forward-looking rules

The set of linearized equations for the world aggregates, given in Table 1, can be reduced to the following two-dimensional system:

\[ E_t x_{t+1}^W = A x_t^W, \quad x_t = \begin{bmatrix} \hat{z}_t^W, \hat{\pi}_t^W \end{bmatrix}. \]

\[ A \equiv \begin{bmatrix} 1 - \frac{\Lambda_1(\sigma \phi + 1)(\mu-1)}{\phi \sigma} & \frac{\sigma(\mu-1)}{\beta} \\ -\frac{\Lambda_1(\phi + \frac{1}{\beta})}{\phi} & \frac{1}{\beta} \end{bmatrix}. \]

Since \( x \) is a column vector of non-predetermined variables, equilibrium determinacy requires that both eigenvalues of \( A \) are outside the unit circle. Then by Proposition C.1 of Woodford (2003) the following result is obtained:

**Proposition 1** Suppose that monetary policy is characterized by a forward-looking interest rate rule with CWID-timing. Then a necessary and sufficient condition for determinacy of the aggregate system is

\[ 1 < \mu < 1 + \frac{2(1+\beta)}{\Lambda_1(\sigma \phi + 1)} \equiv \Gamma_0 \] (29)

The conditions for determinacy of the aggregate system presented in proposition 1 is analogous to the conditions obtained under a forward-looking rule for a similar closed-economy, sticky-price model. This is not surprising since the aggregate system is independent of the open-economy parameters \( \theta \) and \( a \). Therefore, while an active monetary policy is a necessary condition for determinacy, this is not sufficient since the inflation coefficient \( \mu \) is bounded from above. However, for standard parameter values this upper bound is unlikely to bind. For example setting \( \beta = 0.99, \psi = 0.75, \) and \( \sigma = \phi = 1 \) then a \( \mu \geq 24.185 \) is required in order to generate indeterminacy.

The set of linearized conditions for cross-country differences yields a system of the form:
\[ E_t x_{t+1}^R = B x_t^R, \quad x_t = \left[ \hat{Z}_t^R, \hat{\pi}_t^{R(h-f^*)} \right] ', \]

\[
B \equiv \begin{bmatrix}
1 - \frac{(\mu-1)\Lambda_1\Lambda_2}{\beta\kappa_1 \\
\frac{\sigma(2a-1)(\mu-1)}{\beta\kappa_1} \\
- \frac{1}{\beta}
\end{bmatrix},
\]

where \( \lambda_1 = \frac{2(1-a)[1+2a\phi\theta]}{\sigma(2a-1)} + \phi(2a-1) + \frac{1}{\sigma}, \Lambda_2 = 1 + \sigma \phi + 4 \phi a(1-a)(\theta - \sigma) > 0 \) and \( \kappa_1 = 1 \) if domestic price inflation is targeted and \( \kappa_1 = 1 - 2(1-a)\mu \) if consumer price inflation is targeted. As before, determinacy requires that both eigenvalues of the coefficient matrix are outside the unit circle.

**Proposition 2** Suppose that monetary policy targets forward-looking domestic price inflation with CWID-timing. Then a necessary and sufficient condition for determinacy of the difference system is

\[
1 < \mu < 1 + \frac{2(1+\beta)}{\Lambda_1[1 + \sigma \phi + 4 \phi a(1-a)(\theta - \sigma)]} \equiv \Gamma_1 \tag{30}
\]

First note that if \( \theta = \sigma \) then the determinacy conditions of the difference system and the closed-economy are analogous. Consequently for any \( \theta \leq \sigma \), the open-economy introduces no additional requirements for determinacy, such that if (29) is satisfied, then both the aggregate and difference systems are determinate. However, if \( \theta > \sigma \) then the upper bound on the inflation coefficient for the difference system (30) is reduced relative to (29). \(^{11}\) Consequently the potential range of indeterminacy is greater in the open-economy and gets increasingly worse the larger the difference between \( \theta \) and \( \sigma \). The impact that the degree of trade openness has on this upper bound is given by:

\[
\frac{\partial \Gamma_1}{\partial a} = \frac{8(1+\beta)\Lambda_1 \phi(\theta - \sigma)(2a-1)}{\Lambda_1^2[1 + \sigma \phi + 4 \phi a(1-a)(\theta - \sigma)]^2} \geq 0 \tag{31}
\]

where for any \( \theta > \sigma \), (31) > 0 if \( a > 0.5 \) and (31) < 0 if \( a < 0.5 \). Consequently this implies that the range of indeterminacy is greater when preferences over trade tend towards the no-bias case \( a = 0.5 \). \(^{12}\)

\(^{11}\)The discussion on the likely relative size of \( \theta \) and \( \sigma \) is postponed until section 4.1.1 below.

\(^{12}\)The analysis does not consider the case when \( a = 0.5 \) since this would imply that purchasing power parity (PPP) is satisfied and consequently the linearized inflation equation \( \hat{\pi}_t^R = (2a-1)\hat{\pi}_t^{R(h-f^*)} + 2(1-a)\Delta \hat{e}_t \).
Figure 1: Regions of indeterminacy under a forward-looking domestic price inflation rule

depicts the regions in the parameter space \((a, \mu)\) that are associated with determinacy (D) and indeterminacy (I) around the neighborhood of the steady state, for \(\theta = 3\) and \(\theta = 6\). Recalling that given these parameter values, determinacy of the aggregate system requires \(a \mu < 24.185\) it is apparent that indeterminacy is more likely to occur in the open-economy and becomes a more serious problem as \(\theta - \sigma > 0\) increases and as \(a \to 0.5\).

**Proposition 3** Suppose that monetary policy targets forward-looking consumer price inflation with CWID-timing. Then for an active monetary policy \((\mu > 1)\), a necessary and sufficient condition for determinacy of the difference system is

\[
1 < \mu < \min \left\{ \Gamma_A^2 \equiv \frac{1}{2(1-a)}, \Gamma_B^2 \equiv \frac{2(1 + \beta) + \Lambda_1 \Lambda_2}{\Lambda_1 \Lambda_2 + 4(1 + \beta)(1-a)} \right\}
\]

where \(\Lambda_2 = 1 + \sigma \phi + 4 \phi a(1-a)(\theta - \sigma) > 0\).

Under CPI inflation targeting, the inflation coefficient \(\mu\) is constrained by two upper bounds, both of which are increasing with respect to \(a\):

\(\hat{\pi}_t^R = \Delta \hat{e}_t\),

\(13\)While determinacy of the difference system can also be achieved under a passive monetary policy \((\mu < 1)\), such conditions are not reported since the aggregate system is always indeterminate (from Proposition 1).
Thus the range of indeterminacy is potentially greater the higher the degree of trade openness (i.e. the lower is $a$). This is most evidently apparent from the upper bound $\Gamma^A_2$ where only an $a > 0.5$ can be consistent with $\mu > 1$ regardless of the values of $\theta$ and $\sigma$. Figure 2 depicts the regions in the parameter space ($a$, $\mu$) that are associated with determinacy (D) and indeterminacy (I) given the parameter values $\beta = 0.99$, $\psi = 0.75$, and $\sigma = \phi = \theta = 1$. Under a forward-looking CPI rule, each monetary authority setting, for example, a $\mu < 24.185$ can no longer guarantee a unique equilibrium path and the range of indeterminacy increases substantially as the degree of trade openness increases.

\[ \frac{\partial \Gamma^A_2}{\partial a} = \frac{1}{2(1 - a)^2} > 0, \]
\[ \frac{\partial \Gamma^B_2}{\partial a} = \frac{4(1 + \beta) \left[ \Lambda_1 [1 + 2\phi \theta - \phi \sigma - 4\phi(\theta - \sigma)a(1 - a)] + 2(1 + \beta) \right]}{[\Lambda_1 \Lambda_2 + 4(1 + \beta)(1 - a)]^2} > 0. \]

14While the values of $\theta$ and $\sigma$ do influence the upper bound $\Gamma^B_2$, the sensitivity analysis suggests that the impact on the threshold levels for determinacy is small.
### 3.2 Current-looking Rules

The aggregate system under a current-looking rule is given by:

\[
E_t x_{t+1}^W = C x_t^W, \quad x_t = [\tilde{Z}_t^W, \tilde{\pi}_t^W]'^t,
\]

\[
C \equiv \begin{bmatrix}
1 + \frac{\Lambda_1(\phi+1)}{\beta} & \frac{1}{\beta} \\
-\frac{\Lambda_1(\phi+1)}{\beta} & 1
\end{bmatrix}.
\]

If domestic inflation is targeted, then the following difference system is obtained:

\[
E_t x_{t+1}^R = D x_t^R, \quad x_t = [\tilde{Z}_t^R, \tilde{\pi}_t^R, \tilde{\pi}_t^{R(h-f^*)}]'
\]

\[
D \equiv \begin{bmatrix}
1 + \frac{\Lambda_1 \Lambda_2}{\beta} & \sigma(2a - 1) \left( \mu - \frac{1}{\beta} \right) \\
-\frac{\Lambda_1 \Lambda_2}{\beta} & 1
\end{bmatrix}.
\]

where \( \lambda_1 \) and \( \Lambda_2 \) are defined as above. As before, determinacy requires that both eigenvalues of the coefficient matrices \( C \) and \( D \) are outside the unit circle.

**Proposition 4** Suppose that monetary policy is characterized by a current-looking interest rate rule with CWID-timing. Then a necessary and sufficient condition for determinacy of the aggregate system and the difference system under domestic inflation targeting is \( \mu > 1 \).

Proposition 4 shows that implementing a current-looking rule has no impact on the requirements for equilibrium determinacy in the open-economy if each monetary authority targets domestic price inflation. Therefore for any value of \( \theta \) and \( \sigma \) a necessary and sufficient condition for determinacy is for the central bank to follow an active monetary policy.

If consumer price inflation is targeted the difference system can be expressed as:

\[
E_t x_{t+1}^R = E x_t^R, \quad x_t = [\tilde{Z}_t^R, \tilde{\pi}_t^R, \tilde{\pi}_t^R]'
\]

\[
E \equiv \begin{bmatrix}
1 + \frac{\Lambda_1(1+4(1-a)\phi+\phi(2a-1)^2)}{\beta} & \sigma(2a-1)\mu - \frac{2\mu(1-a)}{\beta} \\
-\frac{\Lambda_1(2a-1)\Lambda_2}{\beta} & 2(1-a)\mu + \frac{2\mu(1-a)}{\beta}
\end{bmatrix}.
\]

Thus determinacy requires that one eigenvalue of the coefficient matrix is inside the unit circle and the other two are outside the unit circle, since the lagged value of the inflation rate
is predetermined. Then by proposition C.2 of Woodford (2003) the next result is obtained.

**Proposition 5** Suppose that monetary policy targets current-looking consumer price inflation with CWID-timing. Then a necessary and sufficient condition for determinacy of the difference system is \( \mu > 1 \) and either

\[
2(1-a)\mu(1-\beta) + \beta^2 + \frac{\Lambda_1 \Lambda_2 (2a-1)}{2(1-a)} + \frac{\beta(1-\beta)}{2(1-a)\mu} > 1
\]

or

\[
\mu < \frac{1-\beta(2-\beta)}{2(1-\beta)}
\]

where \( \Lambda_2 = 1 + \sigma \phi + 4a(\theta - \sigma) \). The sensitivity analysis suggests then one of these conditions is always satisfied for any value of \( \sigma, \theta \) and \( \mu > 1 \). Therefore the analysis suggests that regardless of the inflation index targeted, indeterminacy is virtually impossible under a current-looking rule.

Consequently given the assumption of CWID-timing the following conclusions emerge.

Under forward-looking rules, cross-country trade can make an economy vulnerable to aggregate instability. If consumer price inflation is targeted, the higher the degree of trade openness, the greater the possibility of inducing sunspot equilibria. If domestic price inflation is targeted, then provided \( \theta > \sigma \), aggregate instability increases as \( a \to 0.5 \). Secondly, in order to minimize aggregate instability, targeting domestic inflation is always preferable to targeting consumer price inflation. This result complements Clarida et al. (2002) conclusion that not only is domestic inflation targeting the optimal monetary policy to implement in a two-country economy, but it is also the most effective index of inflation to reduce real indeterminacy.
4 CWID vs CIA-timing

How robust are the above results to the timing assumption on how money enters the utility function? It is straightforward to show that the aggregate system with CIA-timing under a current (backward)-looking rule can be reduced to the same two dimensional system obtained with CWID-timing under a forward (current)-looking rule. Consequently for the aggregate system the timing-equivalence result of Carlstrom and Fuerst (2001) is replicated: the determinacy conditions for a current (backward)-looking rule with CIA-timing is analogous to the conditions for a forward (current)-looking rule with CWID-timing. This section shows the breakdown of this timing-equivalence result for the difference system, under both domestic and consumer price inflation targeting.

4.1 Current-looking rules

4.1.1 Domestic Price Inflation

We start by considering the determinacy properties of the difference system for current-looking rules with CIA-timing. If domestic inflation is targeted the following difference system is obtained:

$$E_t x_{t+1}^R = F x_t^R,$$

where

$$F = \begin{bmatrix}
1 - \mu - (2a - 1) & \sigma \frac{\lambda_1 \lambda_2}{\beta} \\
- \frac{\lambda_1 \lambda_2}{\beta} & 1 + \mu \frac{\lambda_1 \lambda_2}{\beta} - 2(1-a)
\end{bmatrix},$$

and the requirement for determinacy is that both eigenvalues are outside the unit circle.

Proposition 6 Suppose that monetary policy targets current-looking domestic price inflation with CIA-timing. Then the necessary and sufficient conditions for determinacy of the difference system is $\mu > 1$ and:

either (i) $2\theta a > \sigma(2a-1)$ or (ii) $\mu < \frac{1 - \beta}{\lambda_1 2(1-a) \phi [2a(\sigma - \theta) - \sigma]} \equiv \Gamma_3^A$ (35)
Figure 3: Regions of indeterminacy under a current-looking domestic price inflation rule with CIA-timing ($\sigma = 5$ and $\theta = 1.5$)

and

$$
either (i) \Lambda_3 > 1 \text{ or } (ii) \mu < \frac{2(1 + \beta) + \Lambda_1 \Lambda_2}{\Lambda_1 [1 - \Lambda_3]} \equiv \Gamma^B_3 \tag{36}
$$

where $\Lambda_2 = 1 + \sigma \phi + 4\phi a(1 - a)(\theta - \sigma) > 0$ and $\Lambda_3 = \phi [4a(1 - a)(\theta - \sigma) + \sigma(3 - 4a)]$.

First consider the upper bounds given by (35)(ii) and (36)(ii). Comparing these upper bounds with $\Gamma_0$ of condition (29) for the aggregate system, it is straightforward to verify that $\Gamma^B_3 > \Gamma_0 > \Gamma^A_3$. Thus if condition (35)(i) is satisfied the difference system places no additional restrictions on equilibrium determinacy. Hence, if the aggregate system is determinate then the difference system is also determinate. Now consider the case when condition (35)(i) does not bind which requires $a > 0.5$ and $\sigma > \theta$. Then from condition (35)(ii)

$$\frac{\partial \Gamma^A_3}{\partial a} = \frac{2(1 - \beta)\Lambda_1 \phi [4a(\sigma - \theta) - \sigma]}{[\Lambda_1 2(1 - a)\phi [2a(\sigma - \theta) - \sigma]]^2} < 0$$

which implies that the upper bound $\Gamma^A_3$ decreases, the lower the degree of trade openness.

Figure 3 depicts the regions in the parameter space $(a, \mu)$ that are associated with determinacy (D) and indeterminacy (I) given the parameter values $\beta = 0.99$, $\psi = 0.75$, $\phi = 1$, $\sigma = 5$ and $\theta = 1.5$. Note that the aggregate system is determinate for these parameter values provided $\mu < 8.7282$. Figure 3 suggests that for these parameter values, the upper bound on $\mu$ to sustain determinacy is remarkably small. For a low degree of trade openness
indeterminacy exists until \( a \) becomes sufficiently low for condition (35)(i) to bind. Thus this is an example where relatively closed economies can be more prone to indeterminacy than relatively open economies.

Therefore the timing of money matters in the open economy. If the monetary authority targets domestic inflation under a forward-looking rule with CWID-timing, proposition 2 suggests that the difference system could induce additional determinacy restrictions when \( \theta > \sigma \), whereby indeterminacy gets progressively worse as \( a \to 0 \).

5. Under CIA-timing and a current-looking rule, the difference system places additional restrictions on determinacy when \( \theta < \sigma \) and \( a > 0 \).

4.1.2 Consumer Price Inflation

If consumer price inflation is targeted then the difference system is given by:

\[
E_t \hat{x}^R_{t+1} = G \hat{x}^R_t,
\]

\[
\hat{x}_t = \left[ \hat{Z}^R_t, \hat{\pi}^R_t, \hat{\pi}^R_{t-1} \right]',
\]

where \( \lambda_1, \lambda_2 \) and \( \Lambda_2 \) are defined as above. Determinacy requires that one eigenvalue is inside the unit circle and the other two eigenvalues are outside the unit circle.

**Proposition 7** Suppose that monetary policy targets current-looking consumer price inflation with CIA-timing. Then for an active monetary policy \( (\mu > 1) \) the necessary and sufficient conditions for determinacy of the difference system is, either:

\[
(i) \quad 4(1-a)(1+\beta) \Lambda_1 \Lambda_4 \quad \text{or} \quad (ii) \quad \mu < \frac{2(1+\beta) + \Lambda_1 \Lambda_2}{\Lambda_1 \Lambda_4 - 4(1-a)(1+\beta)} \equiv \Gamma_4
\]

15Empirical studies offer no clear conclusion on the size of \( \theta \) and \( \sigma \). For instance the literature suggests that \( \sigma \) be between 1 and 10 (e.g. Gali et al. (2002)). For \( \theta \), evidence suggests that it can take a value anywhere between 1 and 7 (e.g. Treffer and Lai (1999)). Thus it is empirically plausible for \( \sigma \geq \theta \).
and either

\[(i) \quad |A_2| > 3\]

or

\[(ii) \quad \frac{2(1-a)}{\beta} \mu [2(1-a)\mu(1-\beta) + \Lambda_1 \Lambda_2 (\mu - 1) - \Lambda_1 \mu 2(1-a)[1 + 2a\theta \phi] - 1] + (1 - \beta) + 2\beta(1-a)\mu + \Lambda_1 \mu 2(1-a)[1 + 2a\theta \phi] > 0 \]

(38)

where \(\Lambda_2 = 1 + \sigma \phi + 4\phi a(1-a)(\theta - \sigma)\), \(\Lambda_4 = \phi \sigma + 4a - 3 - 4\phi a(1-a)(\theta + \sigma)\) and

\[|A_2| \equiv 1 + \frac{1}{\beta} + 2(1-a)\mu + \frac{\Lambda_1 \Lambda_2 (2a-1)}{\beta} - \frac{(\mu-1)\Lambda_1 \Lambda_2}{\beta}.\]

First consider the upper bound \(\Gamma_4\) on the inflation coefficient given by (37)(ii). Comparing this upper bound with condition (29) for the aggregate system yields \(\Gamma_4 > \Gamma_0\). Hence if condition (38) is satisfied the difference system places no additional restrictions on equilibrium determinacy. Using standard parameter values for \(\beta\), \(\psi\) and \(\phi\) to calculate the eigenvalues numerically, the sensitivity analysis suggests that condition (38) is always satisfied for any value of \(\sigma\), \(\theta\) and \(\mu > 1\). Furthermore, the numerical exercise highlights that condition (37)(i) is very likely to bind unless the degree of trade openness is very low. Using values of \(\beta = 0.99\), \(\psi = 0.75\) and \(\phi = 1\) requires \(a > 0.98\) for (37)(i) not to bind with \(\theta = 3\), \(\sigma = 1\) or \(a > 0.95\) with \(\theta = 1.5\) and \(\sigma = 5\).

Therefore if the monetary authority targets CPI inflation with CIA-timing, under a current-looking rule, the difference system introduces no additional requirements for equilibrium determinacy. This is in stark contrast to a forward-looking rule with CWID-timing, where the range of indeterminacy increases substantially as the degree of trade openness increases. It is also apparent from propositions 6 and 7, that CPI inflation targeting in this example is superior to domestic inflation targeting, in minimizing aggregate instability. This is in stark contrast to the results obtained with CWID-timing, whereby the domestic price index was shown to be the preferable.

4.2 Backward-looking Rules

Here we briefly consider the determinacy properties of the difference system for backward-looking inflation rules with CIA-timing. For domestic inflation targeting the difference
system is given by:

\[ E_t \mathbf{x}_{t+1}^R = \mathbf{H} \mathbf{x}_t^R, \quad \mathbf{x}_t = \left[ \hat{Z}_t^R, \hat{\pi}_t^R(\mathbf{h} - f^*) \right]_t^R \]

\[ \mathbf{H} = \begin{bmatrix}
1 + \frac{\Lambda_1}{\beta} & \sigma \left[ \mu - \frac{2(a-1)}{\beta} \right] & -\sigma \mu \left[ 2(1-a) + \frac{2(a-1)\Lambda_1\Lambda_2}{\beta \sigma} \right] \\
-\frac{\Lambda_1\Lambda_2}{\beta} & 1 & \frac{\mu \Lambda_1 \Lambda_2}{\beta} \\
0 & 1 & 0
\end{bmatrix}, \]

where \( \lambda_1, \lambda_2 \) and \( \Lambda_2 \) are defined as before. Thus determinacy requires that one eigenvalue is inside the unit circle and the other two are outside the unit circle, since the lagged value of the domestic inflation rate is predetermined.

**Proposition 8** Suppose that monetary policy targets backward-looking domestic price inflation under CIA-timing. Then for an active monetary policy \((\mu > 1)\) the necessary and sufficient conditions for determinacy of the difference system is, either:

(i) \( \Lambda_3 < 1 \) or

(ii) \( \mu < \frac{2(1+\beta) + \Lambda_1 \Lambda_2}{\Lambda_1 [\Lambda_3 - 1]} \equiv \Gamma_5 \) \quad (39)

and either

(i) \( \frac{2(\beta - 1)}{\Lambda_2} < \Lambda_1 \) or

(ii) \( \frac{\phi \mu \Lambda_1^2 (1-a)}{\beta} [2a\theta - \sigma(2a-1)] [\Lambda_1 \mu 2(1-a) + 1 + \beta + \Lambda_1 \Lambda_2] \)

\[ + 1 - \beta + \Lambda_1 \mu [1 + \phi \sigma(2a-1)] > 0 \] \quad (40)

where \( \Lambda_2 = 1 + \sigma \phi + 4 \phi a (1-a)(\theta - \sigma) \) and \( \Lambda_3 = \phi [4a(1-a)(\theta - \sigma) + \sigma(3-4a)] \).

First consider condition (39)(i). This condition binds for any value of \( \theta \) and \( \sigma \) provided \( a > 0.5 \). From condition (39)(ii):

\[ \frac{\partial \Gamma_5}{\partial a} = \frac{4\phi \Lambda_1 \Lambda_4 (\sigma - \theta)(1 - 2a) + 4\phi \Lambda_1 [2(1+\beta) + \Lambda_1 \Lambda_2]}{\Lambda_1^2 [\Lambda_3 - 1]^2} \geq 0 \] \quad (41)

where \( \Lambda_4 = 2(1+\beta) + \Lambda_1 [\Lambda_2 - \Lambda_3 + 1] > 0 \). Given \( a < 0.5 \), if \( \sigma > \theta \) then (41) \( > 0 \) whereas if \( \theta > \sigma \) then (41) \( \geq 0 \). This is illustrated in Figure 4, which depicts the regions in the parameter space \((a, \mu)\) associated with determinacy (D) or an explosive solution (N), given the parameter values \( \beta = 0.99, \psi = 0.75 \) and \( \phi = 1 \). For the case when \( \sigma > \theta \), the upper
Figure 4: Regions of determinacy under a backward-looking domestic price inflation rule with CIA-timing

Figure 5: Regions of determinacy under a backward-looking consumer price inflation rule with CIA-timing
bound on the inflation coefficient required for determinacy increases as the degree of trade openness decreases. However, when $\theta < \sigma$, the upper bound first decreases for low values of $a$ and then increases thereafter. Figure 5 depicts the regions of determinacy under a backward-looking consumer price inflation rule, where by inspection, a determinate solution only exists for a very low degree of trade openness. Comparing these results with propositions 4 and 5, it is apparent that regardless of the index of inflation targeted, once again, the timing-equivalence result fails for open economies.

### 4.3 Discussion

The results from the above analysis suggest that for open economies there is no equivalence between the policy-rule targeted and the way money is modeled. Therefore the assumptions made on how money balances enter the utility function has serious implications for the conditions for equilibrium determinacy. This follows from the fact that regardless of the timing assumption imposed on how money enters the utility function, this has no effect on the timing of interest rates in the no-arbitrage condition between home and foreign bonds, which is a key equation in solving for cross-country relative variables (20). Consequently different timing conventions result in different behavioral implications for the real exchange rate. Take, for example, the case of CPI inflation targeting. Despite the fact that the local dynamics of the two aggregate systems are identical, indeterminacy is not possible in the difference system under a current-looking rule with CIA-timing, while indeterminacy can arise under a forward-looking rule with CWID-timing. The intuition for this rests with the degree of influence the terms of trade exerts on the adjustment dynamics of the CPI inflation rate. By imposing the definition of the terms of trade into the CPI index (4), it is straightforward to show that the CPI inflation rate (in terms of the percentage deviation from its steady state value) depends on both the domestic inflation rate and the terms of trade position

$$\hat{\pi}_{t+1} = \hat{\pi}_{t+1}^h + (1 - a) \left( \hat{T}_{t+1} - \hat{T}_t \right).$$

(42)

Under the assumption of CWID-timing, now suppose that, in response to a non-fundamental shock, the agent in the home country believes that home CPI inflation rate is expected to rise. Under an active, interest rate policy, the domestic monetary authority responds by
increasing the *home* real interest rate which lowers domestic current consumption, domestic real marginal cost and thus the domestic inflation rate. In the closed economy this negates the possibility of sunspot equilibria since the domestic and CPI inflation rates are the same. In an open economy, the relative increase in the domestic real interest rate implies a current nominal appreciation (i.e. $\hat{e}_{t+1}$ rises relative to $\hat{e}_t$) from the interest parity condition (20) and thus an improvement in the terms of trade (i.e. $\hat{T}_{t+1}$ increases relative to $\hat{T}_t$). From (42), this puts upward pressure on the *home* CPI inflation rate.

Since the degree of openness determines the influence of the terms of trade on the CPI inflation rate, if this effect is strong enough, the CPI inflation rate can actually rise despite domestic inflation falling, thus validating the initial inflationary belief. However, under CIA-timing the nominal interest rate is now negatively related to the real exchange rate (22). This exerts additional downward pressure on real marginal cost and hence domestic inflation, the effect of which is stronger, the higher the degree of trade openness. Consequently this counterbalances the impact of upward pressure exerted on CPI inflation brought about by an improvement in the terms of trade and thus can prevent the validation of the initial inflationary belief.

Table 2 summarizes the key results of this analysis using the following parameter values: $eta = 0.99$, $\psi = 0.75$, $\phi = 1$, $\mu = 5$ and for $\sigma > \theta$ we set $\sigma = 5$ and $\theta = 1.5$ and for $\sigma < \theta$ we set $\sigma = 1$ and $\theta = 3$. The results from the above analysis therefore suggest that the inter-relationship between the degree of trade openness, the index of inflation targeted and the determinacy conditions are affected by the timing assumption on money imposed. Since
the timing equivalence result fails to hold in open economies there is no straightforward results for determinacy because the above issues interact. If CWID-timing is assumed then indeterminacy can be prevented if the monetary authority avoids implementing a forward-looking rule. Otherwise, indeterminacy is more likely to exist the more open is the economy. If CIA-timing is assumed, then indeterminacy can be prevented under a current-looking rule by targeting the consumer price inflation. Otherwise, indeterminacy is more likely to exist, the more closed the economy. With the failure of the timing equivalence result, the belief that domestic inflation targeting is preferable to CPI targeting in reducing multiple equilibria does not stand up under closer inspection.

5 Conclusion

We have considered the implications of designing interest rate rules in order to prevent policy induced aggregate instability in a two country open-economy framework. It has been shown that the timing equivalence result derived for a closed-economy, no longer applies to open economies. The basic assumptions of monetary models can thus have much wider implications for (in)determinacy of interest-rate rules, once we allow for international trade in both goods and assets. In the closed-economy, the timing assumption on money balances affects the pricing equation for the nominal interest rate. In open economies the timing assumption also affects the behavior of the real exchange rate, which has serious implications for the relationship between the degree of trade openness, the index of inflation targeted and the emergence of sunspot equilibria. Consequently monetary authorities face much greater challenges in designing interest rate rules for open economies without unintentionally generating aggregate instability.
References


