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Sudden Stops, Productivity and the Optimal Level of International Reserves for Small Open Economies

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Abstract

This paper contributes to the theory of optimal international reserves by extending the Jeanne and Rancière (2011) endowment small open economy (SOE) model to a SOE with capital and production that explicitly accounts for the main sources of economic growth. A first version of our set-up considers capital as the sole factor of production in the spirit of the AK model of endogenous growth with constant population, implying increasing returns to scale and justified on the grounds of its ability to generate sustained long-run growth, as observed empirically. Under a plausible calibration for typical emerging market countries facing the risk of sudden stops in capital inflows, we find that the optimal ratio of international reserves to output is 1.7%, which is quite lower than that in Jeanne and Rancière (2011), of 9.1%, even if calibrated to the same sample of 34 middle-income countries. A richer version then introduces also labour as a second factor in a conventional labour-augmenting Cobb-Douglas production function with constant returns to scale and exogenous population growth, consistent with a long-run balanced growth path and the sustained per capita income growth in the data. Under this alternative technology and the same calibration, we find that the optimal reserves-to-output ratio for emerging market SOEs is 5.5%, almost identical to the 6% obtained by Bianchi *et al.* (2018) in a different, sovereign debt model without capital and production. We conclude that our results are explained by the role of capital accumulation as precautionary saving: the accumulated capital stock can potentially be used as a pledge to external creditors in obtaining borrowing, therefore insuring better a SOE against sudden stops.

Keywords: optimal international reserves, small open economies, sudden stops, production technology, capital accumulation, precautionary saving, insurance contracts

JEL classification: E21, E23, F32, F34, F41, O40

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Contents

1 Introduction

International reserves have increased substantially in recent years across the globe. Figure 1 shows that middle-income countries account for nearly a half of this increase.

[Figure 1 about here]

Consequently, the accumulation of international reserves in emerging market economies (EMEs) has become one of the most debated issues in open-economy macroeconomics (Chinn *et al.*, 1999; Aizenman and Marion, 2003; Dooley *et al.*, 2004; Jeanne and Rancière, 2006, 2011; Caballero and Panageas, 2007, 2008; Alfaro and Kanczuk, 2009; Durdu *et al.*, 2009; Calvo *et al.*, 2012; Dominguez *et al.*, 2012). Have many EMEs, in fact, accumulated excessive rather than adequate reserves? And what is an optimal ratio of reserves to output in a small open economy (SOE)? There is no consensus in the recent literature, which offers contradictory explanations on these questions of immediate policy relevance. In particular, there is a long-lasting debate whether reserve accumulation is driven by self-insurance against abrupt capital flow reversals or new mercantilism (see, e.g., Aizenman and Lee, 2007; Aizenman and Sun, 2009).

Figure 2, however, makes it clear that the overall trend of rising international reserves, when expressed in terms of external debt, is driven by the Asian EMEs, especially after their bitter experience with the Asian financial crisis of 1997-1998. This observation points to the hypothesis that international reserves are likely to be accumulated in Asia with different motives and magnitudes compared to the other EMEs, and therefore may need a different modelling approach.

[Figure 2 about here]

On the other hand, when expressed relative to broad money or GDP, as in figures 3 and 4, the increasing trend in international reserves looks more uniform across EMEs, and perhaps may potentially be well explained by a common theoretical framework.

[Figures 3 and 4 about here]

Basically, two main benefits of large reserve holdings have been emphasized: (i) international reserves provide liquidity to smooth consumption (e.g., Jeanne and Rancière, 2006, 2011); (ii) international reserves give a flexibility to manage sizable capital outflows in periods of crises (e.g., Aizenman *et al.*, 2007). Moreover, it has also been argued that reserve policies can help guard away an economy from a crisis or contribute to a recovery after a crisis (Aizenman and Marion, 2004; Dominguez *et al.*, 2012).

The issue of reserve accumulation has been discussed under two main approaches: (i) one of them rationalizes why EMEs hold a high level of reserves as a form of self-insurance against 'sudden stops'[1] in capital inflows (Chinn *et al.*, 1999; Greenspan, 1999; Eichengreen and Mathieson, 2000; Aizenman and Marion, 2003, 2004; Dooley *et al.*, 2004; Aizenman *et al.*, 2007; Dominguez *et al.*, 2012); (ii) the other examines what the determinants of reserve holdings are and, furthermore, what the optimal level of reserves is (Jeanne and Rancière, 2006, 2011; Caballero and Panageas, 2007, 2008; Alfaro and Kanczuk, 2009; Durdu *et al.*, 2009; Calvo *et al.*, 2012).

Jeanne and Rancière (2006, 2011), in particular, have considered the role of optimal international reserves as an insurance against sudden stops in capital inflows in an endowment SOE, abstracting from physical capital accumulation through investment. Yet, the literature has not analyzed this important role reserves play in a richer SOE set-up that models production and investment explicitly. Our contribution with this theoretical paper consists in filling in the gap.

Indeed, most studies on international reserves have focused on other reserve-related issues, such as active reserve management (e.g., Aizenman *et al.*, 2007) or the new type of monetary mercantilism (e.g., Aizenman and Lee, 2007). While Jeanne and Rancière (2006, 2011) do consider optimal reserves in a SOE framework, essential features in neoclassical growth theory as well as in reality, such as production technology and production factors, remain outside the scope of their model. In concluding, these authors admit that their analysis is based on a stylized framework and one way to make it more realistic would be to add productive capital and investment. They suggest that the effects of such an extension are *a priori* ambiguous: on one hand, investment offers a new margin to smooth consumption, which would tend to reduce the optimal level of international reserves in terms of output; on the other hand, there will be a new benefit from reserves, namely, to smooth domestic investment and output, that may tend to increase the optimal reserves-to-GDP ratio. Which one of these two effects will dominate is not obvious before an explicit careful investigation. This unexplored research question of central interest motivates our present work, also aiming to better understand the key determinants of the optimal level of international reserves in models of production SOEs.

[Figure 5 about here]

Figure 5 shows a scatter plot of the relationship in 2013 between the ratios of investment and international reserves, respectively, to GDP in the sample of 34 middle-income EMEs used in Jeanne and Rancière (2011), referred to henceforth as JR. This figure could serve as another general motivation, in particular for the theoretical nature of the study we undertake here. It is insightful to note that while there is a statistically significant (p-value of 0.006) positive (0.865) slope coefficient in the JR sample when regressing the reserve-to-output ratio on a constant and the investment-to-output ratio, just eliminating the two

[1] Calvo (1998) seems to have coined and interpreted first this term in a published title. However, Bianchi and Mendoza clarify (in a footnote) that anecdotal evidence suggests that a comment from the audience in a presentation by Rudiger Dornbusch used the phrase referring to the Mexican crisis of 1994 and quoting Douglas Adams (a British comic writer whose works satirize contemporary life) in the sense that in sharp current-account reversals "it is not the fall that kills you, it is the sudden stop at the end".

obvious outliers (Botswana and China, for which formal statistical tests also confirmed that) results in statistical insignificance at all conventional levels (p-value of 0.14). This fact demonstrates the sensitivity of simple empirical regressions to the outliers in a sample and, hence, the importance of analytical results that can be derived in relevant theoretical environments.

Along these lines or argument, and beyond, our paper proposes an extension to a production economy of the endowment SOE model in JR. More fundamentally, in doing so we bring together two strands of literature that have evolved independently and separately from each other over many years, namely neoclassical growth theory of the 1950s and 1960s and the open-economy theory of capital flows under the risk of sudden stops since the late 1990s. JR have developed an 'insurance model' of optimal international reserves where the representative consumer can smooth consumption during sudden stops if the central bank in an endowment SOE holds a stock of international reserves. The authors derive a closed-form expression for the optimal level of reserves relative to the level of output, and quantify it at 9.1%. They agree that this value cannot account for the rising reserve levels in EMEs since the early 1990s, especially in East Asia, which can clearly be seen by looking back into Figure 4.

Following their approach and with the aim to study explicitly the effects of capital, labour and production on optimal reserves in EMEs, we consider two different model versions of a production SOE. First, we derive optimal reserves-to-output where capital is the sole factor of production, and we refer to this version as the one-factor production SOE AK model, or simply the AK model (of endogenous growth); this version implies increasing returns to scale (IRS) and is justified on the grounds of the ability of the AK-model to generate endogenously, via the influence of government policies – such as subsidies or taxes on investment – on capital accumulation, sustained long-run growth (Acemoglu, 2009, p. 55; Jones and Vollrath, 2013, p. 216). We, then, derive the same ratio in a two-factor production model where labour is also included, and we refer to this version as the two-factor production SOE Cobb-Douglas (CD) model with labour-augmenting technological progress and exogenous population growth, or simply the CD model (of exogenous growth); in turn, this version implies constant returns to scale (CRS), but with diminishing returns to scale (DRS) for each of the two factors, capital and labour, and is justified on the grounds of being consistent with convergence to long-run balanced growth path (BGP) in exogenous growth models (Acemoglu, 2009, p. 59) and with sustained per capita income growth in these models (Jones and Vollrath, 2013, pp. 36-37).

We find that both the AK and CD model extensions imply a negative relationship between the optimal reserve-to-output ratio and capital-augmenting (in fact, sole-factor) or labour-augmenting (in a two-factor production) technological progress. For a plausible calibration of the JR sample of 34 middle-income countries, but updated by 11 years, up to 2014, our AK model quantifies the optimal ratio of reserves to output at 1.7%, which is quite lower than that in the Jeanne and Rancière (2011) endowment SOE benchmark.

The richer CD model under the same calibration similarly quantifies the optimal reserves-to-output ratio lower – but not as much, being 5.5% – than the endowment case. This is, roughly, the level quantified in Bianchi *et al.* (2018), 6%, in their endowment model of optimal reserves when sovereign debt can realistically be repaid in multiple periods – for which our model, as most other we shall survey hereafter, does not allow, aiming to focus on capital and labour as production factors largely ignored in the literature thus far. We conclude that our results are theoretically explained by the role of capital accumulation as precautionary saving: the accumulated capital stock can potentially be used as a pledge to external creditors in obtaining borrowing, therefore insuring better a SOE against sudden stops. But differently from the AK technology with constant population, labour-augmenting technological progress with increasing population results in a lower per capita pledge to potential foreign insurers and, thus, in a higher optimal reserve-to-output ratio in the CD model relative to the AK model. As we shall see, these two natural extensions of the JR endowment SOE to incorporate investment, capital, labour and production also imply a richer analytical version of the optimal reserves formula driven by a different underlying concept of productivity and the corresponding definition of the saving rate, which affects the reported numerical differences too: the AK extension is based on productive capital, while the CD extension is based on productive labour. Similarly to the JR quantification of the optimal reserves-to-GDP ratio, ours cannot explain the observed data in Figure 4 beyond the 1990s, and suggests further work to be needed, possibly based on other motives for reserves accumulation than precautionary saving or, equivalently, insurance against sudden stops that was in the focus here.

The rest of the paper is structured as follows. Section 2 gives an overview of the literature on the motives behind the accumulation of international reserves and on their optimal level. Section 3 presents our extensions to the JR endowment small open economy, namely the AK model and the Cobb-Douglas model for a production small open economy. The results of our calibration and quantification exercise are discussed and illustrated in section 4, and section 5 concludes. Proofs of propositions and corollaries, containing technical details on the derivations, are provided in the supplementary online appendix (our data and code are available upon request as a zip archive).

2 Literature

There is a comprehensive literature on international reserves since at least the 1960s. It mainly focuses on two approaches. The first approach studies the determinants of reserve holdings, examining the reasons behind reserve accumulation. The second approach is concerned with the management of high levels of international reserves. The present paper contributes to the first approach, and we therefore only review this strand of the literature.

2.1 Earlier Literature on the Determinants of Reserves

In the earlier literature, international reserves were seen essentially as a buffer stock, and the relationship between reserves and liquidity was in the centre of interest (Balogh, 1960; Caves, 1964). Balogh (1960) proposes an economic theory of reserve holdings. According to his view, the level of international reserves depends on the objective of economic policy and provides liquidity to the economy. Caves (1964) defines the liquidity problem as financing the United States (US) deficit, and analyzes the role of international reserves in potential issues related to fixed exchange-rate regimes. Even though both papers try to answer why nations hold reserves, they do not explicitly present motives behind holding reserves.

Such motives were given by Heller (1966), by analogy with the motives for holding money in the Keynesian tradition: (i) a transaction motive; (ii) a precautionary motive; and (iii) a speculative motive. Furthermore, Heller (1966) was the first to propose an optimal reserve function in taking adjustment cost and opportunity cost into account. It depends on the marginal propensity to import, the opportunity cost of reserves and the balance of payments (BoP) volatility.

It is widely accepted in the subsequent literature that the above three motives are the 'traditional factors' of the optimal reserve function. Clark (1970), Kelly (1970), and Hamada and Ueda (1977) developed this 'traditional view' in terms of modifying some assumptions, but their key result remains consistent with Heller (1966): the optimal reserve level increases with BoP volatility and decreases with the propensity to import and the opportunity cost.

Frenkel and Jovanovic (1981) proposed a buffer stock model for the optimal reserve level based on the same two types of costs. The first one is the opportunity cost, which can be defined as a comparison of alternative investment returns. The second one is the cost of adjustment, which is a cost of reserve depletion. In order to determine the optimal level of reserves, both costs are minimized. However, there is a negative relationship between them, so the opportunity cost increases when reserves are at a high level, whereas a high level of reserves is associated with a lower adjustment cost.

With increasing trade and financial liberalization in the course of the 1990s, precautionary demand of holding international reserves has gained more importance in international reserve analysis. Ben-Bassat and Gottlieb (1992) introduce the effect of sovereign risk on the precautionary demand for holding international reserves. They discuss the cost of decreasing the reserve level, which might be a signal of an external payment problem for a country with external debt. The authors also point out that past defaults are important. If a country had experienced a default in the past, it would require holding more reserves to keep its international credibility.

In the wake of the East Asian financial crisis of 1997-1998, researchers and policymakers have also become concerned with the issue of 'reserve adequacy', allowing a safeguard for a country from a sudden stop of capital inflows. Some simple policy rules have been proposed in order to provide insurance for economies which have a risk of vulnerability from such episodes or crises. Feldstein (1999) argues that the accumulation of foreign reserves is an insurance against sudden stops of capital inflows and capital outflows in EMEs. Greenspan (1999) similarly suggests a measure of the 'optimal' level of reserves, according to which a country's reserve level should be equal to its short-term external debt (STED). Chinn *et al.* (1999) compare the Latin American countries and the East Asian countries in terms of an insurance model for a currency crisis. Eichengreen and Mathieson (2000) analyze the importance of the currency composition of international reserves for EMEs along the proposed concepts by Feldstein (1999) and Greenspan (1999).

The accumulation of reserves in the Asian economies after the 1997-1998 financial crisis, in an attempt to prevent future occurrences of similar major disturbances, led to further attention to the potential vulnerabilities of EMEs and the role of reserve holdings to mitigate them. One of the main common features of Asian EMEs after the 1997-1998 crisis is that most of them have been generating current account surpluses. Dooley *et al.* (2004) develop a theory of the determinants of international reserves by also considering the current account (CA). CA surpluses may lead to an appreciation of the domestic currency. Furthermore, the relationship between the CA surplus and the demand for reserves could be either negative or positive. If a central bank keeps on buying foreign exchange reserves during the CA surplus period, it obviously increases the country's reserve level.

On the other hand, a negative relationship between CA surpluses and international reserves is presented by Aizenman *et al.* (2007). According to them a CA surplus may be a signal that a country is less exposed to external shocks, and this might be a reason for a decrease in reserve levels. By contrast, if a country runs a CA deficit, it is expected that the central bank sells their reserves to purchase domestic currency, which causes a decrease in international reserves.

2.2 More Recent Literature on the Optimal Level of Reserves

As just outlined, most studies in the field of international reserves have traditionally focused on the determinants of reserves and the fundamental trade-off between the benefits of holding reserves and its costs. Only a few recent papers have examined the optimal level of international reserves.

Caballero and Panageas (2007) study the relationship between reserve accumulation and sudden stops of capital inflows in a dynamic general equilibrium model for EMEs. Given the fact that EMEs run persistent CA deficits to smooth consumption intertemporally, they need capital inflows from foreign countries, but these inflows are subject to the risk of a sudden stop. The authors calibrate the model under this condition and find that insurance strategies aiming at perfect as well as imperfect risk sharing may both lead to a high reduction of reserve accumulation. Similarly to the insurance model of JR, Caballero and Panageas (2008) set up a quantitative model where the government issues nondefaultible debt indexed to the endowment income-growth shock. They find benefits from introducing financial instruments, which serve as insurance against both the emergence of sudden stops and changes in their probability of occurring.

Durdu *et al.* (2009) propose a dynamic stochastic general equilibrium framework in order to implement a quantitative assessment of the 'new mercantilism' under two theoretical models; a one-sector endowment economy and a two-sector production economy. In their analysis, three key factors are changes in the business cycle volatility of output, financial globalization, and self-insurance against a sudden stop. They derive a formula for the optimal level of reserves and find that financial globalization and the risk of a sudden stop are the main reasons behind reserve accumulation. These authors also find that CA surpluses and undervalued exchange rates are two important factors of the large buildup of reserves in response to financial globalization or sudden stop risk.

Based on a different dynamic SOE model, Alfaro and Kanczuk (2009) argue that the optimal level of reserves may be low, close to zero, and hence may not be of much importance, since a country can protect itself by defaulting on its external debt instead of accumulating reserves. There is a large volume of published studies on international reserve holdings that take the level of external debt as given. In order to examine the implications of the joint decision on holding reserves and sovereign debt, the authors suggest that an alternative option is to decrease the level of sovereign debt and hence reduce the probability, and the negative effect, of a potential crisis.

The issue of the optimal international reserve level has also been explored by Calvo *et al.* (2012) within a statistical model where reserves affect the probability of a sudden stop and output costs. The global financial environment is a key factor in their analysis, as the expected return from reserve holdings is conditional on it. In addition, the opportunity cost of reserve holdings is calculated as the spread of public-sector bonds over the interest earned from holding reserves. The optimal level of reserves is then determined as the one that maximizes expected return net of cost, given global financial conditions. One of the main contributions of the paper is that the authors endogenize the probability of sudden stops and the costs of a crisis.

JR (2006, 2011) present an 'insurance model against sudden stops' for an endowment SOE. In a theoretical context, their approach can be viewed – see, e.g., Bianchi *et al.* (2018) – in the sense that international reserves act as an Arrow-Debreu security held by the nation's benevolent government that pays off in a sudden stop and implies a cost to be purchased from the rest of the world (RoW). The optimal level of reserves depends on the key determinants of reserve holdings, such as the probability and size of sudden stops, consumers' risk aversion, the opportunity cost of reserves, and potential output growth. Their calibration results show that the optimal level of reserves relative to GDP is 9.1%, based on 34 middle-income countries over the period of 1975-2003. However, depending on parameters, the optimal level of reserves could be larger or smaller for individual cases. The JR model differs from the above mentioned models of optimal reserves in several aspects. Firstly, JR provide a closed-form formula for the optimal level of reserves in terms of the level of output, whereas Caballero and Panageas (2007, 2008) and Durdu *et al.* (2009)

solve their models numerically. Secondly, JR analyze the optimal level of reserves as an insurance against sudden stops rather than the mercantilist motive.

The very recent literature has made considerable further progress. Bianchi *et al.* (2018) model international reserves as non-state-contingent assets that provide insurance against rollover risk. Differently from Alfaro and Kanzuk (2009), they assume that the debt may not be repaid in full in the next period, which leads to the emergence of debt rollover risk. In their endowment model – abstracting from capital, labour and production in accordance with the literature we surveyed in this section, except Durdu *et al.* (2009) – they quantify optimal reserves for emerging market economies in terms of income at 6%, a number that is, roughly, the same as our 5.5% in the CD production function model version. Bianchi and Sosa-Padilla (2020) examine a macroeconomic stabilization channel and its interaction with a precautionary motive. They depart from the existing work by incorporating nominal rigidities, which by itself also gives rise to a macroeconomic stabilization hedging role for international reserves. Their key contribution consists in proposing a theory that is quantitatively consistent with the observed levels of reserves in emerging market economies, also linking, both theoretically and empirically, the accumulation of international reserves to the exchange rate regime. Arce *et al.* (2019) study the joint dynamics of private and official capital flows employing the idea that the motive for reserve accumulation derives instead from the correction of an externality, indeed typical for the literature on the mercantilist motive of reserves. The mercantilist motive explains reserve accumulation as a by-product of industrial policies promoting exports when there are growth externalities (see, e.g., Rodrik, 2008; Benigno and Fornaro, 2012). Yet in contrast to that literature, they focus on an externality originating in financial markets and leading to excessive systemic risk. These very recent advances, densely summarized in the present paragraph, have also viewed reserve accumulation as a kind of macroprudential policy. Nevertheless, one of their key simplifying assumptions is that all of them remain embedded in an endowment SOE framework, thus ignoring the crucial role played by capital and labour as productive factors, to which our theoretical study contributes essentially to.

2.3 Literature on Capital, Productivity, Growth and Reserves

The earlier literature (Heller, 1966; Hamada and Ueda, 1977; Frenkel and Jovanovic, 1981; and Ben-Bassat and Gottlieb, 1992) derives optimal reserve formulas using a cost-benefit approach. Moreover, the opportunity cost of holding reserves is described as a difference between the return on capital and on reserves. In other words, the marginal productivity of capital enters the optimal reserve formula through the definition of the opportunity cost of holding international reserves. Edwards (1985) discusses the issue and defines the opportunity cost of holding reserves alternatively, as a difference between the interest rate on the debt of a country and the return on reserves. Ben-Bassat and Gottlieb (1992) accept this idea only when the marginal borrowing rate equals or exceeds the marginal productivity of capital. These authors claim that in reality the marginal productivity of capital exceeds the borrowing cost because of market imperfections. However, with increasing financial globalization, the opportunity cost of reserve holdings is better measured as in Edwards (1985), by the difference between the interest rate paid on external liabilities of a country and the lower return on its reserve holdings (García and Soto, 2004; Rodrik, 2006; JR, 2006, 2011). Accordingly, many recent studies on international reserves optimality ignore the relationship between capital productivity and reserve holdings.

However, another strand of research has indeed been trying to explain the relationship between productivity, capital accumulation, growth and international reserves. Bonfiglioli (2008) analyzes the effects of financial globalization on economic growth in terms of total factor productivity (TFP) and capital accumulation. She shows that financial globalization has a positive effect on productivity. However, her empirical study also finds no direct relationship between financial globalization and capital accumulation. Gourinchas and Jeanne (2006) argue that capital mobility may not succeed in closing the gap between rich and poor countries because differences in living standards originate predominantly in productivity differences and differences in human capital. Kose *et al.* (2009) examine the relationship between financial openness and TFP growth. The authors find that capital account openness has a positive effect on TFP growth.

Mourmouras and Russel (2009) investigate the wisdom behind the observed large reserve holdings in terms of investment, capital liquidation and short-term liabilities for a SOE. The authors suggest that capital liquidation and short-term debt are good for economies in terms of increasing investment and higher real wages for workers[2] in good times. On the other hand, capital inflows may cause more financial instability when a country is hit by a sudden stop. In order to insure a country against a sudden stop, the authors suggest to increase international reserves. By accumulating a high level of reserves, central banks can eliminate or decrease the negative effect of capital liquidation on wage variability and workers' welfare.

Cheng (2012) studies a dynamic open economy model in order to analyze the role of domestic financial underdevelopment in the accumulation of reserves. Showing the needs of domestic saving instruments for emerging market firms which have an external credit constraint, he describes the role of central banks as a financial intermediary: central banks provide liquidity to firms, relaxing their credit constraint. In order to decrease the level of reserves he suggests increasing financial market deepness domestically. His paper is based on three stylized facts on EMEs. Firstly, these economies experienced fast economic growth and accumulated a high level of reserves. The positive relationship between the rate of economic growth and the level of international reserves can be seen as part of a 'catch-up' strategy for EMEs. Secondly, these economies have underdeveloped domestic financial markets. Therefore, they require external financing. Lastly, there is a big persistent difference between gross domestic savings and domestic loans in some EMEs. To shed

[2] Mourmouras and Russel (2009) explain the reason for higher real wages in terms of a higher capital intensity ratio.

light on these facts, Cheng (2012) reports Granger causality tests as evidence that foreign reserves affect economic growth via gross fixed capital formation.

Benigno and Fornaro (2012) present a model for fast growing EMEs which run CA surpluses, hold a high level of reserves and experience capital inflows. The authors analyze the joint behaviour of private and public capital inflows in EMEs indicating differences between the tradable and nontradable sectors. They suggest that by holding a high level of reserves governments have an important instrument for growth strategies in relation to growth externalities and financial stabilization. The key mechanism is that an increase in reserves leads to real depreciation[3] and to a reallocation of production towards the tradable sector, which increases the use of imported inputs, the absorption of foreign knowledge and productivity growth. However, this mechanism depends on the imperfect substitutability of private and public capital flows.

Gourinchas and Jeanne (2013) examine the neoclassical framework of the growth model, which implies that higher productivity growth attracts more foreign capital inflows. However, the authors find the opposite relationship to hold empirically in the data for developing countries. They call this the 'allocation puzzle'[4] and propose a solution, involving the interaction of growth, saving and international reserve accumulation. They show that the allocation puzzle is much more related to saving and the behaviour of capital flows (generally, the accumulation of reserves) than to investment.

From the perspective of the literature cited in the present section, we now can delve naturally and easily, in the next section, into our modelling approach and analytical results, without much need for further discussion of the fundamental concepts and theories reviewed above for that purpose.

3 Theory: Optimal Reserves-to-Output Ratio in a Production SOE

In this section, we describe the assumptions of our theoretical framework and derive an optimal reserves formula considering a production SOE. We follow JR to set up the environment, but extend it with the aim to highlight how the conventional production factors from neoclassical theory, namely capital and labour, play a key role in the determination of optimal reserves. Whenever possible, we use the notation of JR to preserve comparability.

Naturally, in the transition from the JR endowment economy to a corresponding production economy, some of the JR model assumptions need to be changed. To introduce physical investment and capital accumulation in our two production SOE model versions, we employ, respectively, two common production function specifications from the economic growth literature, as mentioned. While the JR model features an endowment SOE, our

[3] Rodrik (2008) provides evidence on this mechanism. He shows that real depreciations stimulate economic growth in developing countries.

[4] Gourinchas and Jeanne (2013) argue that public capital flows and the accumulation of reserves play an important role in creating the allocation puzzle.

extension incorporates investment and capital accumulation in two model versions of a production SOE that have been popular in neoclassical growth theory: (i) a one-factor AK model, implying IRS and perpetual growth; and (ii) a two-factor labour-augmenting CD model, implying CRS and convergence to a BGP. The reason is that these two models have found most empirical support, but while the AK model can account for sustained long-run growth, the labour-augmenting technology in the CD model can account for the 'catch-up' hypothesis. In this way, we examine the contributions of additional channels and parameters, such as those related to investment, labour and productivity, to the derivation of a closed-form expression for optimal reserves.

The next two subsections present our two production SOE model versions.

3.1 AK Technology with Constant Population

In this subsection, an AK-type growth model is employed to examine the effects of productive capital and investment on international reserve holdings. All assumptions of the JR model are maintained, but now an AK production function is added. This extension allows to consider physical capital accumulation under IRS and constant population, featuring perpetual growth consistent with the data, and the theoretical influence of capital productivity in a modified optimal reserves formula extending that of JR.

Environment We focus on the optimal level of reserves relative to the level of output that is perceived as insurance, for a production SOE in our case, against losing access to the international credit market. A representative domestic agent, or a private sector, is assumed, as well as a domestic government. There is also an international representative agent, referred to as foreign insurers or the rest of the world (RoW), who provide international reserves to the country. The representative domestic agent in the SOE produces a single (composite) good, which is consumed or invested as physical capital domestically as well as consumed abroad (as SOE exports). The model is set out in discrete time with infinite horizon, using the time subscript $t = 0, 1, 2, \ldots$ Apart from the risk of sudden stops in capital inflows, there is no other source of uncertainty. In that sense, the country faces a risk of international liquidity problems in an otherwise deterministic setting, as in JR.

Following JR, the domestic private sector consists of a continuum of atomistic and identical infinitely-lived consumers. Their intertemporal utility U_t is written as

$$U_t = E_t \left[\sum_{i=0,\ldots,+\infty} (1+r)^{-i} u\left(C_{t+i}\right) \right], \tag{1}$$

where r denotes the constant world interest rate, the period utility function $u\left(C_{t+i}\right)$ is assumed to be of the constant relative risk aversion (CRRA) type, with CRRA parameter $\sigma \geq 0$ and C being aggregate consumption,

$$u(C) = \frac{C_t^{1-\sigma}}{1-\sigma}, \quad \sigma \neq 1, \tag{2}$$

with $u(C_t) = \log(C_t)$ for $\sigma = 1$.

The consumer's budget constraint now includes investment in physical capital:

$$C_t = Y_t - I_t + L_t - (1+r) L_{t-1} + Z_t, \tag{3}$$

where Y_t is domestic output, I_t is investment in physical capital domestically in order to increase the capital stock and next-period output, L_t is newly-contracted external debt in t with a one-period maturity only and Z_t is a net transfer from the government in t. As in JR, external debt accumulated in $t-1$ has to be repaid in t at r, captured by $(1+r) L_{t-1}$, and default in paying back external debt as well as foreign lending by the SOE are assumed away. Differently from JR, investment in physical capital provides a third channel of saving in any period t, in addition to the net indebtedness of the SOE to the RoW, $L_t - (1+r) L_{t-1}$, and to the domestic government (or the public sector), entering via the net transfer, Z_t. It is perhaps easier to see the implications of our extension to a production SOE by writing disposable income of the domestic private sector in t, DY_t, compactly as:

$$DY_t \equiv Y_t + L_t - (1+r) L_{t-1} + Z_t.$$

Then, the SOE private-sector budget constraint (3) can be re-written as

$$C_t = DY_t - I_t; \tag{4}$$

and, hence, the SOE private-sector saving in physical capital is defined, as standard, by

$$I_t = S_t \equiv DY_t - C_t. \tag{5}$$

As in neoclassical growth theory, it is common to assume that all firms have an identical production function. With AK technology, the aggregate production function is

$$Y_t = F(K_t) = A_K K_t. \tag{6}$$

Furthermore, following neoclassical growth theory, let: (i) investment in physical capital be a constant proportion of total output,

$$\frac{I_t}{Y_t} = s, \tag{7}$$

where s is a constant saving rate; and (ii) the increase in the physical capital stock (net investment) in any current period t equals the difference between new investment and depreciated capital,

$$\triangle K_{t+1} = I_t - \delta K_t, \tag{8}$$

where a constant proportion of the capital stock δ is assumed to depreciate each period.

In this first production SOE model version featuring endogenous growth under AK technology, we assume that there is no population growth in this model. We shall consider exogenous population growth in our second production SOE extension to the JR endowment set-up in section 3.2.

From (7), using (8) to write investment and (6) to write output, we can express the (constant) domestic saving rate in physical capital as,

$$s = \frac{\triangle K_{t+1} + \delta K_t}{A_K K_t} \tag{9}$$

In line with the AK model, we further assume that: (i) capital grows at a constant net rate g_K,

$$K_{t+1} = (1 + g_K) K_t, \tag{10}$$

and (ii) the growth rate of economy (i.e., its output) equals the growth rate of capital, $g_Y = g_K$, The latter assumption is shown to be the condition for sustainable growth in the AK model.

Using the AK technology to replace output and equations (8) and (10), we obtain

$$K_{t+1} = sA_K K_t + (1 - \delta) K_t.$$

Then, the gross growth rate of the capital stock, as standard in neoclassical growth theory, is

$$\frac{K_{t+1}}{K_t} = 1 + g_K = 1 + sA_K - \delta, \tag{11}$$

and the net growth rate is

$$g_K = sA_K - \delta. \tag{12}$$

We assume that the capital stock does not grow in sudden stop episodes. Hence, denoting the 'sudden stop' by a superscript s, $g_K^s = 0$, so that $K_{t+1}^s : K_t^s = K_t$, which implies that in crisis times in the AK model, $sA_K = \delta$. Then, we obtain a condition for investment in sudden stop episodes,

$$I_t^s = \delta K_t^s \tag{13}$$

In the JR SOE model we here extend to AK production one of the critical assumptions is related to newly-contracted one-period ahead external debt, L_t. How much can a SOE borrow from foreign lenders? There should be some limit on the amount of output that can be guaranteed by the domestic private sector to foreign creditors. As in the JR model, this restriction is given in what follows by the condition that the external debt must be completely paid back in the next period, which requires:

$$(1+r)\,L_t \leq \alpha_t F(K_{t+1})^n, \tag{14}$$

where $F(K_{t+1})^n$ is trend output in period $t+1$ (to be defined shortly), α_t is a time-varying parameter used as a proxy for the pledgeability of domestic output to foreign creditors, and the superscript n denotes 'normal' times. Assuming, as in JR, that the agents know the value of α_t and $F(K_{t+1})^n$ in any current period t, condition (14) states that external debt in period t is default-free as long as (14) is fulfilled. We follow JR in also assuming that the time-varying parameter α_t is an exogenous variable, and it can change with expectations regarding enforcement of creditor rights or penalties on domestic defaulters: because of the possibility of sudden stops, the rigidity of the consumer's external debt borrowing constraint can fluctuate over time.

As in the JR model, there are two states in the economy: the normal – or non-crisis – state (denoted by a superscript n), occurring with probability $1-\pi$; or a crisis state interpreted as a sudden stop (denoted by superscript s), occurring with probability π. In the non-crisis state, output increases by a fixed rate g_K and the economy can guarantee a constant portion of the output,

$$F(K_t)^n = (1+g_K)^t F(K_{t-1}) \tag{15}$$

$$\alpha_t^n = \alpha. \tag{16}$$

On the other hand, when the economy faces a sudden stop, domestic output decreases by a constant fraction γ below its long-run growth path, and guaranteed output goes down to zero:

$$F(k_t)^s = (1-\gamma)\,F(k_t)^n \tag{17}$$

$$\alpha_t^s = 0. \tag{18}$$

Due to normalization, the guaranteed output does not drop below a positive level. The sum of the time-varying parameter and the output loss parameter is assumed lower than unity, $\alpha+\gamma<1$, in order to secure that the domestic private sector does not have difficulty to pay back all the debt during the crisis. The interest rate on external debt repayment is assumed to be higher than the growth rate of the SOE output (itself equal to the growth rate of physical capital), $r > g_K$, to hold the private sector's intertemporal income limited as in JR.

We follow JR in also assuming that after a sudden stop the capital inflow converges to its pre-crisis pattern within a certain number of periods, υ. Moreover, the country returns to the normal state, n, in period $t+\upsilon+1$. In reality, a country would gain access to international liquidity as in its pre-crisis level in more than one year, if a sudden stop hits the economy in the current period t. Therefore a 'sudden stop episode' can be defined as the length $[t, t+\upsilon]$, as in the JR model. In other words, matching the various times of a crisis stage $s_t = s^0, s^1, \ldots . s^\upsilon$, in a specific period t the country might be either in the non-crisis state, $s_t = n$, or in some of the crisis states.

As in the JR model, the dynamics of output and external credit in a sudden stop episode starting at t are given by:

$$F(k_{t+\tau})^s = [1-\gamma(\tau)]\, F(k_{t+\tau})^n, \tag{19}$$

$$\alpha^s_{t+\tau} = \alpha(\tau), \tag{20}$$

where $\tau = 0, 1, \ldots, \upsilon$. In both equations (19) and (20) $\gamma(\tau)$ and $\alpha(\tau)$ are exogenous functions of τ. For $\tau = 0$ in (19) and (20), we see that $\gamma(0) = \gamma$ and $\alpha(0) = 0$, as in the JR model. Furthermore, we similarly assume that the economy returns to its trend path in a monotonic way, in the sense that both are non-negative but $\alpha(\tau)$ is increasing in τ, while is $\gamma(\tau)$ is decreasing in τ. When the crisis is over, the private sector can be financed by international liquidity as in pre-crisis periods, so there will be no restriction to access foreign markets, hence, $\alpha(\upsilon) = \alpha$, as in JR.

In our model version with physical capital and AK technology outlined thus far, sudden stops have negative effects on both consumption and investment decisions of domestic consumers, and therefore reduce their welfare. Economic crises reduce trend consumption because consumers' elasticity of intertemporal substitution in consumption is bounded. Moreover, sudden stop episodes cause a reduction of domestic output which implies a decrease in the consumers' intertemporal income. Eventually, consumption increases as foreign capital flows return into the economy after the sudden stop. However, our AK-production SOE model reveals a new feature in the adjustment of the economy, absent in JR: it takes more than one year for investment to recover to its pre-crisis level. Investment continues to decrease after the sudden stop year. Figure 6 illustrates this in a 5-year event window. There might be many possible explanations for the persistent effect of sudden stops on investment, such as increasing costs of investment, difficulty to find foreign funds for investment, or the preference of external creditors to invest in more stable economies.

[Figure 6 about here]

The second domestic agent in the SOE is the government – or, equivalently, the monetary(-fiscal) authority, which plays a critical role in the JR model and in our extensions. The task of the government in this set-up is to provide smooth domestic consumption between normal and crisis states. To implement such a policy, the government has as a tool what JR term 'reserve insurance contracts'. Introducing investment in physical capital in our extensions does not affect the government, and we therefore keep all assumptions related to it and its transfers as in JR. Yet, for completeness, we briefly describe the behaviour of the government next.

Following the JR model, a reserve insurance contract is a simple contract between the government and foreign insurers. The aim of the government is to protect domestic agents from the case of a sudden reversal in capital flows; therefore, the government forgoes some funds today in order to gain capital access during the crisis.[5] In this sense, reserve insurance contracts embody the trade-offs in reserve management, and the mechanism is as follows. Firstly, the government announces a settlement with external creditors in period 0. Then, the external fund providers receive a payment X_t from the monetary authority in period t. This process continues until a crisis occurs. Once the crisis starts at time t, the economy obtains a fund R_t. The monetary authority might sign a new reserve insurance deal with foreign insurers when the sudden stop episode ends.[6]

The government's role can be seen in the budget constraint (3) since it shifts the funds coming from the agreement with foreign investors to the private sector as follows; if the country is in the non-crisis stage,

$$Z_t^n = -X_t; \tag{21}$$

however, if a sudden stop occurs, the government secures a payment in the form of

$$Z_t^s = R_t - X_t. \tag{22}$$

Equation (22) shows the government's gain during the sudden stop of capital inflows. The economy earns R_t from foreign insurers, but should also effect the last payment of the reserve insurance contract, X_t, within the duration of the sudden stop.

There is no change either in foreign insurers' participation condition once we incorporate physical capital and investment. Therefore, all assumptions regarding foreign insurers are kept as in JR. For completeness, we briefly describe their behaviour next.

The role of external creditors is to supply international liquidity to the economy during the sudden stop via the reserve insurance contracts. This definition requires a condition that foreign creditors should agree on the price of the government contracts. This is a critical parameter in the JR model, which enters the condition for foreign insurers' participation. The marginal utility of funds for the investors at date t is denoted by μ_t. As in JR, it is more expensive in the crisis than in the normal state:

$$\mu_t^s \geq \mu_t^n. \tag{23}$$

[5] This could be seen as the cost of reserves and JR show that this kind of insurance should be financed by long-term liabilities.

[6] Since the time of the crisis is unknown, an insurance contract signed in period 0 must be specified as an infinite sequence of conditional payments $(X_t, R_t)_{t=1,\dots,+\infty}$ (see JR).

The price of insurance depends on the ratio between μ_t^s and μ_t^n. For simplicity, the JR model assumes that the price parity of funds in normal times to funds in the sudden stop episode is fixed and equal or less than one, which we follow:

$$p = \frac{\mu_t^n}{\mu_t^s} \leq 1. \tag{24}$$

The JR model considers external investors as being perfectly competitive and as sharing the same time discount rate with the domestic private sector. Under these assumptions foreign insurers supply any 'reserve insurance contract' $(X_t, R_t)_{t=1,\ldots,+\infty}$ whose present discounted value is non-negative, of the form

$$\sum_{t=1}^{+\infty} \beta^t (1-\pi)^{t-1} \left[(1-\pi) X_t \mu_t^n - \pi (R_t - X_t) \mu_t^s\right] \geq 0. \tag{25}$$

Optimal Reserves-to-Output The production SOE relies on self-insurance against sudden stops by choosing the right amount of international reserves. The advantage of the intentional parsimony of the AK model version introduced thus far is that it allows for a closed-form solution for optimal reserves relative to output and the ensuing analytical insights, provided that the borrowing constraint (14) is binding.[7]

Due to the above assumption, the economy's short-term debt to output ratio is constant in non-crisis time. The ratio of short-term external debt (STED) to GDP is denoted by λ,

$$\lambda = \frac{L_t^n}{A_K K_t^n} = \frac{1+g_K}{1+r}\alpha. \tag{26}$$

To provide smooth consumption, the government maximizes private sector's intertemporal utility (1) under the CRRA assumption (2) subject to constraints (3), (7), (13), (21), (22) and the binding credit constraints (14) and external creditors' participation condition (25). The Lagrangian function for this constrained optimization problem can be written as:

$$\mathcal{L} = \sum_{t=1}^{+\infty} \beta^t (1-\pi)^t \left\{(1-\pi) u\left(C_t^n\right) + \pi u\left(C_t^s\right) + v\left[(1-\pi) X_t \mu_t^n - \pi (R_t - X_t) \mu_t^s\right]\right\}, \tag{27}$$

where v is the shadow cost of constraint (25), and the normal state consumption is given by

$$C_t^n = A_K K_t - (1+g_K) K_t + (1-\delta) K_t + \frac{\alpha}{1+r} F\left(k_{t+1}\right) - \alpha F\left(k_t\right) - X_t,$$

[7] If the constraint is not binding, a closed-form solution is not possible. Moreover, condition (14) implies that there is no precautionary savings in the model, since the reserve insurance contract plays a substitution role to the precautionary savings. In essence, the insurance motive and preacutionary saving are equivalent, and we use them synonymously.

or

$$C_t^n = \left[1 - s - \lambda \frac{r - (sA_K - \delta)}{1 + (sA_K - \delta)}\right] A_K K_t - X_t \tag{28}$$

while the sudden stop episode consumption is given by

$$C_t^s = (1 - \gamma) A_K K_t - \delta K_t - \alpha A_K K_t + R_t - X_t$$

or

$$C_t^s = \left[-\gamma - \frac{\delta}{A_K} - \lambda \frac{r - (sA_K - \delta)}{1 + (sA_K - \delta)}\right] A_K K_t + R_t - X_t. \tag{29}$$

The first-order conditions imply

$$u'\left(C_t^n\right) = pu'\left(C_t^s\right). \tag{30}$$

Equation (30) shows that domestic consumption can be substituted at the same rate between the normal and crisis state by the private sector and external creditors, as in the JR model. If we re-write (30) and the external creditors' binding condition (14), we can describe the government transfers X_t, in form of,

$$X_t = \frac{\pi}{\pi + p(1 - \pi)} R_t.$$

Now we can solve the first-order condition if the borrowing constraint (14) is always binding.

Proposition 1 ***(Optimal reserves-to-output ratio in a SOE with AK technology)*** *Assuming the described AK-production SOE environment with the external credit constraint (14) always binding, the optimal level of the ratio of international reserves to output,* $\rho_{AK} \equiv \frac{R_t}{A_K K_t}$*, is constant and given by:*

$$\rho_{AK}^* = \frac{\gamma + \lambda - \left(1 - p^{\frac{1}{\sigma}}\right)\left[1 - \lambda \frac{r - (sA_K - \delta)}{1 - (sA_K - \delta)}\right] + \frac{\delta}{A_K} - p^{\frac{1}{\sigma}} s}{1 - \frac{\pi}{\pi + p(1 - \pi)}\left(1 - p^{\frac{1}{\sigma}}\right)}, \tag{31}$$

where, γ *is the output loss in the first period of capital outflows,* λ *is the ratio of short-term external debt to GDP,* p *is the price ratio of funds in different states (normal times and sudden stop episodes),* r *is the world interest rate,* δ *is the depreciation rate of physical capital,* A_K *is the technology level of the economy,* π *is the probability of a crisis, and* σ *is the risk aversion.*

Proof of Proposition 1: See Appendix B.1.

The optimal level of international reserves in terms of output for a SOE derived under an AK production function, equation (31), features some determinants that are common with the original JR endowment SOE model. For example, it is positively related to the STED-GDP ratio, λ; to the output cost of sudden stops, γ; to the probability of a sudden stop, π.

In addition to these determinants, we have three new ones which are related to the production structure of the economy. One such new determinant under an AK technology is the investment rate of economy, s. It affects negatively the optimal reserves-output ratio. A second additional determinant is the depreciation rate of capital, δ. It influences positively the reserve ratio. The third additional determinant under the AK technology is the productivity level, A_K, which influences the optimal reserve ratio negatively. As we discussed while presenting the model assumptions, the growth rate of the economy is equal to the growth rate of the capital stock, and equation (31) gives the parameters of the growth rate of capital stock where it equals investment rate of the economy times capital productivity minus the depreciation rate of physical capital. Therefore, if we follow the joint sign of these parameters, we can see that the endogenous growth rate of the AK technology is also negatively related with the optimal reserves-output ratio.

Corollary 1 ***(Relative optimal reserves-to-output ratio in a SOE with AK technology)*** *In order to judge about the magnitude of the optimal reserves-to-output ratio derived in Proposition 1, ρ^*_{AK}, relative to the output cost of a sudden stop, γ, and the STED-to-output ratio, λ, we follow JR in re-writing (31) as:*

$$\gamma + \lambda - \rho^*_{AK} = \frac{\left(1 - p^{\frac{1}{\sigma}}\right)\left[1 - \alpha - \gamma + (\gamma + \lambda)\frac{p(1-\pi)}{\pi + p(1-\pi)}\right] - \frac{\delta}{A_K} + p^{\frac{1}{\sigma}} s}{1 - \frac{\pi}{\pi + p(1-\pi)}\left(1 - p^{\frac{1}{\sigma}}\right)}. \tag{32}$$

Proof of Corollary 1: See Appendix B.1.

By inspection of equation (32), we can see similarities with the analogous expression in JR, their equation (19); yet, our extended model derives an additional term, $-\frac{\delta}{A_K} + p^{\frac{1}{\sigma}} s$, which results from adding investment and productive capital under our AK technology assumption. This term makes the production SOE crucially different from the endowment SOE benchmark, because in the case of $p = 1$, which implies that external insurers do not have preferences between the crisis and non-crisis states, our richer equation (32) does not reduce in the same way as equation (19) in JR does, making the right-hand side (RHS) zero, so that in this special case of full insurance, the economy's reserve ratio is equal to the aggregate size of the output loss and the STED-GDP ratio, $\gamma + \lambda$. In our case, production also matters in this special case, as we obtain that $\rho^*_{AK} = \gamma + \lambda + \frac{\delta}{A_K} - s$. The added terms in the production case on the RHS of (32) imply that optimal reserves depend positively on the depreciation rate of capital, δ, but negatively on capital productivity, A_K, and the investment rate, s. Without modelling a production SOE, as we did here, the optimal reserves formula omits these important parameters for growing economies that we observe in the real world.

The key difference between the JR endowment benchmark and our AK production model lies in the role of investment and productive capital, captured in (32) by the depreciation rate of physical capital, δ, the investment rate the of economy, s, and the productivity level, A_K. Firstly, a higher depreciation rate in SOEs requires a quicker replacement of the capital stock. Then, these economies (as production mostly depends on imported goods) might need to finance their production and, hence, need access to external borrowing. In order to provide insurance for the private sector during the sudden stop, a higher depreciation rate implies a higher reserves-to-GDP ratio.

Secondly, an increase in the investment rate s leads to a decrease in optimal reserve holding. Since the investment rate equals the saving rate and if investment is financed by a higher proportion of domestic savings, then the economy needs less foreign capital to insure itself against a sudden stop of capital flows. This theoretical result also attests that investment still could be thought of as an opportunity cost of reserves (see Rodrik, 2006).

Thirdly, a higher productivity of capital A_K leads to a lower reserves-to-GDP ratio. Therefore, not only investment decreases the reserves-to-GDP ratio, but also productivity (of capital, in the AK model) plays an important role in reducing optimal reserve holding for SOEs relative to the endowment benchmark in JR. We could think of a mechanism working in the opposite direction of capital stock depletion via the depreciation rate. As long as the private sector employs a sufficiently productive AK technology, this might decrease the need for international borrowing compared to a less productive AK technology or to the JR endowment benchmark.

3.2 Labour-Augmenting Cobb-Douglas Technology with Exogenous Population Growth

In the preceding subsection, labour was normalized and productive capital was the only variable of interest, featuring the AK model as one of the prominent endogenous growth models in neoclassical theory. In this subsection, we also introduce labour and examine the effect of a more general production function on optimal reserve holdings under exogenous population growth, in the tradition of neoclassical growth theory. More precisely, we now employ a CRS labour-augmenting[8] Cobb-Douglas production function. As we are seeking a solution for the balanced growth path (BGP) of the SOE model in the long run, we employ this particular production function rather than the alternatives such as Hicks-neutral technology and Solow-neutral technology. Harrod-neutral technology is the only one that is consistent with a solution for the BGP in the long run (Acemoglu, 2009). It also allows us to check robustness of the AK model results implying IRS with the CRS in the CD model.

All assumptions of the preceding subsection hold through, except that the production function hereafter takes a different form. The latter requires a few additional assumptions, which are first discussed. Then, a second optimal reserves formula is derived within the CD model version of our production SOE.

[8] Known also as Harrod-neutral technology.

Environment This model version differs from the one with AK technology in that it introduces labour, as a second factor of production. The aggregate production function is

$$Y_t = F\left(K_t^{\theta}, A_N N_t\right) = K_t^{\theta}(A_N N_t)^{1-\theta}. \tag{33}$$

N_t is total employment at time t and A_N is now a parameter interpreted in the neoclassical tradition as labour-augmenting technology. $0 < \theta < 1$ measures the capital share in production and assumes CRS.

The standard features of this production function are assumed: continuity, twice-differentiability with respect to each argument, positive diminishing returns to each factor and constant returns to scale to both factors – see, e.g., Acemoglu (2009).

Introducing labour, N_t, into the production SOE model requires a description of population growth, assumed to be exogenously given, as in neoclassical growth theory and, more recently, in Gourinchas and Jeanne (2013),

$$\frac{\triangle N_{t+1}}{N_t} = g_N; \;\; N_t = (1+g_N)^t N_0, \tag{34}$$

where N_0 is the population level in a base period and g_N is the constant population growth rate.

There is no change in the definition of the budget constraint (3) but domestic output, equation (6) in the preceding subsection featuring AK technology, is now replaced by equation (33) featuring labour-augmenting Cobb-Douglas technology. Assuming again, as in neoclassical growth theory, that the saving-to-output ratio is constant, s, we can now define (see for more detailed steps Appendix B.2) the rate of growth of capital per capita, $k_t = \frac{K_t}{Y_t}$:

$$g_k = \frac{\triangle k_{t+1}}{k_t} = s\frac{y_t}{k_t} - \delta - g_N \tag{35}$$

Then, as in neoclassical growth theory (see Appendix B.2), it turns out that the solution for the BGP in effect imposes a constant capital-output ratio, which is a function of four parameters:

$$k_t = k = \frac{s}{g_K + \delta + g_N} = const \text{ (along BGP)}. \tag{36}$$

In this model capital accumulation is modified since we introduce labour and population growth. It can be written as

$$\triangle K_{t+1} = sY_t - (\delta + g_N)\, K_t. \tag{37}$$

Investment is, then,

$$\begin{aligned} I_t^n &= sY_t = \triangle K_{t+1} + (\delta + g_N) K_t \\ &= (g_K + \delta + g_N) K_t. \end{aligned} \tag{38}$$

If we replace output by the production function in order to see the effect of its components on the optimal level of reserves, we obtain

$$K_t^\theta (A_N N_t)^{1-\theta} = \frac{g_K + \delta + g_N}{s} K_t,$$

where output is proportional to the capital stock.

The neoclassical BGP concept, then, implies that all key variables grow at the same rate:

$$g_Y = g_K = g_N = g_A. \tag{39}$$

We assume that the capital-labour ratio (or per capita capital) does not grow in sudden stops, equivalent to writing:

$$k_{t+1}^s = k_t^s.$$

Again, the economy can be in two states, normal times and a crisis state. The SOE still obeys the restriction on the pledgeable output. The role of the monetary authority and external creditors' participation condition are unchanged.

Optimal Reserves-to-Output We continue to assume that the external credit constraint (14) is always binding, which allows for a closed-form solution of this simple insurance problem of the production SOE. However, the parameter λ now takes the form (for more detailed steps, see Appendix B.2)

$$\lambda = \frac{L_t^n}{K_t^\theta (A_N N_t)^{1-\theta}} = \frac{1 + (1-\theta) g_N + \theta g_K}{1+r} \alpha, \tag{40}$$

since the country keeps a constant STED-to-output ratio when (14) is always binding. λ is the same parameter as in the previous AK production SOE model version, but the definition of output, (33), in the denominator of (40) has now changed.

Using the CD production function, consumption in the non-crisis state can be written as (see Appendix B.2)

$$C_t^n = \left\{ 1 - s - \lambda \frac{r - [(1-\theta) g_N + \theta g_K]}{1 + (1-\theta) g_N + \theta g_K} \right\} K_t^\theta (A_N N_t)^{1-\theta} - X_t. \tag{41}$$

By analogy, consumption in the sudden stop episode can be written as (see Appendix B.2)

$$C_t^s = \left\{ -\gamma - \frac{(\delta + g_N)\,K_t}{K_t^\theta (A_N N_t)^{1-\theta}} - \lambda \frac{r - [(1-\theta)\,g_N + \theta g_K]}{1 + (1-\theta)\,g_N + \theta g_K} \right\} K_t^\theta (A_N N_t)^{1-\theta} + R_t - X_t. \quad (42)$$

Since there is no change in the role of monetary(-fiscal) authority, it enters a reserve insurance contract as described above in order to maximize the private sector's utility subject to the relevant constraints.

The optimal reserves-to-output ratio, ρ_{CD}^*, is then constant, as ρ_{AK}^* was in the AK-technology set-up, but now given by a richer expression, as stated formally in the next proposition.

Proposition 2 ***(Optimal reserves-to-output ratio in a SOE with labour-augmenting Cobb-Douglas technology)*** *Assuming the described labour-augmenting Cobb-Douglas production SOE environment with the external credit constraint (14) always binding, the optimal level of the ratio of international reserves to output, $\rho_{CD}^* \equiv \frac{R_t}{K_t^\theta (A_N N_t)^{1-\theta}}$, is constant and given by:*

$$\rho_{CD}^* = \frac{\gamma + \lambda - \left(1 - \lambda \frac{r - [(1-\theta) g_N + \theta g_K]}{1 + (1-\theta) g_N + \theta g_K}\right)\left(1 - p^{\frac{1}{\sigma}}\right) + (\delta + g_N)\left(\frac{k}{A_N}\right)^{1-\theta} - p^{\frac{1}{\sigma}} s}{1 - \frac{\pi}{\pi + p(1-\pi)}\left(1 - p^{\frac{1}{\sigma}}\right)}. \quad (43)$$

Proof of Proposition 2: See Appendix B.2.

As seen in (43), the optimal level of reserves in terms of GDP with deterministic constant returns to scale labour-augmenting Cobb-Douglas production function in our SOE set-up has many common parameters with the AK version considered earlier. For example, the optimal reserves-to-output ratio depends again on parameter determinants such as; γ, the output loss in the first period of capital outflows; λ, the STED-to-output ratio; p, the price ratio of the funds in non-crisis and crisis states; r, the world interest rate; g_K, the growth rate of the capital stock; δ, the depreciation rate of physical capital; π, the probability of a sudden stop; and σ, the coefficient of relative risk aversion. However, there are also important differences between the two production SOE extensions to the JR endowment SOE benchmark we compare here – among the key parameters now are also: the growth rate of labour, g_N; the capital-labour ratio, k; the capital share in income, θ; and the technology level, where now in the CD production SOE A_N is interpreted as labour-augmenting technology.

Equation (43) implies relationships that are the same as in equation (31) earlier, namely, that the optimal reserve ratio is a positive function of: (i) the output cost of a sudden stop, γ; (ii) the level of short external term debt, λ; (iii) the probability of a sudden stop, π; (iv) the depreciation rate of the installed capital stock, δ; (v) the world interest rate, r; and (vi) risk aversion, σ; whereas it is a negative function of the investment rate of the economy, s.

Differently from the AK-production SOE, the additional determinants in this CD-production version influence the optimal reserves-to-output ratio as follows: (i) population growth, g_N, positively, as expected; (ii) the capital share in income, θ, negatively; (iii) the capital-labour ratio, k, positively; (iv) labour-augmenting productivity, A_N, negatively; (v) the growth rate of capital, g_K, negatively.

The differences in the AK IRS model of endogenous growth versus the CD CRS model of exogenous growth arise from the modelling of investment and the respective production functions. A first difference is the growth rate of the economy. Unlike the JR endowment SOE model, in both the previous subsection with AK technology and the present subsection with CD technology we analyze components of the growth rate of the economy rather than a single deterministic endowment growth parameter (g, in JR's notation). However, our two production SOE versions implied a different relation between the growth rate of the economy, g_Y, and the component growth rates: (i) $sA_K - \delta$ in the AK model and (ii) $(1-\theta)\,g_N + \theta g_K$ in the CD model. Secondly, the CD model assumes production in a richer context than the AK model since it includes labour input and population growth. Therefore, we have the additional determinants of optimal reserves-to-output: the capital-labour ratio, k, and the capital share in income, θ. A final difference relates to the definition of productivity. In each model version, AK and CD, productivity (no matter whether it is capital productivity, A_K, or labour-augmenting productivity, A_N) decreases the reserves-to-GDP ratio. However, this difference leads to different productivity results in magnitude for the respective model versions of our production SOE, which we discuss further down in the calibration section 4.

Corollary 2 ***(Relative optimal reserves-to-output ratio with labour-augmenting Cobb-Douglas technology)*** *In order to judge about the magnitude of the optimal reserves-to-output ratio derived in Proposition 2, ρ^*_{CD}, relative to the output cost of a sudden stop, γ, and the STED-to-output ratio, λ, we follow JR in re-writing (43) as:*

$$\gamma + \lambda \; - \rho^*_{CD} = \frac{\left(1 - p^{\frac{1}{\sigma}}\right)\left[1 - \alpha - \gamma + (\gamma + \lambda)\,\frac{p(1-\pi)}{\pi + p(1-\pi)}\right] - (\delta + g_N)\left(\frac{k}{A_N}\right)^{1-\theta} + p^{\frac{1}{\sigma}} s}{1 - \frac{\pi}{\pi + p(1-\pi)}\left(1 - p^{\frac{1}{\sigma}}\right)} \tag{44}$$

Proof of Corollary 2: See Appendix B.2.

It can easily be seen that if the terms $-(\delta + g_N)\left(\frac{k}{A_N}\right)^{1-\theta} + p^{\frac{1}{\sigma}} s$ are ignored, the CD-production SOE optimal reserve formula in (44) would reduce to that in the JR endowment benchmark. Therefore, both models include many similar determinants. However, as we noted already for the AK-technology, modelling explicitly production in a SOE, here by labour-augmenting CD technology, avoids the reduction of the special case of $p = 1$, to the Greenspan-Guidotti rule: $\rho^*_{CD} = \gamma + \lambda$. In our richer model, even in this case $\rho^*_{CD} = \gamma + \lambda + (\delta + g_N)\left(\frac{k}{A_N}\right)^{1-\theta} - s$, so that, in addition to the parameters we emphasized already in the AK case, now also population growth, g_N, the steady-state capital-labour ratio, k, and the share of labour in the production process, $1 - \theta$, tend to increase optimal reserves in terms of output, whereas now labour-augmenting technology, A_N, replaces capital productivity, A_K, in tending to decrease it, together with the investment rate, s, as it was for the AK-production model version.

4 Calibration: Quantification and Interpretation of Our Analytical Results

In this section, we analyze some quantitative implications of our production SOE model versions comparing them to the respective findings in the JR endowment SOE. To make the comparison as direct and sharp as possible, we employ data for the same 34 middle-income countries.[9] In accordance with the JR method, we construct a benchmark calibration based on the average sudden stop in our sample, as updated to 2014. We then discuss to what extent our two model versions of a production SOE are able to explain the recent trend toward a buildup of international reserves in EMEs.

Our calibration of the key parameters determining the optimal reserve-to-output ratio according to propositions 1 and 2 (and the respective corollaries 1 and 2) are given in Table 1.

[Table 1 about here]

We recomputed some of the JR model parameters based on our updated sample, such as the output loss, the size of the sudden stop, and the crisis probability, since they play a modified role in our AK- and CD-technology model extensions. We did not change some other JR parameters, such as the risk-free interest rate, the relative price of a non-crisis dollar and the CRRA, as they have no distinct novel role in our model but are necessary for a comparison across these model versions.

To analyze the behaviour of the model economy, we decompose domestic consumption, C_t, in terms of domestic output, Y_t, less investment, I_t, the financial account, FA_t, income transfers from abroad, IT_t, and reserves decumulation, $-\Delta R_t$,[10]

$$C_t = Y_t - I_t + FA_t + IT_t - \Delta R_t. \tag{45}$$

As in JR, a sudden stop is defined by an unexpected abrupt fall in the financial account. *Ceteris paribus*, it leads to a drop in domestic consumption. This effect can be amplified by a simultaneous drop in output, but can also be mitigated by decumulating reserves.

[9] We use the Jeanne and Rancière (2011) sample of 34 countries, but extending the original sample period, 1975-2003, by 11 years, to 2014. JR applied the World Bank's classification to define their middle-income countries. However, after the publication of their paper this classification has changed: in effect, the sample now includes 7 high-income countries, i.e., Argentina, Chile, Czechia, Hungary, Korea, Poland and Uruguay. Following JR, we also exclude major oil-producing countries from the dataset.

[10] Equation (45) can also be interpreted as decomposing domestic absorption since domestic absorption equals the sum of domestic consumption and investment, $DA_t = C_t + I_t$.

4.1 AK Technology SOE Model

One can see the correspondence between the national accounting identity (45) and our AK-production SOE model extension by observing that the representative consumer's budget constraint in a sudden stop becomes

$$\underbrace{C_t^s}_{C_t} = \underbrace{(1-\gamma)\,Y_t^n}_{Y_t} - \left\{ \underbrace{\delta K_t^s}_{I_t} + \underbrace{(-L_{t-1})}_{FA_t} + \underbrace{[-r_t L_{t-1} - (\pi+\omega)\,R_t]}_{IT_t} - \underbrace{(-R_t)}_{\Delta R_t} \right\}, \tag{46}$$

where ω denotes a pure risk premium and might be interpreted as an opportunity cost of holding reserves.[11] As JR have emphasized, such a decomposition is useful in allowing to infer the magnitude of the shocks hitting the economy in a sudden stop episode, that is, λ and γ, from the empirical behaviour of the terms on the RHS of (46). Furthermore, our extended model to AK and CD production highlights the dynamics of output as resulting from investment and capital accumulation as well as employment, key macrovariables that are omitted in the JR endowment SOE model.

Following Guidotti *et al.* (2004) and Jeanne and Rancière (2011), a sudden stop in year t is identified in our sample as a drop in the ratio of capital inflows to GDP exceeding 5% relative to the preceding year. The countries in our sample and the years in which they went through sudden stop episodes are listed in Table 2.

[Table 2 about here]

Even though we use the same sample of countries as JR (2011), our sudden stop years were defined applying their methodology to our updated dataset, and therefore some minor differences in the sudden stop episodes by country are observed. Moreover, when we calculate capital inflows in our dataset mostly World Bank's *World Development Indicators* (*WDI*, online) was used, whereas JR relied on IMF's *International Financial Statistics* (*IFS*).

[Figure 6 about here]

Figure 6 illustrates this novel feature of output dynamics in our extension to production, now driven obviously by investment dynamics relative to the JR benchmark. It depicts the average behavior of consumption and the contribution of the various components on the RHS of (46) in a five-year event window centred around a sudden stop year, where the middle observation '0' labels the latter (output is normalized to 100 in that year of the sudden stop). Although all components of equation (45) display a similar pattern with the

[11] Because it has no role in affecting productivity and investment, the opportunity cost of holding reserves is not described in our extended set-up. However, in order to enable comparisons between our AK- and CD-production model vesrions and the JR endowment benchmark, we follow their methodology in expressing $X_t = (\pi+\omega)\,R_t$.

JR model, investment adds inertia in its own adjustment and, hence, in the adjustment of output. Both investment and output in our AK model continue to decrease after the sudden stop period, featuring higher persistence, whereas all other components of equation (45) start recovery after period '0', as is the case with output when investment and capital are not modelled in JR. The difficulties in accessing international borrowing facilities after the sudden stop and the capital outflows during the crisis make the private sector vulnerable, and this affects investment decisions. Therefore, a recovery may not be seen in investment and output in the first year after the sudden stop.

The unconditional probability of a crisis, π, is 9.8% per year, in our updated calibration and thus remains consistent with JR (2011), 10%. The STED-GDP ratio, interpreted as the size of a sudden stop, λ, is calibrated at the average level of the ratio of capital inflows to GDP, $\frac{FA_t}{Y_t}$, over our sample of crisis episodes, and is almost 9.9%, again consistent with JR, 10%. Output loss, γ, was calibrated at the average difference between the GDP growth rate one period before the crisis and the growth rate in the first year of the capital outflows. We observed in our updated sample a 2% decrease in GDP growth rates on average in the first year of capital outflows and a 4% decrease when we restrict the sample to countries that suffered an output reduction; however, it shows large variation across countries. JR set this loss to 6.5%, and we use their calibration in order to allow for a more consistent comparison.[12] The risk-free short-term world interest rate, r, the risk aversion parameter, σ, and the price ratio of funds in dollars across states,[13] p, are calibrated as in Jeanne and Rancière (2011) at 5%, 2, and 0.855, respectively.

The role played by the investment rate, s, the depreciation rate of physical capital, δ, and the AK-technology parameter, A_K, are the three additional determinants in extending the optimal reserve formula in JR to a production SOE, first using an AK production function, as discussed here. We calibrate the investment rate to be equal to 24%, which is the sample average of the investment share in total income in our data from the Penn World Tables (PWT, 7.0). Following the growth accounting literature, we set the depreciation rate of physical capital to 6% per annum (Caselli, 2005; Gourinchas and Jeanne, 2013). Our AK-technology parameter, A_K, is calibrated based on a proxy as suggested in Caselli (2005): $A_K = \frac{y_t}{k_t}$ = where $y_t = \frac{Y_t}{N_t}$ is GDP per worker in the data and $k_t = \frac{K_t}{N_t}$ is capital per worker in the data. Our sample shows average GDP per worker $\overline{y} = \$15141$ from PWT (7.0). In line with Caselli (2005), we found that average capital per worker, $\overline{k}$, is 2.49 times higher than GDP per worker. Consequently, we calculated $A_K = 0.4$.

Our extended SOE model highlighting an AK technology results in a lower optimal reserve-to-output ratio, 1.7%, relative to the JR endowment SOE benchmark, 9.1%, and the trends in Figure 4. Thus, when taking into account investment and productive capital, countries would need (much) less reserves. Our AK-model implies that the optimal level

[12] JR calculate output decreases by 4% on average in the first year of sudden stops and by 9% when they only focus attention on subset of the countries in which output fell. Then they take the average of two estimates and set output loss to 6.5%.

[13] Which is based on the calculation of the opportunity cost of reserves in JR.

of international reserves relative to GDP is a decreasing function of the investment rate, s. Furthermore, higher productivity A_K (of capital, here in the AK SOE model) implies a lower reserve-to-output ratio too. In other words, a higher saving rate and a higher capital productivity both reduce the optimal level of international reserves in terms of GDP relative to the JR endowment SOE benchmark.

[Figure 7 about here]

Figure 7 illustrates the relationship between the optimal level of international reserves and its determinants for the AK model. This figure also shows the sensitivity of our results to the key determinants of the optimal reserves-to-output ratio. It is based on the optimal reserves formula (31) and the reported calibration in Table 1. As can be seen in the respective panels of Figure 7, our results suggest a positive relationship between the reserves-to-GDP ratio and some of its key determinants, such as the size λ, the output cost γ, and the probability π of a sudden stop, the risk-free world interest rate r, the coefficient of relative risk aversion σ and – in our extension to an AK production – also on the depreciation rate of capital δ. On the other hand, and perhaps most importantly given the aims of the present paper, we demonstrate here a few novel findings, as follows. The optimal reserves-to-GDP ratio depends negatively on other determinants, notably those that arise when modelling a production economy, here under an AK technology, namely (i) the investment rate s and (ii) capital productivity A_K. Figure 7 also shows the negative relationship between the (endogenous via policy) growth rate of the capital stock and of the economy, g_K, and the optimal reserves-to-output ratio. Although this latter link does not appear in the optimal reserves formula (31) explicitly, it can be easily derived from model assumptions and equation (12).

The intuition for our main result we propose is that, with the capital stock now accumulated via investment and potentially used as a pledge to external creditors in obtaining borrowing and therefore insuring better against sudden stops, the optimal reserves-to-output ratio is lower in the endogenous growth AK model relative to an otherwise similar endowment economy. Moreover, it is much lower because of the increasing returns assumption embodied in the AK technology the SOE employs.

4.2 Labour-Augmenting Cobb-Douglas Technology SOE Model

In order to make direct comparisons between our two production SOE model versions and the JR endowment SOE benchmark, we use the same dataset and the same calibration strategy as was described in the preceding subsection. However, the investment component in the decomposition of consumption has a different analytical representation from the parallel equation (46) in the AK setup, which is now under the labour-augmenting Cobb-Douglas version written as

$$\underbrace{C_t^s}_{C_t} = \underbrace{(1-\gamma)\, Y_t^n}_{Y_t} - \left\{ \underbrace{(\delta + g_N)\, K_t}_{I_t} + \underbrace{(-L_{t-1})}_{FA_t} + \underbrace{[-r_t L_{t-q} - (\pi + \omega)\, R_t]}_{IT_t} - \underbrace{(-R_t)}_{\Delta R_t} \right\}. \quad (47)$$

While in the AK model investment in sudden stops was described simply as δK_t, it includes as well dependence on labour via population growth g_N in the CD version we consider in the present subsection, and equals $(\delta + g_N)\, K_t$.

Our CD set-up has some common parameters with our AK set-up, and these are calibrated in the same way – see again Table 1. Yet, the CD model includes a differently defined technology parameter, A_N. Following Caselli (2005) and Gourinchas and Jeanne (2013), in calibrating it we assume that $A_N = \left(\frac{y}{k^\theta}\right)^{\frac{1}{1-\theta}}$, where y is GDP per worker and k is capital per worker. As in the preceding section, the average GDP per worker is $\overline{y} = \$15141$ for our dataset from PWT (7.0) and the average capital per worker, $\overline{k}$, is 2.49 times higher than GDP per worker. With the capital share in output taking its standard calibration value of 0.3, e.g., as in Gourinchas and Jeanne (2013), we calculate A_N to be equal to 10241. The average growth rate of population, g_N, is found to be 1.5% in our updated dataset. We used it in the calculation of g_K, which is equal to 9.8%, since $g_Y = (1-\theta)\, g_N + \theta g_K$.

Based on our formula (43), the optimal ratio of reserves to output is quantified at 5.5% in the richer two-factor CRS Cobb-Douglas production SOE model, i.e., roughly three times higher than in the AK set-up in the preceding subsection, but at the same time 60% lower than that in the JR endowment SOE model. The intuition we suggest to explain this percentage is that, differently from the AK technology, adding labour and constant population growth, consistent with a long-run BGP in neoclassical models of exogenous growth and with sustained per capita income growth, results in a lower per capita pledge to potential creditors in international financial markets and, hence, in a higher optimal reserves-to-output ratio in the CD model relative to the AK model. Moreover, it is not that much lower than the JR endowment SOE benchmark as under the AK technology because of the constant returns assumption embodied in the CD technology with regard to the two production factors taken altogether, and the decreasing returns to each of the two production factors when taken individually.

[Figure 8 about here]

Figure 8 depicts the relationship between the optimal level of international reserves and its determinants for the CD model. Similarly to Figure 7 for the AK version, Figure 8 illustrates that the optimal reserves-to-GDP ratio for the CD version depends positively on some of its key determinants, as was in the JR endowment SOE benchmark, such as: (i) the size of the sudden stop; (ii) the output cost of a sudden stop; (iii) the probability of a sudden stop; (iv) the world interest rate; (v) the coefficient of relative risk aversion; (vi) the depreciation rate of capital and – in our extension to a CD production in the present subsection – also on (vii) population growth and (viii) the capital-labour ratio. On the other hand, and notably given the objective of the present subsection, our analysis revealed some novel results, as follows: the optimal reserves-to-GDP ratio depends negatively on: (i) the investment rate, as in the preceding subsection under the AK technology; and now also on the additional determinants highlighted by the CD production SOE model of the present subsection, namely; (ii) the labour-augmenting technology parameter; (iii) the growth rate of capital; (iv) the capital share in output.

5 Concluding Comments

This paper aimed to highlight the role of the neoclassical production factors on the optimal level of international reserve holdings by small open economies facing the risk of sudden stops. To do so, we extended the Jeanne and Rancière (2011) endowment SOE model by adding to it, in turn, two different aggregate production technologies, each justified from earlier theoretical and empirical work on long-run economic growth.

When, in a first model version, we employed an AK production function we were able to analytically show and then quantify in our sample by using plausible calibration that capital-augmenting productivity decreases, by perhaps too much, the optimal level of international reserves relative to GDP: 1.7% versus 9.1% in the endowment SOE of JR, due to the increasing returns to scale the AK technology assumes. Secondly, when in turn employing a more general production function in order to add the effect of labour in the model, i.e., a labour-augmenting Cobb-Douglas production function, we similarly found that the CD model preserved the negative relationship between the optimal reserve-to-output ratio and now labour-augmenting productivity. Nevertheless, the optimal ratio of reserves to output under CD technology increased to 5.5%, yet still remaining 60% lower than that in the JR endowment SOE model. It is worth noting, nevertheless, that this is, roughly, the same magnitude as the one obtained by Bianchi *et al.* (2018), 6%, in their endowment model of optimal reserves when sovereign debt can realistically be repaid in multiple periods, which is quite reassuring. We interpreted our ratios, derived and computed in two alternative production SOEs, as follows. Differently from the AK technology exhibiting IRS and perpetual growth, labour-augmenting technological progress featuring CRS results in a lower per capita pledge to potential foreign insurers and, thus, in a higher optimal reserves-to-output ratio in the CD model relative to the AK model. Moreover, capital accumulation in the CD version has to make up not only for the depreciation of capital but also for population growth. However, as was the case with the JR endowment SOE benchmark, our extensions here to production SOE could not rationalize, in quantitative terms and irrespective of the theoretical value added, the drive toward reserve accumulation, especially in East Asian EMEs, since the late 1990s.

One of the drawbacks of our analysis, which potentially explains the latter conclusion, is that the JR endowment SOE model, which we extended to production, relies on an insurance motive for holding international reserves rather than on the competing mercantilist motive. Alternatively, our model versions can be refined by introducing explicitly an exchange rate for the SOE and studying its implications for reserves accumulation. Further, instead of the assumption of one single good, the model could be generalized to two goods, with various disaggregation of production structure by sectors, including tradables and nontradables. Finally, the model versions we derived and quantified by calibration can be subjected to empirical validation, as in a sequel paper by Nasir (2020).

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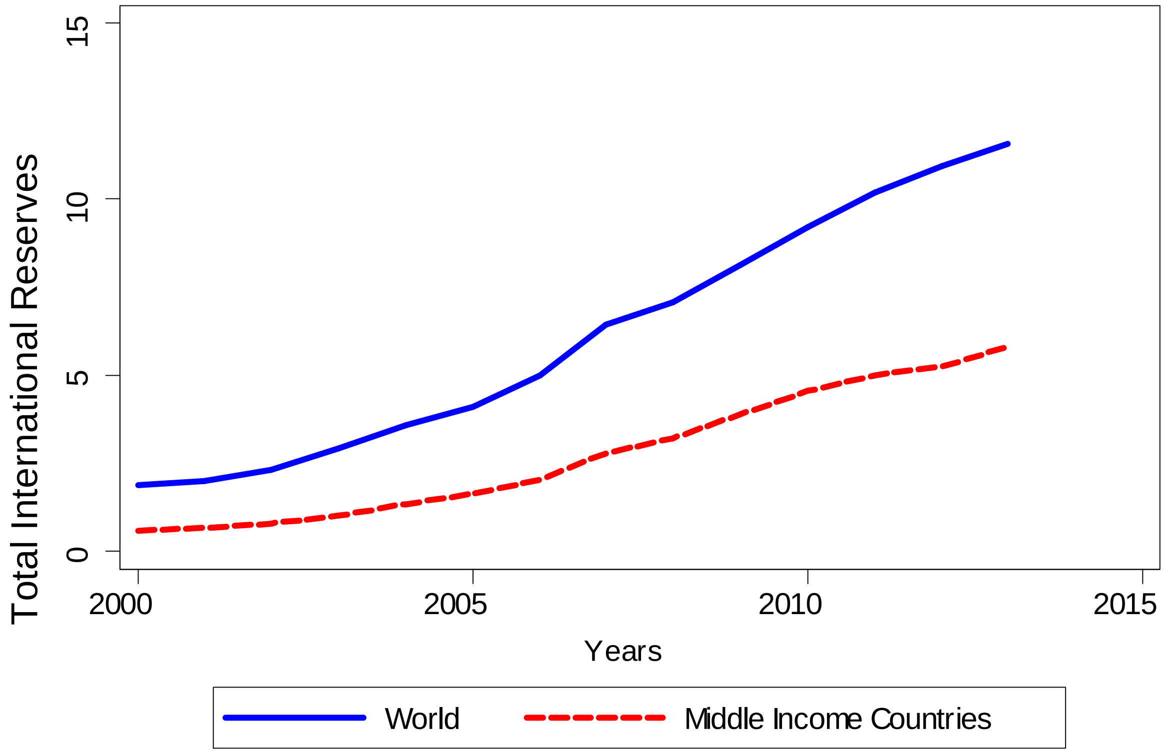

Figure 1: Global International Reserves, in trillions US Dollars. *Source:* Data is for our/JR sample of 34 countries listed in Table 2 from IMF's *International Financial Statistics* and World Bank's *World Development Indicators*.

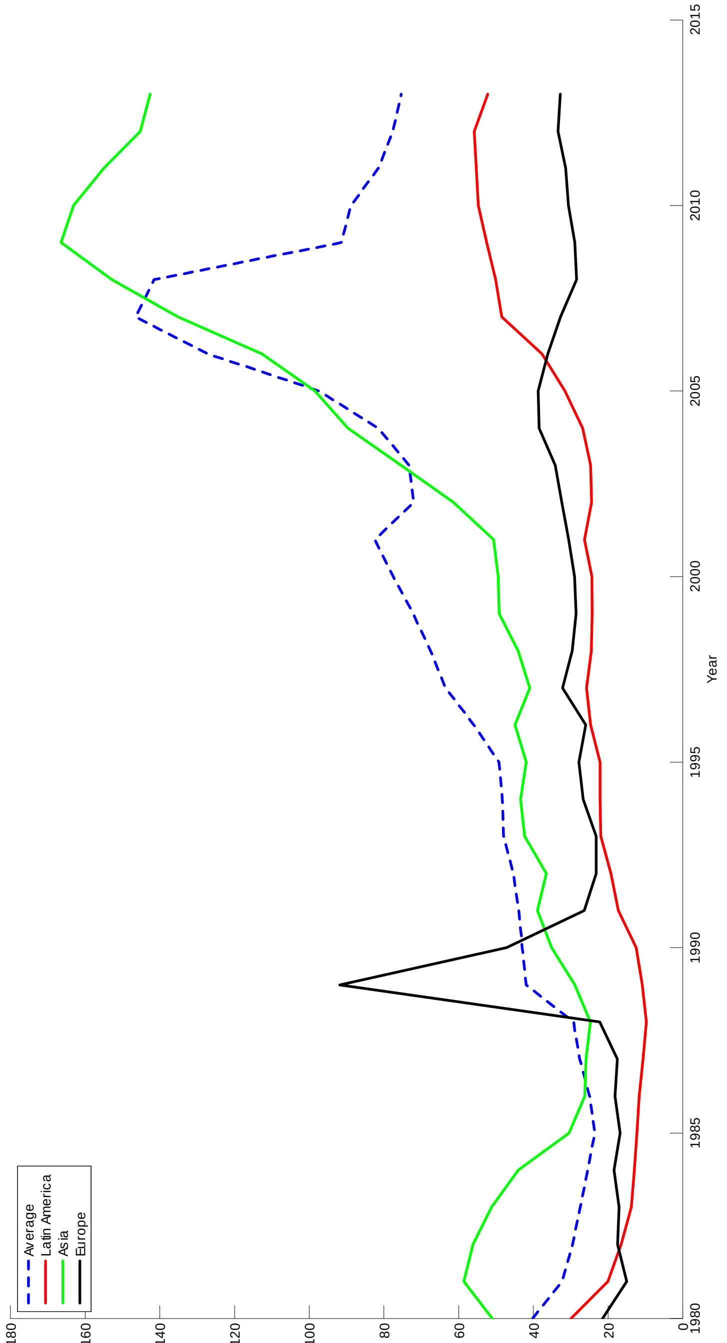

Figure 2: International Reserves as % of Total External Debt by World Regions. *Source:* Data is for our/JR sample of 34 countries listed in Table 2, grouped by continent, from IMF's *International Financial Statistics* and World Bank's *World Development Indicators.*

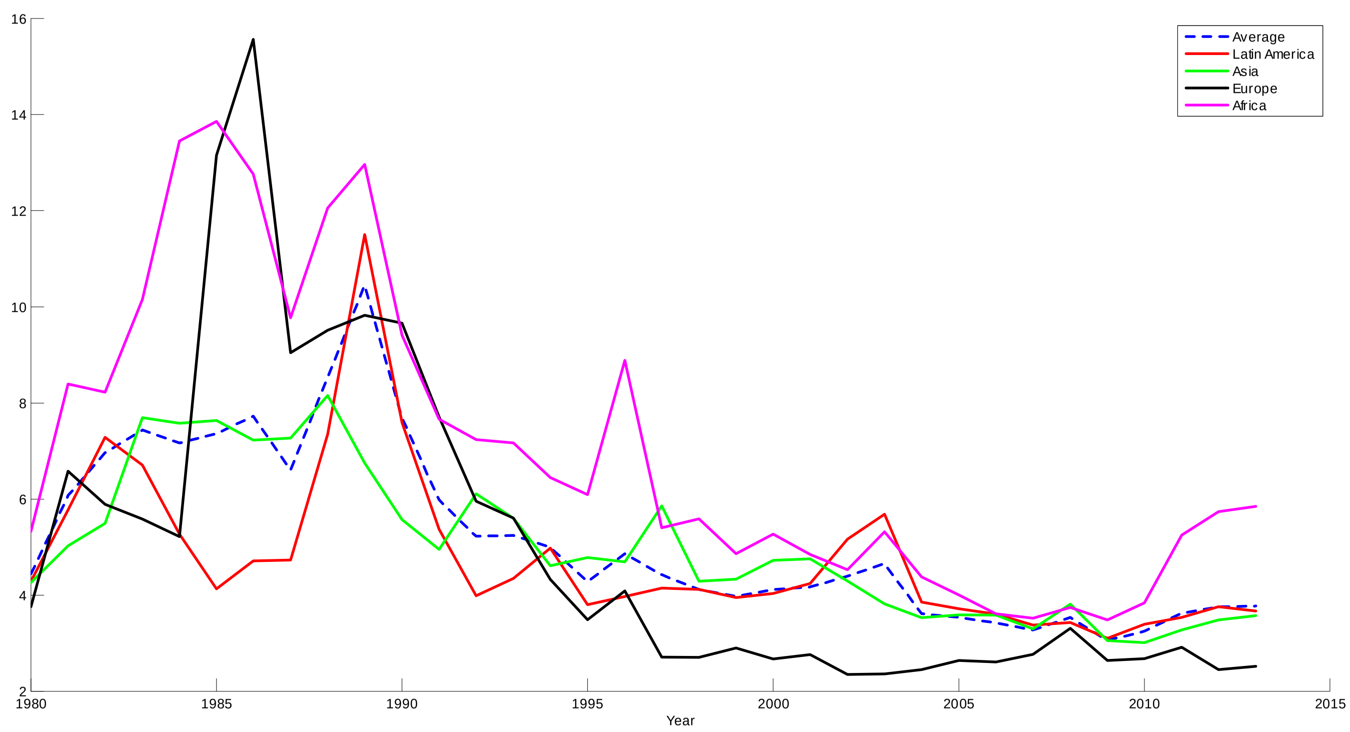

Figure 3: Broad Money as % of International Reserves by World Regions. *Source:* Data and grouping as in Figure 2.

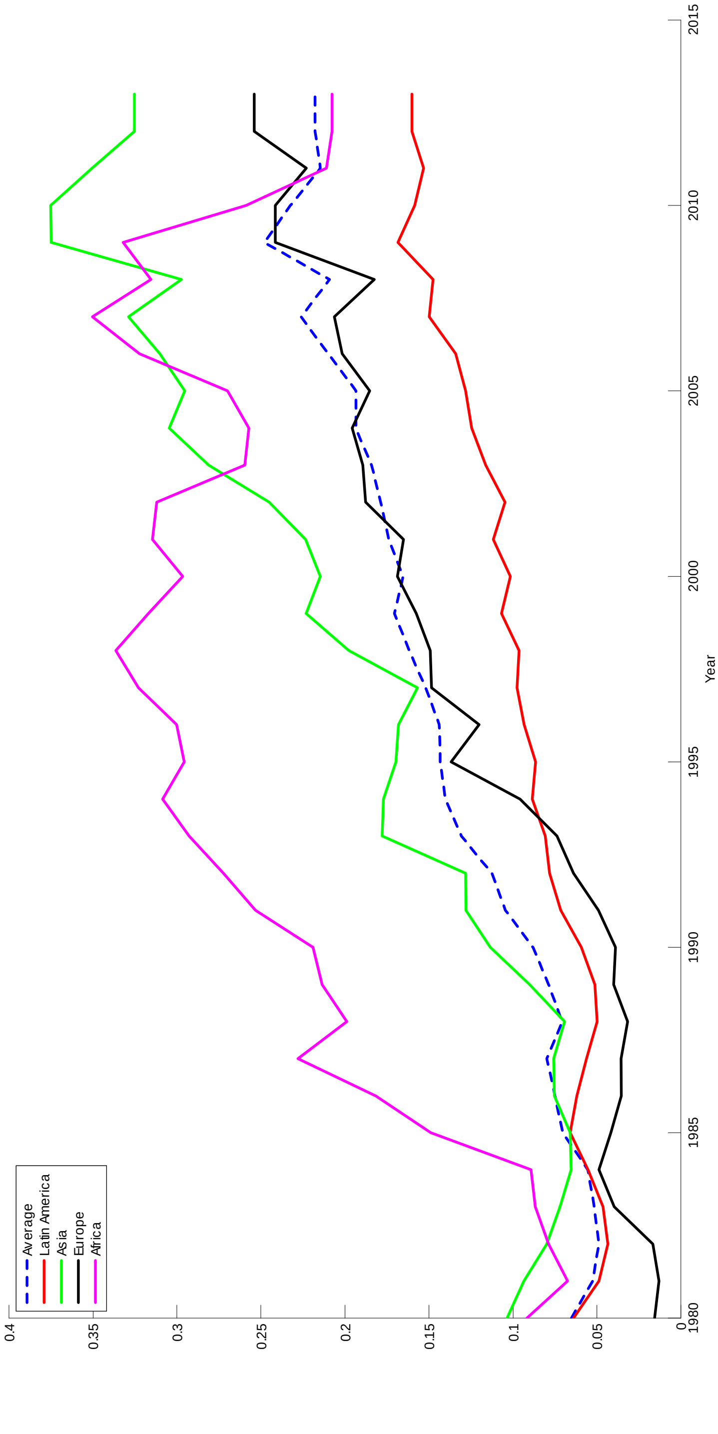

Figure 4: International Reserves as % of GDP by World Regions. *Source:* Data and grouping as in Figure 2.

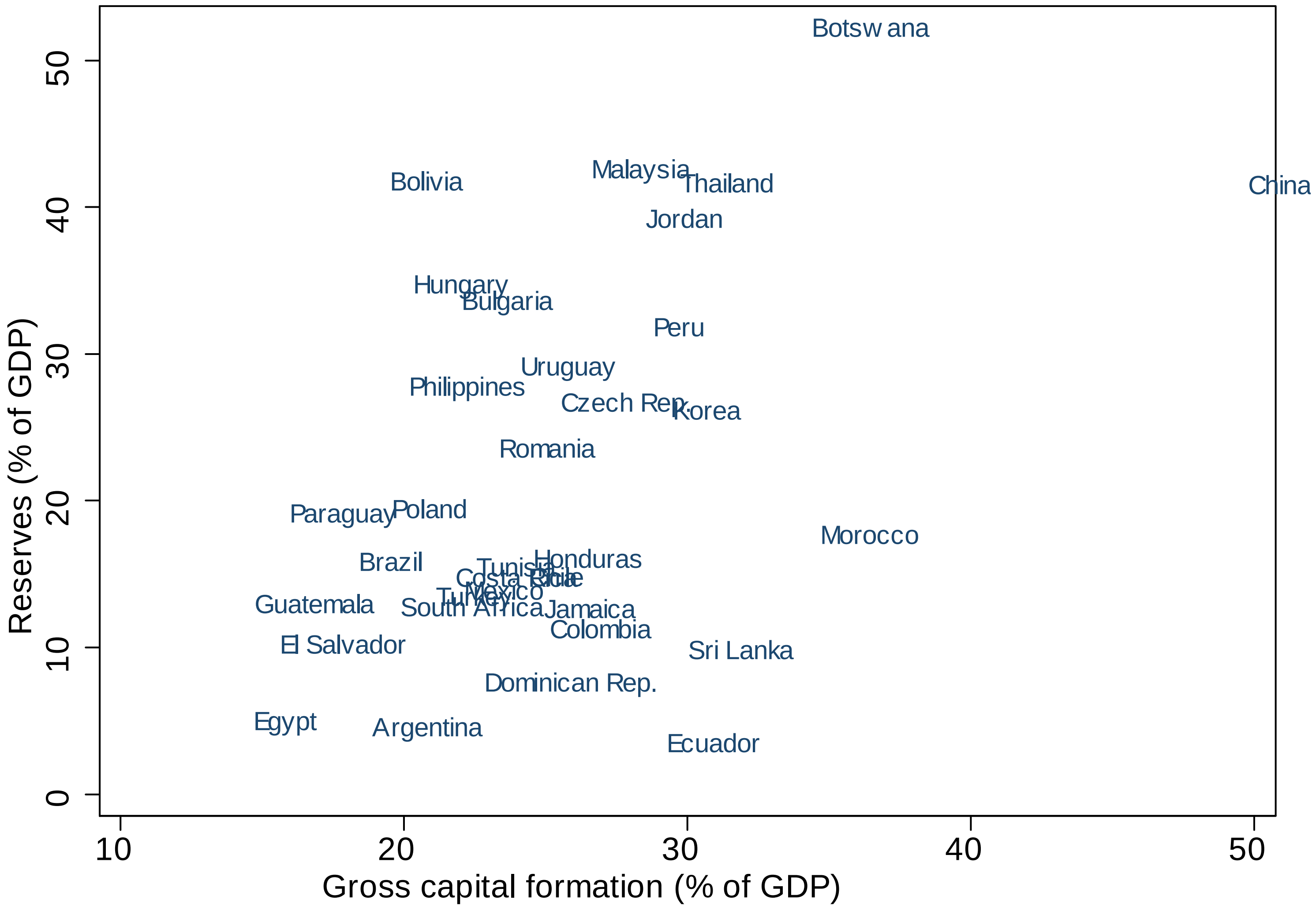

Figure 5: International Reserves and Investment as % of GDP by Middle-Income Country. *Source:* World Bank, *World Development Indicators* (online), and authors' calculations. Data on reserves to GDP ratios and gross capital formation are for 2013 for the 34 middle income countries in the our/JR sample, as listed in Table 2.

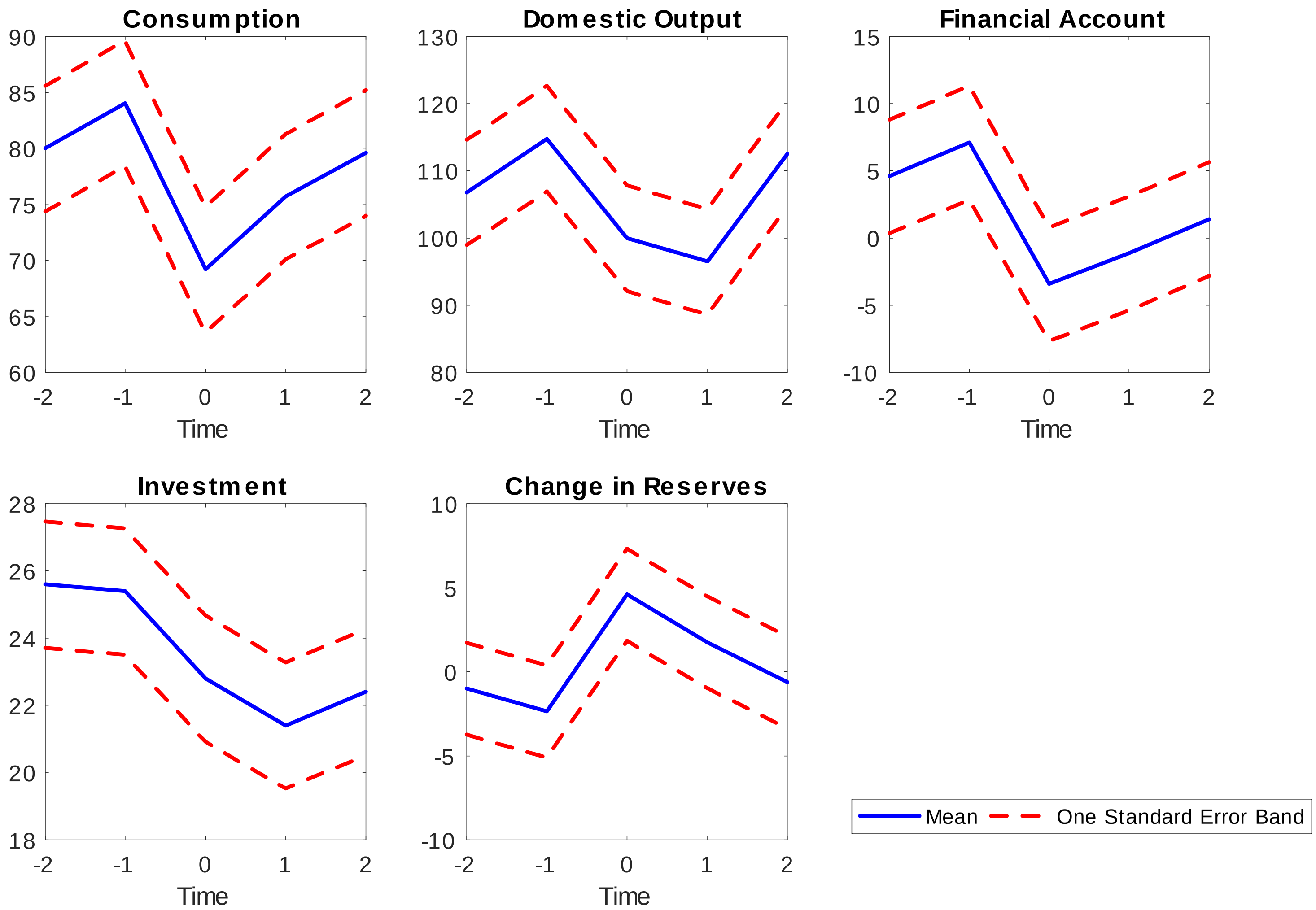

Figure 6: Average Dynamics of Key Model Variables in Sudden Stops, 1980-2014. Output is normalized to 100 in the year of the sudden stop, i.e., time 0. *Source:* Authors' calculations using data from IMF's *International Financial Statistics* and World Bank's *World Development Indicators*.

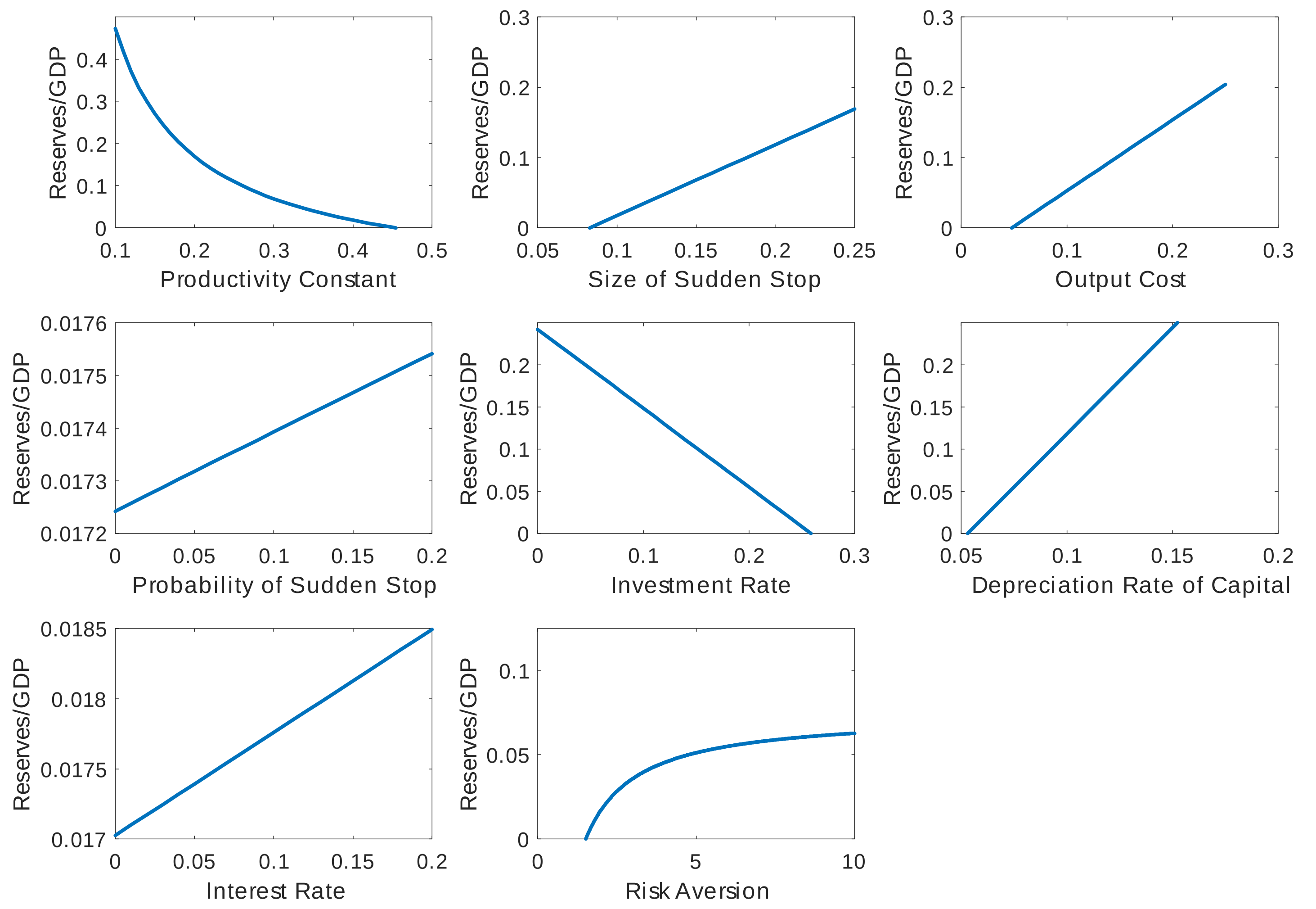

Figure 7: Optimal Reserves-to-GDP Ratio as a Function of Its Key Determinants in the AK SOE Model. *Source:* Authors' calculations using data from IMF's *International Financial Statistics*, Penn World Table 7.0 and World Bank's *World Development Indicators*.

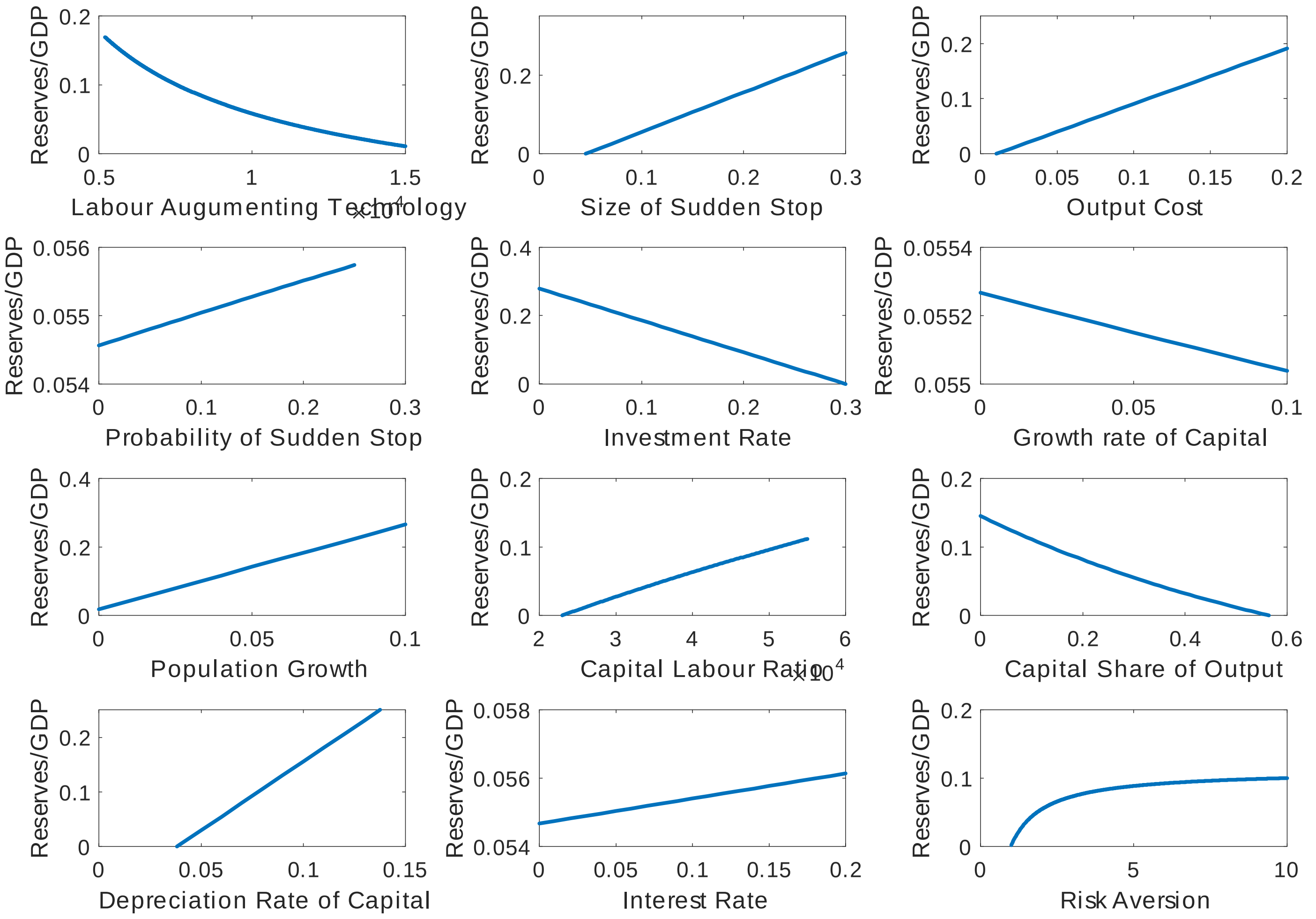

Figure 8: Optimal Reserves-to-GDP Ratio as a Function of Its Key Determinants in the Labour-Augmenting Cobb-Douglas SOE Model.
Source: Authors' calculations using data from IMF's *International Financial Statistics*, Penn World Table 7.0 and World Bank's *World Development Indicators.*

Parameters	AK-Technology	CD-Technology	Range of Variation
Size of a sudden stop	$\lambda = 0.10$	same	$[0,\ 0.3]$
Probability of a sudden stop	$\pi = 0.10$	same	$[0,\ 0.25]$
Output loss	$\gamma = 0.065$	same	$[0,\ 0.2]$
Risk-free world interest rate	$r = 0.05$	same	–
Coefficient of relative risk aversion	$\sigma = 2$	same	$[1,\ 10]$
Price of a non-crisis dollar	$p = 0.885$	same	–
Technology	$A_K = 0.40$	$A_N = 10241$	–
Investmnet rate	$s = 0.24$	same	$[0,\ 0.48]$
Depreciation rate of the capital stock	$\delta = 0.06$	same	$[0,\ 1]$
Output per worker	$y = 15141$	same	–
Capital-labour ratio	$k = 37701$	same	–
Growth rate of the population	$g_N = 0$	$g_N = 0.015$	$[0,\ 0.11]$
Growth rate of the capital stock	$g_K = sA_K - \delta$	$g_K = 0.098$	–
Capital share in output	$\theta = 1$	$\theta = 0.3$	

Source: Authors' calculations using data from IMF's *International Financial Statistics*, Penn World Tables 7.0 and World Bank's *World Development Indicators*.

Table 1: Calibration of Parameters

Country	Years of Sudden Stops
Argentina	1989, 1994, 2001, 2002, 2008
Bolivia	1980, 1982, 1983, 1985, 2000, 2003, 2006
Botswana	1977, 1987, 1993, 2001
Brazil	2008
Bulgaria	1989, 1990, 1994, 1996, 2008
Chile	1982, 1983, 1998, 2007
China	–
Colombia	–
Costa Rica	1981
Czechia	1996, 2003
Dominican Republic	2002
Ecuador	1983, 1999, 2000, 2006
Egypt	1987, 1990, 1999, 2006
El Salvador	2004, 2007
Guatemala	–
Honduras	–
Hungary	1994, 1996
Jamaica	1985, 1986, 2002, 2003
Jordan	1976, 1979, 1980, 1984, 1989, 1992, 1993, 1998, 2001, 2003
Korea	1997, 2008
Malaysia	1987, 1994, 1996, 1997, 1998, 2005, 2008
Mexico	1982, 1995
Morocco	1978, 1995
Paraguay	1985, 1988
Peru	1983, 1998
Philippines	1983, 1997, 1998, 2008
Poland	1994
Romania	2008
South Africa	–
Sri Lanka	–
Thailand	1998, 2007
Tunisia	–
Turkey	1994, 2001
Uruguay	1983, 2002, 2004

Notes: The sample includes countries classified as middle-income by the World Bank, plus 7 high-income countries: Argentina, Chile, Czechia, Hungary, Korea, Poland and Uruguay. A country-year observation is identified as a sudden stop if the ratio of capital inflows to gross domestic product falls by more than 5%. Capital inflows are measured as the current account deficit minus reserves accumuation.
Source: Authors' calculations using data from IMF's *International Financial Statistics* and World Bank's *World Development Indicators.*

Table 2: Countries and Years of Sudden Stops

Supplementary Online Appendix to "Sudden Stops, Productivity and the Optimal Level of International Reserves for Small Open Economies"

Alexander Mihailov* and Harun Nasir†

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Abstract

This supplementary online appendix provides the sources and definitions of our data (in section A) as well as the proofs of our propositions 1 and 2 and corollaries 1 and 2 in the main text, in detailed steps of algebraic derivation (in section B). For replication purposes, a zip file archive is available upon request that contains our dataset, code and the respective input and output files (using the software packages STATA and MATLAB).

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A Data Sources and Definitions

This Appendix A provides details on our data sources and definitions.

Our calculations for the calibrated quantification of the optimal level of international reserves in terms of output are based on data from IMF's *International Financial Statistics* (*IFS*), Penn World Table (PWT) 7.0 and World Bank's *World Development Indicators* (*WDI*), as specified in Table 1.

Variable or parameter	Notation	Data source
International reserves	R	*WDI*
Short-term external debt-to-GDP (size of a sudden stop)	λ	*WDI*
Probability of a sudden stop	π	*WDI*
Output loss in a sudden stop	γ	*WDI*
Capital-labour ratio	k	PWT, 7.0
GDP per worker	y	PWT, 7.0
Investment (and saving) rate	s	PWT, 7.0
AK technology parameter	A_K	PWT, 7.0
Labour-augmenting Cobb-Douglas technology parameter	A_N	PWT, 7.0
Growth rate of output	g_Y	*WDI*
Growth rate of the capital stock	g_K	*WDI* and PWT, 7.0
Growth rate of population	g_N	*WDI*
Growth rate of the capital-labour ratio	g_k	*WDI* and PWT, 7.0
Price of a non-crisis dollar in terms of a crisis dollar	p	Jeanne and Rancière (2011)
Depreciation rate of capital	δ	Caselli (2005)
Risk-free world interest rate	r	Jeanne and Rancière (2011)
Coefficient of relative risk aversion	σ	Jeanne and Rancière (2011)
Capital share in output	θ	Gourinchas and Jeanne (2013)
Current account deficit	CA	*WDI*

Table 1: List of Variables with Their Notation and Data Source

B Technical Appendix: Derivations and Proofs

This appendix B provides further technical details on some derivations and the proofs of propositions 1 and 2 and corollaries 1 and 2 in the main text.

More Detailed Steps in the Optimization Problem of the SOE Government

The foreign insurers' contract is given by

$$\sum_{t=1}^{+\infty} \beta^t (1-\pi)^{t-1} \left[(1-\pi) X_t \mu_t^n - \pi (R_t - X_t) \mu_t^s\right] \geq 0,$$

and if it is binding every period t, as assumed, it holds with equality so that one can write

$$(1-\pi) X_t \mu_t^n = \pi (R_t - X_t) \mu_t^s$$

$$p = \frac{\mu_t^n}{\mu_t^s} = \frac{\pi (R_t - X_t)}{(1-\pi) X_t}$$

$$p = \frac{\pi (R_t - X_t)}{(1-\pi) X_t}$$

$$\frac{p(1-\pi)}{\pi} = \frac{R_t - X_t}{X_t}$$

$$\frac{p(1-\pi)}{\pi} = \frac{R_t}{X_t} - 1$$

$$\frac{R_t}{X_t} = \frac{p(1-\pi)}{\pi} + 1$$

$$R_t = X_t \frac{p(1-\pi)+\pi}{\pi}$$

$$X_t = \frac{\pi}{p(1-\pi)+\pi} R_t.$$

The SOE government chooses the paths $(X_t, R_t)_{t=1,\ldots,+\infty}$ so as to maximize domestic welfare (1) in the main text subject to the budget constraints (3), (21), (22), the binding credit constraint (14) and the foreign insurers' participation constraint (25). The Lagrangian function for the constrained optimization problem can then be written as:

$$\mathcal{L} = \sum_{t=1}^{+\infty} \beta^t (1-\pi)^t \left\{(1-\pi) u(C_t^n) + \pi u(C_t^s) + v\left[(1-\pi) X_t \mu_t^n - \pi (R_t - X_t) \mu_t^s\right]\right\},$$

where v is the shadow cost of constraint (25). The first-order necessary conditions imply

$$u'\left(C_t^n\right) = pu'\left(C_t^s\right)$$

$$p = \frac{u'\left(C_t^n\right)}{u'\left(C_t^s\right)}.$$

With the assumed isoelastic, or CRRA, period utility,

$$u\left(C_t\right) = \frac{C_t^{1-\sigma}}{1-\sigma},$$

we further obtain

$$p = \frac{\left(C_t^n\right)^{-\sigma}}{\left(C_t^s\right)^{-\sigma}} = \left(\frac{C_t^n}{C_t^s}\right)^{-\sigma}$$

and finally

$$p^{\frac{1}{\sigma}} C_t^n = C_t^s. \tag{1}$$

B.1 Proof of Proposition 1 and Corollary 1: AK Technology

Proof of Proposition 1.

With the AK technology, output is given by

$$Y_t = F\left(K_t; A_K\right) = A_K K_t.$$

The 'normal time' budget constraint (superscript n) of the private sector is given by

$$C_t^n = Y_t^n\left(\cdot\right) - I_t^n + L_t^n - \left(1+r\right) L_{t-1}^n + Z_t^n$$

and investment by

$$I_t^n = sY_t^n\left(\cdot\right),$$

with capital accumulation written as

$$K_{t+1}^n : K_{t+1} = I_t^n + \left(1-\delta\right) K_t.$$

Using the AK technology to replace $Y_t^n\left(\cdot\right) = F\left(K_t; A_K\right) = A_K K_t$ in the assumption for normal-time investment above, we obtain

$$K_{t+1} = sA_K K_t + \left(1-\delta\right) K_t.$$

Then, the gross growth rate of the capital stock is, as standard in neoclassical growth theory,

$$1 + g_K = \frac{K_{t+1}}{K_t} = 1 + sA_K - \delta.$$

And, therefore, this is also the gross growth rate of the output in the AK model.

Note that the short-term external debt (STED) ratio to output remains constant as in JR but is now given by

$$\lambda \equiv \frac{1 + (sA_K - \delta)}{1 + r}\ \alpha.$$

Replacing investment, we can write the normal-time budget constraint of the private sector as

$$C_t^n = Y_t^n(\cdot) - sY_t^n(\cdot) + L_t^n - (1 + r)\, L_{t-1}^n + Z_t^n.$$

With AK technology, output $Y_t^n = F(K_t; A_K) = A_K K_t$ and using $L_t^n = \frac{1+(sA_K-\delta)}{1+r}\alpha Y_t^n(\cdot) = \lambda Y_t^n(\cdot)$ and $Z_t^n = -X_t$, we further obtain (successively):

$$C_t^n = A_K K_t - sA_K K_t + \frac{1 + sA_K - \delta}{1 + r}\alpha A_K K_t - (1 + sA_K - \delta)\alpha A_K K_{t-1} - X_t$$

$$C_t^n = A_K K_t - sA_K K_t + \lambda A_K K_t - \alpha A_K K_t - X_t$$

$$C_t^n = A_K K_t - sA_K K_t + \lambda A_K K_t - \lambda \frac{1 + r}{1 + sA_K - \delta} A_K K_t - X_t$$

$$C_t^n = \left[1 - s + \lambda\left(1 - \frac{1 + r}{1 + sA_K - \delta}\right)\right] A_K K_t - X_t$$

$$C_t^n = \left[1 - s + \lambda \frac{(sA_K - \delta) - r}{1 + (sA_K - \delta)}\right] A_K K_t - X_t$$

$$C_t^n = \left[1 - s - \lambda \frac{r - (sA_K - \delta)}{1 + (sA_K - \delta)}\right] A_K K_t - X_t.$$

The 'sudden stop' budget constraint (superscript s) of the private sector is given by

$$C_t^s = (1 - \gamma)\, Y_t^n(\cdot) - I_t^s - (1 + r)\, L_{t-1}^s + Z_t^s,$$

and we assume that the capital stock does not grow in sudden stops, $g_K = 0$, so that

$$K_{t+1}^s : K_{t+1} = K_t,$$

which implies, in the AK model,

$$1 + g_K = \frac{K_{t+1}}{K_t} = 1 + sA_K - \delta = 1$$

so that

$$sA_K = \delta$$

and

$$I_t^s = \delta K_t.$$

Note that in the AK model capital grows only if $sA_K > \delta$.

Replacing sudden-stop investment, we can also write the above budget constraint as

$$C_t^s = (1-\gamma)\, Y_t^n(\cdot) - \delta K_t + L_t^s - (1+r)\, L_{t-1}^s + Z_t^s.$$

With AK technology, output is given by $Y_t = F(K_t; A_K) = A_K K_t$, and using $L_t^s = 0$, $L_{t-1}^s = \frac{1+(sA_K-\delta)}{1+r}\alpha Y_{t-1}^n(\cdot) = \lambda Y_{t-1}^n(\cdot)$ and $Z_t^s = R_t - X_t$, we further obtain (successively):

$$C_t^s = (1-\gamma)\, A_K K_t - \delta K_t - (1 + sA_K - \delta)\, \alpha A_K K_{t-1} + R_t - X_t$$

$$C_t^s = (1-\gamma)\, A_K K_t - \delta K_t - \alpha A_K K_t + R_t - X_t$$

$$C_t^s = (1-\gamma)\, A_K K_t - \delta K_t - \frac{1+r}{1+sA_K-\delta}\lambda A_K K_t + R_t - X_t \tag{2}$$

$$C_t^s = \left[-\gamma - \frac{\delta}{A_K} + 1 - \frac{1+r}{1+sA_K-\delta}\lambda\right] A_K K_t + R_t - X_t$$

$$C_t^s = \left[-\gamma - \frac{\delta}{A_K} + \frac{1+sA_K-\delta-1-r}{1+sA_K-\delta}\lambda\right] A_K K_t + R_t - X_t$$

$$C_t^s = \left[-\gamma - \frac{\delta}{A_K} + \frac{(sA_K-\delta)-r}{1+(sA_K-\delta)}\lambda\right] A_K K_t + R_t - X_t$$

$$C_t^s = \left[-\gamma - \frac{\delta}{A_K} - \lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right] A_K K_t + R_t - X_t.$$

Therefore, from the first-order condition (1) above, the optimal level of reserves as a ratio to output can be expressed as:

$$p^{\frac{1}{\sigma}}\left\{\left[1 - s - \lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right] A_K K_t - X_t\right\} = \left[-\gamma - \frac{\delta}{A_K} - \lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right] A_K K_t + R_t - X_t$$

$$p^{\frac{1}{\sigma}}\left[1-s-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]A_KK_t-p^{\frac{1}{\sigma}}X_t=\left[-\gamma-\frac{\delta}{A_K}-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]A_KK_t+R_t-X_t$$

$$p^{\frac{1}{\sigma}}\left[1-s-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]A_KK_t-\left[-\gamma-\frac{\delta}{A_K}-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]A_KK_t=R_t-X_t+p^{\frac{1}{\sigma}}X_t$$

$$\left[p^{\frac{1}{\sigma}}(1-s)-p^{\frac{1}{\sigma}}\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}+\gamma+\frac{\delta}{A_K}+\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]A_KK_t=R_t-\left(1-p^{\frac{1}{\sigma}}\right)X_t$$

$$\begin{aligned}&\left[p^{\frac{1}{\sigma}}(1-s)+\left(1-p^{\frac{1}{\sigma}}\right)\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}+\gamma+\frac{\delta}{A_K}\right]A_KK_t\\=\;&R_t-\left(1-p^{\frac{1}{\sigma}}\right)\frac{\pi}{\pi+p(1-\pi)}R_t\end{aligned}$$

$$\begin{aligned}&\left[p^{\frac{1}{\sigma}}(1-s)+\left(1-p^{\frac{1}{\sigma}}\right)\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}+\gamma+\frac{\delta}{A_K}\right]A_KK_t\\=\;&\left[1-\left(1-p^{\frac{1}{\sigma}}\right)\frac{\pi}{\pi+p(1-\pi)}\right]R_t\end{aligned}$$

And, finally, we obtain the optimal reserves-to-output ratio under the AK technology case, ρ^*_{AK}, in the production SOE we analyzed:

$$\rho^*_{AK}=\frac{R_t}{A_KK_t}=\frac{\gamma+\frac{\delta}{A_K}+p^{\frac{1}{\sigma}}(1-s)+\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\left(1-p^{\frac{1}{\sigma}}\right)}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}. \tag{3}$$

An alternative equivalent expression where the parameter λ appears also additively, as in the original JR optimal reserves formula, can be obtained as follows.
Since, from (2) above,

$$C_t^s=(1-\gamma)A_KK_t-\delta K_t-\frac{1+r}{1+sA_K-\delta}\lambda A_KK_t+R_t-X_t$$

then

$$C_t^s=\left[(1-\gamma)-\frac{\delta}{A_K}-\frac{1+r}{1+sA_K-\delta}\lambda\right]A_KK_t+R_t-X_t$$

$$C_t^s=\left[(1-\gamma)-\frac{\delta}{A_K}-\frac{1+r+[1+(sA_K-\delta)-1-(sA_K-\delta)]}{1+sA_K-\delta}\lambda\right]A_KK_t+R_t-X_t$$

$$C_t^s = \left[(1-\gamma) - \frac{\delta}{A_K} - \frac{1+(sA_K-\delta)}{1+(sA_K-\delta)}\lambda - \frac{1+r-1-(sA_K-\delta)}{1+(sA_K-\delta)}\lambda\right] A_K K_t + R_t - X_t$$

$$C_t^s = \left[(1-\gamma) - \frac{\delta}{A_K} - \lambda - \frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\lambda\right] A_K K_t + R_t - X_t.$$

Now, the optimal reserves-to-output formula can be obtained, in an alternative version to (3) above, as below:

$$\begin{aligned} & p^{\frac{1}{\sigma}}\left\{\left[1-s-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right] A_K K_t - X_t\right\} \\ = \; & \left[(1-\gamma) - \frac{\delta}{A_K} - \lambda - \frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\lambda\right] A_K K_t + R_t - X_t \end{aligned}$$

$$\begin{aligned} & p^{\frac{1}{\sigma}}\left[1-s-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right] A_K K_t - p^{\frac{1}{\sigma}} X_t \\ = \; & \left[(1-\gamma) - \frac{\delta}{A_K} - \lambda - \frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\lambda\right] A_K K_t + R_t - X_t \end{aligned}$$

$$\begin{aligned} & p^{\frac{1}{\sigma}}\left[1-s-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right] A_K K_t - \left[(1-\gamma) - \frac{\delta}{A_K} - \lambda - \frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\lambda\right] A_K K_t \\ = \; & R_t - X_t + p^{\frac{1}{\sigma}} X_t \end{aligned}$$

$$\begin{aligned} & \left[p^{\frac{1}{\sigma}}(1-s) - p^{\frac{1}{\sigma}}\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)} - (1-\gamma) + \lambda + \frac{\delta}{A_K} + \lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right] A_K K_t \\ = \; & R_t - \left(1-p^{\frac{1}{\sigma}}\right) X_t \end{aligned}$$

$$\begin{aligned} & \left[p^{\frac{1}{\sigma}} - p^{\frac{1}{\sigma}} s + \left(1-p^{\frac{1}{\sigma}}\right)\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)} - 1 + \gamma + \lambda + \frac{\delta}{A_K}\right] A_K K_t \\ = \; & R_t - \left(1-p^{\frac{1}{\sigma}}\right)\frac{\pi}{\pi+p(1-\pi)} R_t \end{aligned}$$

$$\begin{aligned} & \left[-\left(1-p^{\frac{1}{\sigma}}\right) - p^{\frac{1}{\sigma}} s + \left(1-p^{\frac{1}{\sigma}}\right)\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)} + \gamma + \lambda + \frac{\delta}{A_K}\right] A_K K_t \\ = \; & R_t - \left(1-p^{\frac{1}{\sigma}}\right)\frac{\pi}{\pi+p(1-\pi)} R_t \end{aligned}$$

$$\begin{aligned}&\left[\gamma+\lambda+\frac{\delta}{A_K}-p^{\frac{1}{\sigma}}s-\left(1-p^{\frac{1}{\sigma}}\right)\left(1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right)\right]A_KK_t\\ =\ &\left[1-\left(1-p^{\frac{1}{\sigma}}\right)\frac{\pi}{\pi+p(1-\pi)}\right]R_t\end{aligned}$$

$$\rho^*_{AK}=\frac{R_t}{A_KK_t}=\frac{\gamma+\lambda-\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]+\frac{\delta}{A_K}-p^{\frac{1}{\sigma}}s}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)},$$

which is equation (31) in Proposition 1 in the main text. $sA_K>\delta$ is the condition for the AK economy to increase its capital stock, and hence to grow over time. Note that $(sA_K-\delta)>0$ and $r-(sA_K-\delta)>0$ by assumption (as in JR) – and, then $0<\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}<1$ so that $\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]>0$ and $\rho^*_{AK}>0$ as long as $\gamma+\lambda+\frac{\delta}{A_K}>\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]+p^{\frac{1}{\sigma}}s.$

This completes our proof. ■

Proof of Corollary 1.

Re-writing our final expression above, i.e., equation (31) in the main text,

$$\rho^*_{AK}=\frac{\gamma+\lambda-\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]+\frac{\delta}{A_K}-p^{\frac{1}{\sigma}}s}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)},$$

one could also represent ρ^*_{AK} in relative terms to $\gamma+\lambda$, as in JR. We do so next, presenting the detailed steps, as follows.

$$\rho^*_{AK}\left[1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]=\gamma+\lambda-\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{1-(sA_K-\delta)}{1+(sA_K-\delta)}\right]+\frac{\delta}{A_K}-p^{\frac{1}{\sigma}}s$$

$$\rho^*_{AK}-\rho^*_{AK}\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)=\gamma+\lambda-\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]+\frac{\delta}{A_K}-p^{\frac{1}{\sigma}}s$$

$$\gamma+\lambda-\rho^*_{AK}=\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]-\frac{\delta}{A_K}+p^{\frac{1}{\sigma}}s-\rho^*_{AK}\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)$$

$$\begin{aligned}\gamma+\lambda-\rho^*_{AK}=&\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]-\frac{\delta}{A_K}+p^{\frac{1}{\sigma}}s\\ &-\frac{\gamma+\lambda-\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]+\frac{\delta}{A_K}-p^{\frac{1}{\sigma}}s}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\end{aligned}$$

$$\gamma+\lambda-\rho_{AK}^{*}=\frac{\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]-\frac{\delta}{A_K}+p^{\frac{1}{\sigma}}s}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}\left[1-\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)\right]$$
$$-\frac{\gamma+\lambda-\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]+\frac{\delta}{A_K}-p^{\frac{1}{\sigma}}s}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)$$

$$\gamma+\lambda-\rho_{AK}^{*}$$
$$=\frac{\begin{array}{c}\left\{\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]-\frac{\delta}{A_K}+p^{\frac{1}{\sigma}}s\right\}\left[1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]\\-\left\{\gamma+\lambda-\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]+\frac{\delta}{A_K}-p^{\frac{1}{\sigma}}s\right\}\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\end{array}}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}$$

$$=\frac{\left[\left(1-p^{\frac{1}{\sigma}}\right)-\left(1-p^{\frac{1}{\sigma}}\right)\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}-\frac{\delta}{A_K}+p^{\frac{1}{\sigma}}s\right]\left[1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}$$
$$-\frac{(\gamma+\lambda)\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)-\left\{\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]+\frac{\delta}{A_K}-p^{\frac{1}{\sigma}}s\right\}\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}$$

$$=\frac{\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]-\left[\left(1-p^{\frac{1}{\sigma}}\right)\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}+\frac{\delta}{A_K}-p^{\frac{1}{\sigma}}s\right]\left[1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}$$
$$-\frac{(\gamma+\lambda)\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)-\left\{\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]+\frac{\delta}{A_K}-p^{\frac{1}{\sigma}}s\right\}\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}$$

Simplify the numerator above, as below:

$$\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]-\left(1-p^{\frac{1}{\sigma}}\right)\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\left[1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]$$
$$-\frac{\delta}{A_K}+p^{\frac{1}{\sigma}}s+\frac{\delta}{A_K}\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)-p^{\frac{1}{\sigma}}s\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)-(\gamma+\lambda)\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)$$
$$+\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-\left(sA_K-\delta\right)}{1+\left(sA_K-\delta\right)}\right]\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)$$
$$-\frac{\delta}{A_K}\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)+p^{\frac{1}{\sigma}}s\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)$$

$$\begin{aligned}
= & \left(1-p^{\frac{1}{\sigma}}\right)\left[1-\tfrac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]-\left(1-p^{\frac{1}{\sigma}}\right)\lambda\tfrac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\left[1-\tfrac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right] \\
& -\frac{\delta}{A_K}+p^{\frac{1}{\sigma}}s-(\gamma+\lambda)\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right) \\
& +\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)
\end{aligned}$$

Using the simplification above, re-write the full equation:

$$\begin{aligned}
& \gamma+\lambda-\rho^*_{AK}= \\
& \frac{\left(1-p^{\frac{1}{\sigma}}\right)\left\{\begin{array}{c} 1-\tfrac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)-\lambda\tfrac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\left[1-\tfrac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right] \\ -(\gamma+\lambda)\tfrac{\pi}{\pi+p(1-\pi)}+\left[1-\lambda\tfrac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]\tfrac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right) \end{array}\right\}}{1-\tfrac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)} \\
& +\frac{-\tfrac{\delta}{A_K}+p^{\frac{1}{\sigma}}s}{1-\tfrac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}
\end{aligned}$$

Now simplify the expression in the curly brackets above, as below:

$$\begin{aligned}
& 1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\left[1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right] \\
& -(\gamma+\lambda)\frac{\pi}{\pi+p(1-\pi)}+\left[1-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right]\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)
\end{aligned}$$

$$\begin{aligned}
= \;& 1-(\gamma+\lambda)\frac{\pi}{\pi+p(1-\pi)}-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right) \\
& -\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\left[1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right] \\
& +\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)
\end{aligned}$$

$$\begin{aligned}
= \;& 1-(\gamma+\lambda)\frac{\pi}{\pi+p(1-\pi)}-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\left[1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right] \\
& -\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)
\end{aligned}$$

$$\begin{aligned}
= \;& 1-(\gamma+\lambda)\frac{\pi}{\pi+p(1-\pi)}-\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}+\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right) \\
& -\lambda\frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)
\end{aligned}$$

$$= 1 - (\gamma + \lambda) \frac{\pi}{\pi + p(1-\pi)} - \lambda \frac{r - (sA_K - \delta)}{1 + (sA_K - \delta)}$$

Using the simplification above, re-write the full equation:

$$\gamma + \lambda - \rho_{AK}^* = \frac{\left(1 - p^{\frac{1}{\sigma}}\right)\left[1 - (\gamma + \lambda)\frac{\pi}{\pi + p(1-\pi)} - \lambda \frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right] - \frac{\delta}{A_K} + p^{\frac{1}{\sigma}} s}{1 - \frac{\pi}{\pi+p(1-\pi)}\left(1 - p^{\frac{1}{\sigma}}\right)}$$

$$\gamma + \lambda - \rho_{AK}^* = \frac{\left(1 - p^{\frac{1}{\sigma}}\right)\left[1 - (\gamma + \lambda)\frac{\pi}{\pi + p(1-\pi)} - \lambda \frac{r-(sA_K-\delta)}{1+(sA_K-\delta)} - (\lambda + \gamma) + (\lambda + \gamma)\right] - \frac{\delta}{A_K} + p^{\frac{1}{\sigma}} s}{1 - \frac{\pi}{\pi+p(1-\pi)}\left(1 - p^{\frac{1}{\sigma}}\right)}$$

$$\gamma + \lambda - \rho_{AK}^* = \frac{\left(1 - p^{\frac{1}{\sigma}}\right)\left[1 + (\gamma + \lambda)\left(1 - \frac{\pi}{\pi + p(1-\pi)}\right) - \lambda \left(1 + \frac{r-(sA_K-\delta)}{1+(sA_K-\delta)}\right) - \gamma\right] - \frac{\delta}{A_K} + p^{\frac{1}{\sigma}} s}{1 - \frac{\pi}{\pi+p(1-\pi)}\left(1 - p^{\frac{1}{\sigma}}\right)}$$

$$\gamma + \lambda - \rho_{AK}^* = \frac{\left(1 - p^{\frac{1}{\sigma}}\right)\left[1 + (\gamma + \lambda)\frac{p(1-\pi)}{\pi + p(1-\pi)} - \lambda \frac{1+(sA_K-\delta)+r-(sA_K-\delta)}{1+(sA_K-\delta)} - \gamma\right] - \frac{\delta}{A_K} + p^{\frac{1}{\sigma}} s}{1 - \frac{\pi}{\pi+p(1-\pi)}\left(1 - p^{\frac{1}{\sigma}}\right)}$$

$$\gamma + \lambda - \rho_{AK}^* = \frac{\left(1 - p^{\frac{1}{\sigma}}\right)\left[1 + (\gamma + \lambda)\frac{p(1-\pi)}{\pi + p(1-\pi)} - \alpha - \gamma\right] - \frac{\delta}{A_K} + p^{\frac{1}{\sigma}} s}{1 - \frac{\pi}{\pi+p(1-\pi)}\left(1 - p^{\frac{1}{\sigma}}\right)}$$

$$\gamma + \lambda - \rho_{AK}^* = \frac{\left(1 - p^{\frac{1}{\sigma}}\right)\left[1 - \alpha - \gamma + (\gamma + \lambda)\frac{p(1-\pi)}{\pi + p(1-\pi)}\right] - \frac{\delta}{A_K} + p^{\frac{1}{\sigma}} s}{1 - \frac{\pi}{\pi+p(1-\pi)}\left(1 - p^{\frac{1}{\sigma}}\right)},$$

which is equation (32) in Corollary 1 in the main text.

This completes our proof. ■

B.2 Proof of Proposition 2 and Corollary 2: Cobb-Douglas Labour-Augmenting Technology

Proof of Proposition 2.

With the CD labour-augmenting technology, output is given by

$$Y_t = F(K_t, A_N N_t; \theta) = K_t^{\theta}(A_N N_t)^{1-\theta},$$

which can be written alternatively in terms of output per capita (or output per worker, provided the usual implicit assumption in neoclassical growth theory that the labour force participation rate is constant and unitary), $y_t \equiv \frac{Y_t}{N_t}$, and capital per worker or the capital-labour ratio, $k_t \equiv \frac{K_t}{N_t}$:

$$y_t = f(k_t, A_N; \theta) = A_N{}^{1-\theta} \frac{K_t^\theta}{N_t^\theta} = A_N{}^{1-\theta} k_t^\theta.$$

In this model version, exogenous constant population growth at net rate g_N is assumed:

$$\frac{\triangle N_{t+1}}{N_t} = g_N \text{ and } N_t = (1+g_N)^t N_0,$$

where N_0 is the population level in some base period.
To express the production function in % terms, that is, in growth rates, take natural logarithms from both sides in period t and $t-1$, and then subtract to form the respective first log-differences:

$$\ln Y_t = (1-\theta)\ln A_N + (1-\theta)\ln N_t + \theta \ln K_t$$

$$\ln Y_{t-1} = (1-\theta)\ln A_N + (1-\theta)\ln N_{t-1} + \theta \ln K_{t-1}$$

$$\ln Y_t - \ln Y_{t-1} = (1-\theta)(\ln N_t - \ln N_{t-1}) + \theta(\ln K_t - \ln K_{t-1})$$

$$d\ln Y_t = (1-\theta)\, d\ln N_t + \theta d\ln K_t$$

$$g_Y = (1-\theta)\, g_N + \theta g_K,$$

which decomposes the growth rate of output, g_Y, as a weighted average of the growth rates of the population, g_N, and the capital stock, g_K, with the weights defined by the respective contributions of the two productive factors used as inputs, θ for capital and $1-\theta$ for labour, to final output. The last equation could also be written, equivalently, as

$$g_Y = (1-\theta)\, g_N + \theta\left(s\frac{Y_t}{K_t} - \delta\right).$$

Along a *balanced* growth path (BGP) for the population and the capital stock, defined as standard by
$g_N = g_K$ so that $g_Y = g_N = g_K$ too, and thus
$k_t = \frac{K_t}{N_t} = k = const$, hence, $k_t = k$ is a steady state (SS) for k_t
or, equivalently, $g_k = 0$,

$$g_k = \frac{\triangle k_{t+1}}{k_t} = \frac{\triangle K_{t+1}}{K_t} - \frac{\triangle N_{t+1}}{N_t} = s\frac{y_t}{k_t} - \delta - g_N = s\frac{Y_t}{K_t} - \delta - g_N = 0,$$

which – according to neoclassical growth theory (see, e.g., Jones and Vollrath, 2013, p. 28) – implies capital *widening*: namely, capital per worker does not change, $g_k = 0$, but the capital stock grows at the *same* rate as the population,

$$g_K = s\frac{y_t}{k_t} - \delta = s\frac{Y_t}{K_t} - \delta = g_N.$$

If – by contrast – we allow for growth in the capital-labour ratio, $g_k = \frac{\triangle k_{t+1}}{k_t} > 0$, denoted in the literature as capital *deepening*, then

$$g_k = \frac{\triangle k_{t+1}}{k_t} = \frac{\triangle K_{t+1}}{K_t} - \frac{\triangle N_{t+1}}{N_t} = s\frac{y_t}{k_t} - \delta - g_N > 0.$$

Note that

$$g_k = \frac{\triangle k_{t+1}}{k_t} = \frac{\triangle K_{t+1}}{K_t} - \frac{\triangle N_{t+1}}{N_t} = g_K - g_N,$$

so that

$$g_K = g_k + g_N = s\frac{Y_t}{K_t} - \delta - g_N + g_N = s\frac{Y_t}{K_t} - \delta > g_N.$$

In the *normal* state, we assume that g_K is (minimally/marginally) higher than g_N so that g_k grows but (very-very) slowly (and therefore the capital-labour ratio does not explode in a longer-run perspective – due to the deterministic nature of the model in this simplest version). That is:

$$g_k = \frac{\triangle k_{t+1}}{k_t} > 0;$$

then, as shown above,

$$g_Y = dlnY_t = (1-\theta)\, g_N + \theta g_K$$

or, equivalently,

$$g_Y = dlnY_t = (1-\theta)\, g_N + \theta \left(s\frac{Y_t}{K_t} - \delta \right).$$

As assumed (in the main text), domestic private sector saving occurs through investment in physical capital and is a constant fraction of output in the normal state:

$$s = \frac{S_t}{Y_t} = \frac{I_t}{Y_t}.$$

The 'normal time' budget constraint (superscript n) of the private sector is given by

$$C_t^n = Y_t^n\,(\cdot) - I_t^n + L_t^n - (1+r)\, L_{t-1}^n + Z_t^n,$$

and

$$I_t^n = sY_t^n\,(\cdot)\,.$$

The rate of growth of capital per capita, $k_t = \frac{K_t}{N_t}$, is then

$$g_k = \frac{\triangle k_{t+1}}{k_t} = \frac{\triangle K_{t+1}}{K_t} - \frac{\triangle N_{t+1}}{N_t} = s\frac{y_t}{k_t} - \delta - g_N,$$

hence

$$\frac{y_t}{k_t} = \frac{\frac{Y_t}{N_t}}{\frac{K_t}{N_t}} = \frac{Y_t}{K_t} = \frac{g_k + \delta + g_N}{s}.$$

From the last equality above, investment in normal times is

$$sY_t = (g_k + \delta + g_N)K_t.$$

Note that the definition of STED now implies

$$\lambda \equiv \frac{1 + [(1-\theta)\, g_N + \theta g_K]}{1+r}\ \alpha.$$

Replacing investment, we can write the normal-time budget constraint of the private sector as

$$C_t^n = Y_t^n(\cdot) - sY_t^n(\cdot) + L_t^n - (1+r)\, L_{t-1}^n + Z_t^n.$$

With CRS Cobb-Douglas technology, output is given by $Y_t^n(\cdot) = F(K_t, A_N N_t; \theta) = K_t^\theta (A_N N_t)^{1-\theta}$ and using $L_t^n = \frac{1+[(1-\theta)g_N + \theta g_K]}{1+r}\alpha Y_t^n(\cdot) = \lambda Y_t^n(\cdot)$ and $Z_t^n = -X_t$, we further obtain (successively):

$$C_t^n = K_t^\theta (A_N N_t)^{1-\theta} - sK_t^\theta (A_N N_t)^{1-\theta} + \frac{1 + [(1-\theta)\, g_N + \theta g_K]}{1+r}\alpha K_t^\theta (A_N N_t)^{1-\theta}$$
$$- \{1 + [(1-\theta)\, g_N + \theta g_K]\}\, \alpha K_{t-1}^\theta (A_N N_{t-1})^{1-\theta} - X_t$$

$$C_t^n = K_t^\theta (A_N N_t)^{1-\theta} - sK_t^\theta (A_N N_t)^{1-\theta} + \lambda K_t^\theta (A_N N_t)^{1-\theta} - \alpha K_t^\theta (A_N N_t)^{1-\theta} - X_t$$

$$C_t^n = K_t^\theta (A_N N_t)^{1-\theta} - sK_t^\theta (A_N N_t)^{1-\theta} + \lambda K_t^\theta (A_N N_t)^{1-\theta} - \lambda \frac{1+r}{1 + [(1-\theta)\, g_N + \theta g_K]} K_t^\theta (A_N N_t)^{1-\theta} - X_t$$

$$C_t^n = \left[1 - s + \lambda \left(1 - \frac{1+r}{1 + [(1-\theta)\, g_N + \theta g_K]}\right)\right] K_t^\theta (A_N N_t)^{1-\theta} - X_t$$

$$C_t^n = \left\{1 - s + \lambda \frac{[(1-\theta)\, g_N + \theta g_K] - r}{1 + [(1-\theta)\, g_N + \theta g_K]}\right\} K_t^\theta (A_N N_t)^{1-\theta} - X_t$$

$$C_t^n = \left\{1 - s - \lambda \frac{r - [(1-\theta)\, g_N + \theta g_K]}{1 + [(1-\theta)\, g_N + \theta g_K]}\right\} K_t^{\theta} (A_N N_t)^{1-\theta} - X_t.$$

The 'sudden stop' budget constraint (superscript s) of the private sector is given by

$$C_t^s = (1-\gamma)\, Y_t^n\,(\cdot) - \; I_t^s - (1+r)\, L_{t-1}^s + \; Z_t^s,$$

and we assume that the capital-labour ratio (or per capita capital) does not grow in sudden stops which is, by definition, BGP for g_K and g_N, and the SS for g_k, equivalent to writing:

$$k_{t+1}^s = k_t^s$$

$g_k = \frac{\triangle k_{t+1}}{k_t} = 0$ results in $s\frac{y_t}{k_t} - \delta - g_N = 0$,
that is, $s\frac{\frac{Y_t}{N_t}}{\frac{K_t}{N_t}} = \delta + g_N$

$$s\frac{Y_t}{K_t} = \delta + g_N$$

so that

$$sY_t = (\delta + g_N)\, K_t,$$

which implies, in the CD model,

$$I_t^s = (\delta + g_N) K_t.$$

Replacing investment, we can write the sudden-stop budget constraint of the private sector as

$$C_t^s = (1-\gamma)\, Y_t^n\,(\cdot) - (\delta + g_N) K_t + L_t^s - (1+r)\, L_{t-1}^s + Z_t^s.$$

With CRS Cobb-Douglas technology, output is given by $Y_t^n(\cdot) = F(K_t, A_N N_t; \theta) = K_t^{\theta} (A_N N_t)^{1-\theta}$ and using $L_t^s = 0$, $L_{t-1}^s = \frac{1+[(1-\theta) g_N + \theta g_K]}{1+r} \alpha Y_{t-1}^n(\cdot) = \lambda Y_{t-1}^n(\cdot)$ and $Z_t^s = R_t - X_t$, we further obtain (successively):

$$C_t^s = (1-\gamma)\, K_t^{\theta} (A_N N_t)^{1-\theta} - (\delta + g_N)\, K_t - (1 + [(1-\theta)\, g_N + \theta g_K])\, \alpha K_{t-1}^{\theta} (A_N N_{t-1})^{1-\theta} + R_t - X_t$$

$$C_t^s = (1-\gamma)\, K_t^{\theta} (A_N N_t)^{1-\theta} - (\delta + g_N) K_t - \alpha K_t^{\theta} (A_N N_t)^{1-\theta} + R_t - X_t$$

$$C_t^s = (1-\gamma)\, K_t^{\theta} (A_N N_t)^{1-\theta} - (\delta + g_N) K_t - \frac{1+r}{1 + [(1-\theta)\, g_N + \theta g_K]} \lambda\, K_t^{\theta} (A_N N_t)^{1-\theta} + R_t - X_t$$

$$C_t^s = \left[-\gamma - \frac{(\delta + g_N) K_t}{K_t^\theta (A_N N_t)^{1-\theta}} + 1 - \frac{1 + r}{1 + [(1-\theta)\, g_N + \theta g_K]} \lambda \right] K_t^\theta (A_N N_t)^{1-\theta} + R_t - X_t$$

$$C_t^s = \left[-\gamma - \frac{(\delta + g_N) K_t}{K_t^\theta (A_N N_t)^{1-\theta}} + \frac{1 + [(1-\theta)\, g_N + \theta g_K] - 1 - r}{1 + [(1-\theta)\, g_N + \theta g_K]} \lambda \right] K_t^\theta (A_N N_t)^{1-\theta} + R_t - X_t$$

$$C_t^s = \left[-\gamma - \frac{(\delta + g_N) K_t}{K_t^\theta (A_N N_t)^{1-\theta}} + \frac{[(1-\theta)\, g_N + \theta g_K] - r}{1 + [(1-\theta)\, g_N + \theta g_K]} \lambda \right] K_t^\theta (A_N N_t)^{1-\theta} + R_t - X_t$$

$$C_t^s = \left[-\gamma - \frac{(\delta + g_N) K_t}{K_t^\theta (A_N N_t)^{1-\theta}} - \lambda \frac{r - [(1-\theta)\, g_N + \theta g_K]}{1 + [(1-\theta)\, g_N + \theta g_K]} \right] K_t^\theta (A_N N_t)^{1-\theta} + R_t - X_t$$

Therefore, from the first-order condition (1), the optimal level of reserves as a ratio of output can be expressed as:

$$\begin{aligned} & p^{\frac{1}{\sigma}} \left\{ \left[1 - s - \lambda \frac{r - [(1-\theta)\, g_N + \theta g_K]}{1 + [(1-\theta)\, g_N + \theta g_K]} \right] K_t^\theta (A_N N_t)^{1-\theta} - X_t \right\} \\ = & \left[-\gamma - \frac{(\delta + g_N) K_t}{K_t^\theta (A_N N_t)^{1-\theta}} - \lambda \frac{r - [(1-\theta)\, g_N + \theta g_K]}{1 + [(1-\theta)\, g_N + \theta g_K]} \right] K_t^\theta (A_N N_t)^{1-\theta} + R_t - X_t \end{aligned}$$

$$\begin{gathered} p^{\frac{1}{\sigma}} \left[1 - s - \lambda \tfrac{r - [(1-\theta) g_N + \theta g_K]}{1 + [(1-\theta) g_N + \theta g_K]} \right] K_t^\theta (A_N N_t)^{1-\theta} - \left[-\gamma - \tfrac{(\delta + g_N) K_t}{K_t^\theta (A_N N_t)^{1-\theta}} - \lambda \tfrac{r - [(1-\theta) g_N + \theta g_K]}{1 + [(1-\theta) g_N + \theta g_K]} \right] \\ \times K_t^\theta (A_N N_t)^{1-\theta} = R_t - X_t + p^{\frac{1}{\sigma}} X_t \end{gathered}$$

$$\begin{aligned} & \left[p^{\frac{1}{\sigma}} (1 - s) - p^{\frac{1}{\sigma}} \lambda \tfrac{r - [(1-\theta) g_N + \theta g_K]}{1 + [(1-\theta) g_N + \theta g_K]} + \gamma + \tfrac{(\delta + g_N) K_t}{K_t^\theta (A_N N_t)^{1-\theta}} + \lambda \tfrac{r - [(1-\theta) g_N + \theta g_K]}{1 + [(1-\theta) g_N + \theta g_K]} \right] K_t^\theta (A_N N_t)^{1-\theta} \\ = & \; R_t - \left(1 - p^{\frac{1}{\sigma}} \right) X_t \end{aligned}$$

$$\begin{aligned} & \left[p^{\frac{1}{\sigma}} (1 - s) + \left(1 - p^{\frac{1}{\sigma}} \right) \lambda \frac{r - [(1-\theta)\, g_N + \theta g_K]}{1 + [(1-\theta)\, g_N + \theta g_K]} + \gamma + (\delta + g_N) \left(\frac{K_t}{A_N N_t} \right)^{1-\theta} \right] K_t^\theta (A_N N_t)^{1-\theta} \\ = & \; R_t - \left(1 - p^{\frac{1}{\sigma}} \right) \frac{\pi}{\pi + p(1 - \pi)} R_t \end{aligned}$$

$$\left[p^{\frac{1}{\sigma}}(1-s)+\left(1-p^{\frac{1}{\sigma}}\right)\lambda\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}+\gamma+(\delta+g_N)\left(\frac{k_t^s}{A_N}\right)^{1-\theta}\right]K_t^{\theta}(A_N N_t)^{1-\theta}$$
$$=\left[1-\left(1-p^{\frac{1}{\sigma}}\right)\frac{\pi}{\pi+p(1-\pi)}\right]R_t$$

And, finally, we obtain the optimal reserves-to-output ratio, ρ_{CD}^{*}, under the CD technology case in the production SOE we analyzed:

$$\rho_{CD}^{*}=\frac{R_t}{K_t^{\theta}(A_N N_t)^{1-\theta}}=\frac{\gamma+\lambda\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\left(1-p^{\frac{1}{\sigma}}\right)+p^{\frac{1}{\sigma}}(1-s)+(\delta+g_N)\left(\frac{k}{A_N}\right)^{1-\theta}}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}.$$

An alternative equivalent expression where the parameter λ appears also additively as in the original JR optimal reserves expression can be obtained, as follows.

Since

$$C_t^s=(1-\gamma)K_t^{\theta}(A_N N_t)^{1-\theta}-(\delta+g_N)K_t-\frac{1+r}{1+[(1-\theta)g_N+\theta g_K]}\lambda K_t^{\theta}(A_N N_t)^{1-\theta}+R_t-X_t,$$

we could write

$$C_t^s=\left[(1-\gamma)-(\delta+g_N)\left(\frac{K_t}{A_N N_t}\right)^{1-\theta}-\frac{1+r}{1+[(1-\theta)g_N+\theta g_K]}\lambda\right]K_t^{\theta}(A_N N_t)^{1-\theta}+R_t-X_t$$

$$C_t^s=\left[(1-\gamma)-(\delta+g_N)\left(\frac{k_t^s}{A_N}\right)^{1-\theta}-\frac{1+r+[1+(1-\theta)g_N+\theta g_K]-[1+(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\lambda\right]K_t^{\theta}(A_N N_t)^{1-\theta}+R_t-X_t$$

$$C_t^s=\left[(1-\gamma)-(\delta+g_N)\left(\frac{k}{A_N}\right)^{1-\theta}-\frac{1+[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\lambda-\frac{1+r-1-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\lambda\right]K_t^{\theta}(A_N N_t)^{1-\theta}+R_t-X_t$$

$$C_t^s=\left[(1-\gamma)-(\delta+g_N)\left(\frac{k}{A_N}\right)^{1-\theta}-\lambda-\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\lambda\right]K_t^{\theta}(A_N N_t)^{1-\theta}+R_t-X_t.$$

Then, the optimal level of reserves in terms of output can be derived as follows:

$$
\begin{aligned}
& p^{\frac{1}{\sigma}}\left\{\left[1-s-\lambda\frac{r-\left[(1-\theta)\,g_N+\theta g_K\right]}{1+\left[(1-\theta)\,g_N+\theta g_K\right]}\right]K_t^{\theta}(A_N N_t)^{1-\theta}-X_t\right\} \\
= & \left[(1-\gamma)-(\delta+g_N)\left(\frac{k}{A_N}\right)^{1-\theta}-\lambda-\frac{r-\left[(1-\theta)\,g_N+\theta g_K\right]}{1+\left[(1-\theta)\,g_N+\theta g_K\right]}\lambda\right]K_t^{\theta}(A_N N_t)^{1-\theta}+R_t-X_t
\end{aligned}
$$

$$
\begin{aligned}
& p^{\frac{1}{\sigma}}\left[1-s-\lambda\frac{r-\left[(1-\theta)\,g_N+\theta g_K\right]}{1+\left[(1-\theta)\,g_N+\theta g_K\right]}\right]K_t^{\theta}(A_N N_t)^{1-\theta}-p^{\frac{1}{\sigma}}X_t \\
= & \left[(1-\gamma)-(\delta+g_N)\left(\frac{k}{A_N}\right)^{1-\theta}-\lambda-\frac{r-\left[(1-\theta)\,g_N+\theta g_K\right]}{1+\left[(1-\theta)\,g_N+\theta g_K\right]}\lambda\right]K_t^{\theta}(A_N N_t)^{1-\theta}+R_t-X_t
\end{aligned}
$$

$$
\begin{gathered}
p^{\frac{1}{\sigma}}\left[1-s-\lambda\tfrac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\right]K_t^{\theta}(A_N N_t)^{1-\theta}-\left[(1-\gamma)-(\delta+g_N)\left(\tfrac{k}{A_N}\right)^{1-\theta}-\lambda-\tfrac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\lambda\right] \\
\times K_t^{\theta}(A_N N_t)^{1-\theta}=R_t-X_t+p^{\frac{1}{\sigma}}X_t
\end{gathered}
$$

$$
\begin{gathered}
\left[p^{\frac{1}{\sigma}}(1-s)-p^{\frac{1}{\sigma}}\lambda\tfrac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}-(1-\gamma)+\lambda+(\delta+g_N)\left(\tfrac{k}{A_N}\right)^{1-\theta}+\lambda\tfrac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\right] \\
\times K_t^{\theta}(A_N N_t)^{1-\theta}=R_t-\left(1-p^{\frac{1}{\sigma}}\right)X_t
\end{gathered}
$$

$$
\begin{aligned}
& \left[p^{\frac{1}{\sigma}}-p^{\frac{1}{\sigma}}s+\left(1-p^{\frac{1}{\sigma}}\right)\lambda\tfrac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}-1+\gamma+\lambda+(\delta+g_N)\left(\tfrac{k}{A_N}\right)^{1-\theta}\right]K_t^{\theta}(A_N N_t)^{1-\theta} \\
= & R_t-\left(1-p^{\frac{1}{\sigma}}\right)\tfrac{\pi}{\pi+p(1-\pi)}R_t
\end{aligned}
$$

$$
\begin{aligned}
& \left[-\left(1-p^{\frac{1}{\sigma}}\right)-p^{\frac{1}{\sigma}}s+\left(1-p^{\frac{1}{\sigma}}\right)\lambda\tfrac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}+\gamma+\lambda+(\delta+g_N)\left(\tfrac{k}{A_N}\right)^{1-\theta}\right]K_t^{\theta}(A_N N_t)^{1-\theta} \\
= & R_t-\left(1-p^{\frac{1}{\sigma}}\right)\tfrac{\pi}{\pi+p(1-\pi)}R_t
\end{aligned}
$$

$$
\begin{aligned}
& \left[\gamma+\lambda+(\delta+g_N)\left(\tfrac{k}{A_N}\right)^{1-\theta}-p^{\frac{1}{\sigma}}s-\left(1-p^{\frac{1}{\sigma}}\right)\left(1-\lambda\tfrac{r-(1-\theta)g_N+\theta g_K}{1+(1-\theta)g_N+\theta g_K}\right)\right]K_t^{\theta}(A_N N_t)^{1-\theta} \\
= & R_t-\left(1-p^{\frac{1}{\sigma}}\right)\tfrac{\pi}{\pi+p(1-\pi)}R_t
\end{aligned}
$$

$$\begin{aligned}&\left[\gamma+\lambda+(\delta+g_N)\left(\tfrac{k}{A_N}\right)^{1-\theta}-p^{\frac{1}{\sigma}}s-\left(1-p^{\frac{1}{\sigma}}\right)\left(1-\lambda\tfrac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\right)\right]K_t^{\theta}(A_NN_t)^{1-\theta}\\ =\;&\left[1-\left(1-p^{\frac{1}{\sigma}}\right)\tfrac{\pi}{\pi+p(1-\pi)}\right]R_t\end{aligned}$$

$$\rho_{CD}^{*}=\frac{R_t}{K_t^{\theta}(A_NN_t)^{1-\theta}}=\frac{\gamma+\lambda-\left\{1-\lambda\tfrac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\right\}\left(1-p^{\frac{1}{\sigma}}\right)+(\delta+g_N)\left(\tfrac{k}{A_N}\right)^{1-\theta}-p^{\frac{1}{\sigma}}s}{1-\tfrac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)},$$

which is equation (43) in Proposition 2 in the main text. Note that k in $\left(\frac{k}{A_N}\right)^{1-\theta}$ comes from the sudden-stop case and hence the capital-labour ratio is constant, by assumption – this is the reason for simplifying the expressions writing $\left(\frac{k}{A_N}\right)^{1-\theta}$. Note as well that g_N and g_K come from the normal-state case and therefore differ.

This completes our proof. ■

Proof of Corollary 2.

Re-writing our final expression above, i.e., equation (43) in the main text,

$$\rho_{CD}^{*}=\frac{\gamma+\lambda-\left\{1-\lambda\tfrac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\right\}\left(1-p^{\frac{1}{\sigma}}\right)+(\delta+g_N)\left(\tfrac{k}{A_N}\right)^{1-\theta}-p^{\frac{1}{\sigma}}s}{1-\tfrac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)},$$

one could also represent ρ_{CD}^{*} in relative terms to $\gamma+\lambda$, as in JR. We do so next, presenting the detailed steps, as follows.

$$\begin{aligned}&\rho_{CD}^{*}\left[1-\frac{\pi}{\pi+p\,(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]\\ =\;&\gamma+\lambda-\left\{1-\lambda\frac{r-[(1-\theta)\,g_N+\theta g_K]}{1+[(1-\theta)\,g_N+\theta g_K]}\right\}\left(1-p^{\frac{1}{\sigma}}\right)+(\delta+g_N)\left(\frac{k}{A_N}\right)^{1-\theta}-p^{\frac{1}{\sigma}}s\end{aligned}$$

$$\begin{aligned}&\rho_{CD}^{*}-\rho_{CD}^{*}\frac{\pi}{\pi+p\,(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\\ =\;&\gamma+\lambda-\left\{1-\lambda\frac{r-[(1-\theta)\,g_N+\theta g_K]}{1+[(1-\theta)\,g_N+\theta g_K]}\right\}\left(1-p^{\frac{1}{\sigma}}\right)+(\delta+g_N)\left(\frac{k}{A_N}\right)^{1-\theta}-p^{\frac{1}{\sigma}}s\end{aligned}$$

$$\gamma+\lambda-\rho_{CD}^{*}=\left\{1-\lambda\frac{r-\left[\left(1-\theta\right)g_N+\theta g_K\right]}{1+\left[\left(1-\theta\right)g_N+\theta g_K\right]}\right\}\left(1-p^{\frac{1}{\sigma}}\right)-\left(\delta+g_N\right)\left(\frac{k}{A_N}\right)^{1-\theta}$$
$$+\quad p^{\frac{1}{\sigma}}s-\rho_{CD}^{*}\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)$$

$$\gamma+\lambda-\rho_{CD}^{*}=\left\{1-\lambda\frac{r-\left[\left(1-\theta\right)g_N+\theta g_K\right]}{1+\left[\left(1-\theta\right)g_N+\theta g_K\right]}\right\}\left(1-p^{\frac{1}{\sigma}}\right)-\left(\delta+g_N\right)\left(\frac{k}{A_N}\right)^{1-\theta}+p^{\frac{1}{\sigma}}s$$
$$-\quad\frac{\gamma+\lambda-\left\{1-\lambda\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\right\}\left(1-p^{\frac{1}{\sigma}}\right)+\left(\delta+g_N\right)\left(\frac{k}{A_N}\right)^{1-\theta}-p^{\frac{1}{\sigma}}s}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)$$

$$\gamma+\lambda-\rho_{CD}^{*}$$
$$=\quad\frac{\left\{1-\lambda\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\right\}\left(1-p^{\frac{1}{\sigma}}\right)-\left(\delta+g_N\right)\left(\frac{k}{A_N}\right)^{1-\theta}+p^{\frac{1}{\sigma}}s}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}\left[1-\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)\right]$$
$$-\quad\frac{\gamma+\lambda-\left\{1-\lambda\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\right\}\left(1-p^{\frac{1}{\sigma}}\right)+\left(\delta+g_N\right)\left(\frac{k}{A_N}\right)^{1-\theta}-p^{\frac{1}{\sigma}}s}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)$$

$$\gamma+\lambda-\rho_{CD}^{*}$$
$$=\quad\frac{\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]-\left(1-p^{\frac{1}{\sigma}}\right)\lambda\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\left[1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}$$
$$\frac{-\left(\delta+g_N\right)\left(\frac{k}{A_N}\right)^{1-\theta}+p^{\frac{1}{\sigma}}s+\left(\delta+g_N\right)\left(\frac{k}{A_N}\right)^{1-\theta}\left[\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]-p^{\frac{1}{\sigma}}s\left[\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}$$
$$\frac{-\left(\gamma+\lambda\right)\left[\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]+\left(1-p^{\frac{1}{\sigma}}\right)\left\{1-\lambda\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\right\}\left[\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}$$
$$\frac{-\left(\delta+g_N\right)\left(\frac{k}{A_N}\right)^{1-\theta}\left[\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]+p^{\frac{1}{\sigma}}s\left[\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}$$

Simplify the numerator above, as below:

$$
\begin{aligned}
& \left(1-p^{\frac{1}{\sigma}}\right)\left[1-\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)\right] \\
- & \left(1-p^{\frac{1}{\sigma}}\right)\lambda\frac{r-\left[\left(1-\theta\right)g_N+\theta g_K\right]}{1+\left[\left(1-\theta\right)g_N+\theta g_K\right]}\left[1-\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)\right]-\left(\delta+g_N\right)\left(\frac{k}{A_N}\right)^{1-\theta}+p^{\frac{1}{\sigma}}s \\
- & \left(\gamma+\lambda\right)\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right) \\
+ & \left(1-p^{\frac{1}{\sigma}}\right)\left[1-\lambda\frac{r-\left[\left(1-\theta\right)g_N+\theta g_K\right]}{1+\left[\left(1-\theta\right)g_N+\theta g_K\right]}\right]\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)
\end{aligned}
$$

Using the simplification above, re-write the full equation, as below:

$$
\begin{aligned}
& \gamma+\lambda-\rho_{CD}^{*} \\
= & \frac{\begin{array}{c}\left(1-p^{\frac{1}{\sigma}}\right)\left\{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)-\lambda\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\left[1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]\right. \\ \left.-\left(\gamma+\lambda\right)\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)+\left[1-\lambda\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\right]\left[\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)\right]\right\}\end{array}}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)} \\
& -\frac{\left(\delta+g_N\right)\left(\frac{k}{A_N}\right)^{1-\theta}-p^{\frac{1}{\sigma}}s}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}
\end{aligned}
$$

Now simplify the term in the curly brackets above, as below:

$$
\begin{aligned}
& 1-\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)-\lambda\frac{r-\left[\left(1-\theta\right)g_N+\theta g_K\right]}{1+\left[\left(1-\theta\right)g_N+\theta g_K\right]}\left[1-\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)\right] \\
- & \left(\gamma+\lambda\right)\frac{\pi}{\pi+p\left(1-\pi\right)}+\left[1-\lambda\frac{r-\left[\left(1-\theta\right)g_N+\theta g_K\right]}{1+\left[\left(1-\theta\right)g_N+\theta g_K\right]}\right]\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \ 1-\left(\gamma+\lambda\right)\frac{\pi}{\pi+p\left(1-\pi\right)}-\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right) \\
- & \ \lambda\frac{r-\left[\left(1-\theta\right)g_N+\theta g_K\right]}{1+\left[\left(1-\theta\right)g_N+\theta g_K\right]}\left[1-\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)\right] \\
+ & \ \frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)-\lambda\frac{r-\left[\left(1-\theta\right)g_N+\theta g_K\right]}{1+\left[\left(1-\theta\right)g_N+\theta g_K\right]}\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \ 1-\left(\gamma+\lambda\right)\frac{\pi}{\pi+p\left(1-\pi\right)}-\lambda\frac{r-\left[\left(1-\theta\right)g_N+\theta g_K\right]}{1+\left[\left(1-\theta\right)g_N+\theta g_K\right]}\left[1-\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)\right] \\
& -\lambda\frac{r-\left[\left(1-\theta\right)g_N+\theta g_K\right]}{1+\left[\left(1-\theta\right)g_N+\theta g_K\right]}\frac{\pi}{\pi+p\left(1-\pi\right)}\left(1-p^{\frac{1}{\sigma}}\right)
\end{aligned}
$$

$$= \quad 1-(\gamma+\lambda)\frac{\pi}{\pi+p(1-\pi)}-\lambda\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}+\lambda\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)$$
$$-\lambda\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)$$

Using the simplification above, re-write the full equation, as below:

$$\gamma+\lambda-\rho^*_{CD}=\frac{\left(1-p^{\frac{1}{\sigma}}\right)\left[1-(\gamma+\lambda)\frac{\pi}{\pi+p(1-\pi)}-\lambda\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\right]-(\delta+g_N)\left(\frac{k}{A_N}\right)^{1-\theta}+p^{\frac{1}{\sigma}}s}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}$$

$$=\frac{\left(1-p^{\frac{1}{\sigma}}\right)\left[1-(\gamma+\lambda)\frac{\pi}{\pi+p(1-\pi)}-\lambda\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}-(\gamma+\lambda)+(\gamma+\lambda)\right]-(\delta+g_N)\left(\frac{k}{A_N}\right)^{1-\theta}+p^{\frac{1}{\sigma}}s}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}$$

$$=\frac{\left(1-p^{\frac{1}{\sigma}}\right)\left[1+(\gamma+\lambda)\left(1-\frac{\pi}{\pi+p(1-\pi)}\right)-\lambda\left(1+\frac{r-[(1-\theta)g_N+\theta g_K]}{1+[(1-\theta)g_N+\theta g_K]}\right)-\gamma\right]-(\delta+g_N)\left(\frac{k}{A_N}\right)^{1-\theta}+p^{\frac{1}{\sigma}}s}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}$$

$$=\frac{\left(1-p^{\frac{1}{\sigma}}\right)\left[1+(\gamma+\lambda)\frac{p(1-\pi)}{\pi+p(1-\pi)}-\lambda\left(\frac{1+r}{1+[(1-\theta)g_N+\theta g_K]}\right)-\gamma\right]-(\delta+g_N)\left(\frac{k}{A_N}\right)^{1-\theta}+p^{\frac{1}{\sigma}}s}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}$$

$$\gamma+\lambda-\rho^*_{CD}=\frac{\left(1-p^{\frac{1}{\sigma}}\right)\left[1+(\gamma+\lambda)\frac{p(1-\pi)}{\pi+p(1-\pi)}-\alpha-\gamma\right]-(\delta+g_N)\left(\frac{k}{A_N}\right)^{1-\theta}+p^{\frac{1}{\sigma}}s}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)}$$

$$\gamma+\lambda-\rho^*_{CD}=\frac{\left(1-p^{\frac{1}{\sigma}}\right)\left[1-\alpha-\gamma+(\gamma+\lambda)\frac{p(1-\pi)}{\pi+p(1-\pi)}\right]-(\delta+g_N)\left(\frac{k}{A_N}\right)^{1-\theta}+p^{\frac{1}{\sigma}}s}{1-\frac{\pi}{\pi+p(1-\pi)}\left(1-p^{\frac{1}{\sigma}}\right)},$$

which is equation (44) in Corollary 2 in the main text.

This completes our proof. ■